BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(57), 2008, Pages 24–34
ISSN 1024–7696

A closed form asymptotic solution for the FitzHugh-Nagumo model

Adelina Georgescu, Gheorghe Nistor,
Marin-Nicolae Popescu, Dinel Popa

Abstract. By means of a change of unknown function and independent variable, the Cauchy problem of singular perturbation from electrophysiology, known as the FitzHugh-Nagumo model, is reduced to a regular perturbation problem (Section 1). Then, by applying the regular perturbation technique to the last problem and using an existence, uniqueness and asymptotic behavior theorem of the second and third author, the models of asymptotic approximation of an arbitrary order are deduced (Section 2). The closed-form expressions for the solution of the model of first order asymptotic approximation and for the time along the phase trajectories are derived in Section 3. In Section 4, by applying several times the method of variation of coefficients and prime integrals, the closed-form solution of the model of second order asymptotic approximation is found. The results from this paper served to the author to study (elsewhere) the relaxation oscillations versus the oscillations in two and three times corresponding to concave limit cycles (canards).

Mathematics subject classification: 34E99, 34A45.
Keywords and phrases: Asymptotic solution, singular perturbation, FitzHugh-Nagumo model, electrophysiology.

1 Mathematical model

The FitzHugh-Nagumo (F-N) model is the Cauchy problem $x(0) = x^0$, $y(0) = y^0$ for the system of ordinary differential equations

$$\frac{dx}{dt_1} = c\left(x + y - x^3/3\right), \quad \frac{dy}{dt_1} = -\left(x + by - a\right)/c,$$

where $x, y : \mathbb{R} \to \mathbb{R}$, $x = x(t1)$, $y = y(t1)$ are the state functions, $t1$, the time, stands for the independent variable, and $a, b, c \in \mathbb{R}$ are three real parameters. For asymptotically fixed a, b and c, the dynamics generated by this model and its changes with respect to the parameters, i.e. the static, dynamic and perturbed bifurcation, were investigated analytically in several papers among which we quote [1, 2] and numerically, by the methods from [3, 4].

The present study continues these investigations with the asymptotic behavior of the phase portrait of the N-S model as $\mu = c^{-2} \to 0$, when a and b remain asymptotically fixed. For $\mu \neq 0$ the F-N model is a singular perturbation problem, which by the change $(x, y, t_1, c) \to (z, y, t, \mu)$, $z = x$, $y = y$, $t = t1/c$, $\mu = c^{-2}$, reads

$$\mu \frac{dz}{dt} = z + y - z^3/3, \ \frac{dy}{dt} = -z - by + a, \ z(0, \mu) = z^0, \ y(0, \mu) = y^0. \tag{1}$$

The problem (1) is a particular case of the singular perturbation problem

$$\mu \frac{dz}{dt} = F(z, y, \mu), \quad \frac{dy}{dt} = f(z, y, \mu), \quad z(0) = z^0, \quad y(0) = y^0, \tag{2}$$

where z^0 and y^0 are asymptotically fixed. Problems of type (2) were intensively studied by methods of classical qualitative theory of ordinary differential equations by the school of A.N. Tikhonov and A.B. Vasil'eva. They used the boundary layer functions method, which, in applications leads to cumbersome computations. This is why we preferred another way, namely to reduce (1) to a regular perturbation problem, which is developed in the following.

By means of the transform $(z, y) \leftrightarrow (\eta, \varsigma)$, $\varsigma = z + y - z^3/3$, $\eta = z$, the inverse of which reads $y = \varsigma - \eta + \frac{1}{3}\eta^3$, $z = \eta$ and using the chain rule (of differentiation of composite functions)

$$\frac{dz}{dt} = \frac{\partial z}{\partial \varsigma} \cdot \frac{d\varsigma}{dt} + \frac{\partial z}{\partial \eta} \cdot \frac{d\eta}{dt} = \frac{d\eta}{dt}, \quad \frac{dy}{dt} = \frac{\partial y}{\partial \varsigma} \cdot \frac{d\varsigma}{dt} + \frac{\partial y}{\partial \eta} \cdot \frac{d\eta}{dt} = \frac{d\varsigma}{dt} + \left(-1 + \eta^2\right) \frac{d\eta}{dt},$$

problem (1) becomes

$$\begin{cases} \mu \dfrac{d\eta}{dt} = \varsigma, \\[2ex] \dfrac{d\varsigma}{dt} + \left(-1 + \eta^2\right) \dfrac{d\eta}{dt} = -\eta + a - b\left(\varsigma - \eta + \dfrac{1}{3}\eta^3\right), \end{cases}$$

$$\begin{cases} \varsigma(0, \mu) = z^0 + y^0 - \dfrac{1}{3}\left(z^0\right)^3 \equiv \varsigma^0, \\ \eta(0, \mu) = z^0 \equiv \eta^0, \end{cases}$$

or, equivalently,

$$\begin{cases} \mu \cdot \dfrac{d\eta}{dt} = \varsigma, \\[2ex] \mu \cdot \dfrac{d\varsigma}{dt} = a\mu + (-1 + b)\mu\eta + (1 - \mu b)\varsigma - \dfrac{1}{3} b\mu\eta^3 - \varsigma\eta^2, \end{cases} \quad \begin{cases} \varsigma(0, \mu) = \varsigma^0, \\[2ex] \eta(0, \mu) = \eta^0. \end{cases} \tag{3}$$

By means of the change of variable $t = \mu\tau$ and taking into account that $\frac{d}{dt} = \frac{1}{\mu} \cdot \frac{d}{d\tau}$, the singular perturbation problem (3) becomes the problem of regular perturbations

$$\begin{cases} \dfrac{d\eta}{d\tau} = \varsigma, \\[2ex] \dfrac{d\varsigma}{d\tau} = a\mu - (1 - b)\mu\eta + (1 - \mu b)\varsigma - \dfrac{1}{3} b\mu\eta^3 - \varsigma\eta^2, \end{cases} \quad \begin{cases} \varsigma(0, \mu) = \xi^0, \\[2ex] \eta(0, \mu) = \eta^0. \end{cases} \tag{4}$$

2 Models of asymptotic approximation

The use of the time recalling $\tau = t/\mu$ may be embarrassing: it is appropriate to inner asymptotic approximations for singular perturbation two-point problems or to singular perturbation Cauchy problems which possess one asymptotic boundary layer or asymptotic initial layer respectively. The new time τ is the inner independent variable. In these cases t is assumed to be small, namely of order of μ as $\mu \to 0$. For larger t and τ, the inner component of the asymptotic solution looses its importance. In problems of the type (2) there is an infinity of interval layers as t is increased beyond 0.

Therefore $\tau = t/\mu$ can be large as t is other very small but $t >> \mu$, or $t >> 1$. This means that our study is appropriate to t larger than the order used to stretch the boundary or to initial layers. On the other hand, we expect that the "inner" component (i.e. corresponding to τ) of the asymptotic solution be important as τ is increased, because if crosses other and other interval layers. In other words, we expect that the problem involving τ has an "outer" role too, i.e. it takes the role of the problem (1) in t. As far as small τ is concerned, the problem in τ plays the role of a genuine inner problem, corresponding to $t << \mu$ or $t \sim \mu$. These are the reasons for suspecting that (4) is a good approximation of (2) for every t, irrespective of its order. The numerical results based on (4) confirmed this assumption.

Further we use a convenient variant of one (unpublished) result of the second and third authors.

Theorem 1. *Assume that in the Cauchy problem*

$$\frac{dx}{dt} = f\left(t, x, \mu\right), x\left(t_0\right) = x^0, \tag{5}$$

the function $f\left(t, x, \mu\right)$*is continuous and satisfies in* x *(uniformly in* t *and* μ*) the condition Lipschitz on the domain*

$$0 \equiv \left\{(t, x, \mu) \mid \ |t - t_0| \leq a, \ \left|\left|x - x^0\right|\right| \ \leq b, \ |\mu - \mu_0| \leq 0\right\}.$$

Then:

I. 1) there exists a unique continuous solution $x\left(t, \mu\right)$ *of problem (5) on the compact* $\left[t_0, t_0 + H\right] \times \left[\mu_0 - c, \mu_0 + c\right]$, *where* $H = \min\left\{a, \dfrac{b}{\mu}\right\}$, $M = \max\limits_{0} \left|f\left(t, x, \mu\right)\right|$;

2) the solution $x\left(t, \mu_0\right)$ *is defined on* $\left[t_0, t_0 + H_0\right]$, *where* $H_0 = \min\left\{a, \dfrac{b}{M_0}\right\}$, $M_0 = \max\limits_{0} \left|f\left(t, x, \mu_0\right)\right|$. *There exists* $c_0 \in (0, c)$ *such that* $x\left(t, \mu\right)$ *is defined on* $\left[t_0, t_0 + H_0\right] \times \left[\mu_0 - c_{01}\mu_0 - c_0\right]$ *and* $\lim\limits_{\mu \to \mu_0} x\left(t, \mu\right) = x\left(t, \mu_0\right)$, *uniformly in* $t \in \left[t_0, t_0 + H_0\right]$;

II. 1) if, in addition, exist and are continuous f_x, f_μ *on* D, *then there exists the derivative* $\dfrac{\partial x}{\partial \mu}\left(t, \mu\right)$,*denoted by* X, *which is differentiable with respect to* t, *and it is the solution of the Cauchy problem*

$$\frac{dX}{dt} = f_x\left(t, x\left(t, \mu\right), \mu\right), \ X + f_\mu\left(t, x\left(t, \mu\right), \mu\right), \ X\left(t_0\right) = 0. \tag{6}$$

If, in addition, f possesses bounded partial derivatives up to the order $n+1$ in x and μ, then

$$x\left(t, \mu\right) = x\left(t, \mu_0\right) + \left(\mu - \mu_0\right) \frac{\partial x}{\partial \mu}\left(t, \mu_0\right) + \cdots + \frac{\left(\mu - \mu_0\right)^n}{n!} \frac{\partial^n x}{\partial \mu^n}\left(t, \mu_0\right) + \varepsilon_{n+1}\left(t, \mu\right) \tag{7}$$

where $\varepsilon_{n+1}(t,\mu) = O\left(|\mu-\mu_0|^{n+1}\right)$.

Let us use this theorem by denoting $\dfrac{\partial^k x}{\partial \mu^k}(t,\mu_0)$ by $x_k(t)$ and replacing x in (5) by (7) we obtain

$$\begin{aligned}&\frac{d}{dt}\left[x_0(t) + (\mu-\mu_0)x_1(t) + \cdots + \frac{(\mu-\mu_0)^n}{n!}x_n(t) + \varepsilon_{n+1}(t,\mu)\right] = \\ &= f\left(t, x_0(t) + (\mu-\mu_0)x_1(t) + \cdots + \frac{(\mu-\mu_0)^n}{n!}x_n(t), \mu\right) + \varepsilon_{n+1}(t,\mu)\end{aligned} \tag{8}$$

and

$$x_0(t_1) + (\mu-\mu_0)x_1(t_0) + ... + \frac{(\mu-\mu_0)}{n!}x_n(t_0) + \varepsilon_{n+1}(t_0,\mu) = x^0. \tag{9}$$

From (8) and (9), by matching, we deduce the problems satisfied by $x_k(t)$, $k = \overline{0,n}$, (they are the models of regular asymptotic approximation of order k), namely: from (8) for $\mu = \mu_0$, we obtain

$$\frac{dx_0}{dt} = f(t, x_0, \mu_0), \quad x_0(t_0) = x^0;$$

differentiating (8) with respect to μ and taking $\mu = \mu_0$ it follows

$$\frac{dx_1}{dt} = \frac{\partial t}{\partial x}(t, x_0, \mu_0)x_1 + \frac{\partial f}{\partial \mu}(t, x_0, \mu_0), \quad x_1(t_0) = 0;$$

differentiating (8) two times with respect to μ and taking $\mu = \mu_0$, we have

$$\begin{aligned}\frac{dx_2}{dt} &= \frac{\partial f}{\partial x}(t, x_0, \mu_0)x_2 + \frac{\partial^2 f}{\partial x^2}(t, x_0, \mu_0)(x_1, x_1) + \\ &+ \frac{\partial^2 f}{\partial x \partial \mu}(t, x_0, \mu_0)x_1 + \frac{\partial^2 f}{\partial \mu^2}(t, x_0, \mu_0), \quad x_2(t_0) = 0\end{aligned}$$

and so on.

Since, by Theorem 1, the vector field associated with problem (4) is analytic with respect to μ at $\mu = \mu_0 = 0$, the solution of (4) possesses converging series of powers of μ

$$\eta(\tau,\mu) = \sum_{k\geq 0} \frac{\mu^k}{k!}\frac{\partial^k \eta}{\partial \mu^k}(\tau,0), \qquad \varsigma(\tau,\mu) = \textstyle\sum_{k\geq 0} \frac{\mu^k}{k!}\frac{\partial^k \varsigma}{\partial \mu^k}(\tau,0). \tag{10}$$

Denoting $\eta_k(\tau) = \dfrac{\partial^k \eta}{\partial \mu^k}(\tau,0)$, $\varsigma_k(\tau) = \dfrac{\partial^k \varsigma}{\partial \mu^k}(\tau,0)$ and introducing (10) in (4) it follows the Cauchy problem

$$\begin{cases} \dfrac{d}{d\tau}\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\eta_k(\tau)\right] = \sum_{k\geq 0}\dfrac{\mu^k}{k!}\varsigma_k(\tau), \\ \dfrac{d}{d\tau}\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\varsigma_k(\tau)\right] = a\mu - (1-b)\mu\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\eta_k(\tau)\right] + \\ +(1-\mu b)\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\varsigma_k(\tau)\right] - \dfrac{1}{3}b\mu\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\eta_k(\tau)\right]^3 - \\ \qquad -\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\varsigma_k(\tau)\right]\left[\sum_{k\geq 0}\dfrac{\mu^k}{k!}\eta_k(\tau)\right]^2, \end{cases} \tag{11}$$

$$\sum_{k\geq 0}\frac{\mu^k}{k!}\eta_k(0) = \eta^0, \quad \sum_{k\geq 0}\frac{\mu^k}{k!}\varsigma_k(0) = \varsigma^0, \tag{12}$$

whence, by matching, from (12) the models of asymptotic approximation of order k are immediately deduced.

The model of the first approximation reads

$$\begin{cases} \dfrac{d\eta_0}{d\tau} = \varsigma_0, \\ \dfrac{d\varsigma_0}{d\tau} = \varsigma_0 - \varsigma_0\eta_0^2, \end{cases} \qquad \begin{cases} \eta_0(0) = \eta^0, \\ \varsigma_0(0) = \varsigma^0. \end{cases} \tag{13}$$

Like in any regular perturbation problem, (13) could be, formally, deduced by letting $\mu = 0$ in (12). The model of the second order asymptotic approximation (which, formally, follows by taking $\mu = 0$ in the derivative of (12) 1,2 with respect to μ) is

$$\begin{cases} \dfrac{d\eta_1}{d\tau} = \varsigma_1, \\ \dfrac{d\varsigma_1}{d\tau} = a - (1-b)\eta_0 - \varsigma_0 + \varsigma_1 - \dfrac{1}{3}b\eta_0^3 - \varsigma_1\eta_0^2 - \varsigma_0 2\eta_0\eta_1, \end{cases} \qquad \begin{cases} \eta_1(0) = 0, \\ \varsigma_1(0) = 0. \end{cases} \tag{14}$$

Similarly, differentiating (12) 1,2 two times with respect to μ and taking $\mu = 0$ in the obtained equation, we obtain the model of the second order asymptotic approximation

$$\begin{cases} \dfrac{d\eta_2}{d\tau} = \varsigma_2, \\ \dfrac{d\varsigma_2}{d\tau} = -2(1-b)\varsigma_1 - 2b\varsigma_1 + \varsigma_2 - \dfrac{2}{3}b\varsigma\eta_0^2\eta_1 - \varsigma_2\eta_0^2 - 2\varsigma_1\varsigma\eta_0\eta_1 - 2\varsigma_0(\eta_2\eta_1 + \eta_1\eta_1), \end{cases}$$

$$\eta_2(0) = 0, \ \varsigma_2(0) = 0. \tag{15}$$

Apparently, the equations (13) 1,2 are simpler in form than (14) 1,2 and (15) 1,2 but they have cubic nonlinearities. In addition the conditions (13) 3 are nonhomogenous. The equations in (14) and (15) look more complicated but, in fact, they are affine and the associated conditions (14) 3, (15) 3 are homogenous. Hence the model (13) is the most difficult to be solved, at least in principle.

Proposition 1. *As $\mu \to 0$, the limit cycle of the dynamical system associated with the model* (13) *contains two parallel straightlines $y = y_0$.*

Indeed, (13) implies

$$\frac{d}{d\tau}\left(\varsigma_0 - \eta_0 + \eta_0^3/3\right) = 0, \tag{16}$$

therefore $\varsigma_0 - \eta_0 + \eta_0^3/3 {=} \varsigma_0^0 - \eta_0^0 - \eta_0^3/3$, i.e. $y_0 = y^0$. This shows that as $\mu \to 0$, some portions of the trajectory limit cycle, are straightlines. In Section 3 we show that they are situated between the two external (stable) branches of the infinitecline $\varsigma + \eta - \eta^3/3 = 0$ (written, equivalently, as $\varsigma = 0$).

3 Model of the first asymptotic approximation

The dynamical system associated with (13) has an infinity of equilibria; their locus is the y_0 − axis. The closed-form or (algebraically) implicit form of each nonconstant solution of (13) can be found immediately by eliminating ς_0 between the equations of (13). We obtain

$$\frac{d^2\eta_0}{d\tau^2} = \left(1 - \eta_0^2\right)\frac{d\eta_0}{d\tau}, \quad \eta_0|_{\tau=0} = \eta^0, \quad \frac{d\eta_0}{d\tau}|_{\tau=0} = \varsigma^0,$$

or, equivalently, denoting (only in Section 3 and 4) $c = \varsigma^0 - \eta^0 + \dfrac{\eta^{03}}{3}$, integrating this equation and taking into account the initial conditions, it follows

$$\frac{d\eta_0}{d\tau} = \eta_0 - \frac{\eta_0^3}{3} + c, \quad \eta_0|_{\tau=0} = \eta^0. \tag{17}$$

Since, by (16) $\varsigma^0 = 0$ implies $\eta_0 = \eta^0$, i.e. $\left(\varsigma^0, \eta^0\right)$ corresponds to a point $\left(z^0, y^0\right)$ situated on the infinitecline *y=0*, we assume *ς0 ≠0.* Let us also remark that $c = y^0$.

Case $c = 0$. The solution of (17) is

$$\eta_0(\tau) = \frac{\sqrt{3}\eta^0 e^\tau}{\sqrt{|\eta^{02}e^{2\tau} + 3 - \eta^{02}|}}, \tag{18}$$

and from (13)1, it follows

$$\varsigma_0(\tau) = \frac{\sqrt{3}\eta^0\left(3 - \eta^{02}\right)e^\tau}{|\eta^{02}e^{2\tau} + 3 - \eta^{02}|^{3/2}}, \quad \text{with} \quad \varsigma^0 = \eta^0 - \frac{\eta^{03}}{3} \neq 0. \tag{19}$$

The relations (18) and (19) represent the parametic form of the solutions of (13). Whence, as expected by (16), the closed form

$$\varsigma_0 = \eta_0 - \frac{\eta_0^3}{3}, \tag{20}$$

or, coming back to the phase functions y and z, we have

$$\begin{cases} y(t) = 0, \\ z(t) = \dfrac{z^0\sqrt{3}e^{t/\mu}}{\sqrt{z^0 e^{2t/\mu} + 3 - z^{02}}}, \end{cases} \qquad z^0 \neq 0, \pm\sqrt{3}.$$

The form (20) shows that, in the case $c=0$, the trajectory starting at a point of the z-axis is a portion of that axis, namely that one for which $z^0 \neq 0, \pm\sqrt{3}$ and $e^{2t/\mu} \neq \left(z^{02}-3\right)/z^0$.

Case $c \neq 0$. Equation $\eta_0^3 - 3\eta_0 - 3c = 0$, defining the equilibria of (17) has the discriminant $\Delta = -1 + \dfrac{9c^2}{4}$. Hence, for it has three real mutually distinct roots $\eta_{01},;\ \eta_{02}\,;\eta_{03}$, for $\Delta = 0$ it has a double root and a simple root (a triple root is not possible, because it should be null, whereas $c \neq 0$); for $\Delta > 0$, there exists a unique real root η_{00}.

Subcase $\Delta < 0$. Take by convention $\eta_{01} < \eta_{02} < \eta_{03}$. From (17) it follows the (implicit, algebraic) form of its solution

$$\frac{\left|\eta_0 - \eta_{01}\right|^A \ \left|\eta_0 - \eta_{03}\right|^D}{\left|\eta_0 - \eta_{02}\right|^{-B}} = ke^{-\tau/\varepsilon}, \tag{21}$$

or, equivalently, the closed form of τ as a function of η_0 , where

$$A = \frac{1}{\left(\eta_{01} - \eta_{02}\right)\left(\eta_{01} - \eta_{03}\right)} > 0, \quad B = \frac{1}{\left(\eta_{02} - \eta_{01}\right)\left(\eta_{02} - \eta_{03}\right)} < 0,$$

$$D = \frac{1}{\left(\eta_{03} - \eta_{01}\right)\left(\eta_{03} - \eta_{02}\right)} > 0$$

and k is a constant obtained by taking $\tau = 0$ in (21). Since, by (16) $\varsigma_0 - \eta_0 + \dfrac{\eta_0^3}{3} = c$ is a prime integral it follow that ς_0 is a function of η_0. Consequently, it is sufficient to determine only η_0 (because ς_0 follows).

Subcase $\Delta > 0$. Equation (17) is equivalent (in the class of nonconstant solutions) to anyone among the following three forms

$$\frac{d\eta_0}{d\tau} = \eta_0 - \frac{\eta_0^3}{3} + c, \qquad \frac{d\eta_0}{c + \eta_0 - \dfrac{\eta_0^3}{3}} = d\tau, \qquad \int_{\eta^0}^{\eta_0} \frac{d\eta_0}{c + \eta_0 - \dfrac{\eta_0^3}{3}} = \tau - \tau_0.$$

In this way, by expressing τ as a function of η_0, we obtain the closed-form solution. Taking into account the decomposition in simple fractions

$$\frac{1}{\eta_0 + c - \frac{1}{3}\eta_0^3} = \frac{1}{\eta_{00}^2 - 1}\left[\frac{1}{\eta_0 - \eta_{00}} + \frac{\frac{1}{3}\eta_0 + \frac{2}{3}\eta_{00}}{\frac{1}{3}\eta_0^2 + \frac{1}{3}\eta_{00}\eta_0 + \left(\frac{1}{3}\eta_{00}^2 - 1\right)}\right]$$

the solution of (17) becomes successively

$$\tau = \int_{\eta^0}^{\eta_0} \frac{d\eta_0}{\eta_0 + c - \frac{1}{3}\eta_0^3} =$$

$$= \frac{1}{\eta_{00}^2 - 1}\int_{\eta^0}^{\eta_0}\left[\frac{1}{\eta_0 - \eta_{00}} + \frac{1}{2}\frac{\dfrac{2}{3}\eta_0 + \dfrac{1}{3}\eta_{00}}{\dfrac{1}{3}\eta_0^2 + \dfrac{1}{3}\eta_{00}\eta_0 + \left(\dfrac{1}{3}\eta_{00}^2 - 1\right)} + \right.$$

$$\left. + \frac{1}{3}\left(\frac{1}{\dfrac{1}{3}\left(\eta_0 + \dfrac{1}{2}\eta_{00}\right)^2 + \dfrac{1}{4}\eta_{00}^2 - 1}\right)\right] = \frac{1}{\eta_{00}^2 - 1}\ \left[\ln\left[\frac{\eta_0 - \eta_{00}}{\eta^0 - \eta_{00}}\right] + \right.$$

$$+\frac{1}{2}\ln\left|\frac{\frac{1}{3}\eta_0^2+\frac{1}{3}\eta_{00}\eta_0+\frac{1}{3}\eta_0^2-1}{\frac{1}{3}\left(\eta^0\right)^2+\frac{1}{3}\eta_{00}\eta^0+\frac{1}{3}\eta_{00}^2-1}\right|-$$

$$-\frac{3\eta_{00}}{2\sqrt{3\left(\frac{1}{4}\eta_{00}^2-1\right)}}\left(arctg\frac{\left(\eta^0+\frac{1}{2}\eta_{00}\right)}{\sqrt{3\left(\frac{1}{4}\eta_{00}^2-1\right)}}\right)-arctg\frac{\left(\eta_0+\frac{1}{2}\eta_{00}\right)}{\sqrt{3\left(\frac{1}{4}\eta_{00}^2-1\right)}}\Bigg)\Bigg].$$

Subcase $\mathbf{\Delta=0}$. If $\Delta=0$ and the equation $\eta_0+c-\frac{\eta_0^3}{3}=0$ has a simple root η_{01} and a double root η_{02}, hence $\eta_0+c-\frac{\eta_0^3}{3}=\frac{-1}{3}\left(\eta_0-\eta_{01}\right)\left(\eta_0-\eta_{02}\right)^2$, then

$$\frac{1}{\eta_0+c-\frac{1}{3}\eta_0^3}=-\frac{3}{\left(\eta_{01}-\eta_{02}\right)^2}\cdot\frac{1}{\eta_0-\eta_{01}}\cdot\frac{-3}{\left(\eta_{01}-\eta_{02}\right)^2}\cdot\frac{\eta_0-\eta_{01}+2\eta_{02}}{\left(\eta_0-\eta_{02}\right)^2},$$

implying the closed-form solution τ as a function of η_0

$$\tau=\int_{\eta^0}^{\eta_0}\frac{d\eta_0}{\eta_0+c-\frac{1}{3}\eta_0^3}=\int_{\eta^0}^{\eta_0}\frac{-3}{\left(\eta_{01}-\eta_{02}\right)^2}\cdot\frac{1}{\eta_0-\eta_{01}}d\eta_0-$$

$$-\frac{3}{\left(\eta_{01}-\eta_{02}\right)^2}\int_{\eta^0}^{\eta_0}\left[\frac{1}{\eta_0-\eta_{02}}+\frac{-\eta_{01}+3\eta_{02}}{\left(\eta_0-\eta_{02}\right)^2}\right]d\eta_0=$$

$$=-\frac{3}{\left(\eta_{01}-\eta_{02}\right)^2}\left[\ln\frac{\eta^0-\eta_{01}}{\eta_0-\eta_{01}}+\ln\left[\frac{\eta^0-\eta_{02}}{\eta_0-\eta_{02}}\right]-\right.$$

$$\left.-\left(-\eta_{01}+3\eta_{02}\right)\cdot\left(\frac{1}{\eta_0-\eta_{02}}-\frac{1}{\eta^2-\eta_{02}}\right)\right].$$

4 Model of the second order asymptotic approximation

The system (14), can be successively written as

$$\frac{d^2\eta_1}{d\tau^2}=a-\eta_0-b\left(\varsigma_0-\eta_0+\frac{\eta_0^3}{3}\right)+\varsigma_1\left(1-\eta_0^2\right)-2\varsigma_0\eta_0\eta_1=$$

$$=a-\eta_0-bc+\left(1-\eta_0^2\right)\frac{d\eta_1}{d\tau}-2\eta_0\frac{d\eta_0}{d\tau}\eta_1=$$

$$=a-\eta_0-bc+\left(1-\eta_0^2\right)\frac{d\eta_1}{d\tau}-\eta_1\frac{d\eta_0^2}{d\tau}=$$

$$=a-\eta_0-bc+\left(1-\eta_0^2\right)\frac{d\eta_1}{d\tau}+\eta_1\frac{d\left(1-\eta_0^2\right)}{d\tau},$$

hence

$$\frac{d^2\eta_1}{d\tau^2} = a - \eta_0 - bc + \frac{d\left[\eta_1\left(1-\eta_0^2\right)\right]}{d\tau}. \tag{22}$$

Further we solve the Cauchy problem for this equation by applying several times the method of variation of coefficients. Thus, the linear equation corresponding to (22) is

$$\frac{d^2\eta_1}{d\tau^2} = \frac{d\left[\eta_1\left(1-\eta_0^2\right)\right]}{d\tau}, \tag{23}$$

which can be written as

$$\frac{d\eta_1}{d\tau} = \eta_1\left(1-\eta_0^2\right) + C_1, \tag{24}$$

where C_1 is an arbitrary constant. The linear equation corresponding to (24) is

$$\frac{d\eta_1}{d\tau} = \eta_1\left(1-\eta_0^2\right),$$

whence $\ln|\eta_1| = \int\left(1-\eta_0^2\right)d\tau = \int\frac{d\varsigma_0}{d\tau}\cdot\frac{1}{\varsigma_0}d\tau = \int\frac{d\varsigma_0}{\varsigma_0} = \ln|\varsigma_0| + C_2$, i.e. $\eta_1 = K_2\varsigma_0$. Then, the method of variation of coefficients applied to (24), where C_1 is a constant and K_2 is a function of τ, implies $K_2(\tau) = \int\frac{C_1 d\tau}{\varsigma_0} + C_3$, where C_3 is a constant. Therefore, $\eta_1(\tau) = \varsigma_0\left[C_1\int\frac{d\tau}{\varsigma_0} + C_3\right]$ is the general solution of (23). In order to find the solution of (22) we apply again the method of variation of coefficients to find

$$\frac{d\eta_1}{d\tau} = \frac{d\varsigma_0}{d\tau}\left[C_1\int\frac{d\tau}{\varsigma_0} + C_3\right] + \varsigma_0\left[\frac{dC_1}{d\tau}\int\frac{d\tau}{\varsigma_0} + \frac{dC_3}{d\tau} + \frac{C_1}{\varsigma_0}\right] =$$

$$= \frac{d\varsigma_0}{d\tau}\left[C_1\int\frac{d\tau}{\varsigma_0} + C_3\right] + \frac{dC_1}{d\tau}\varsigma_0\int\frac{d\tau}{\varsigma_0} + \varsigma_0\frac{dC_3}{d\tau} + C_1 =$$

$$= \varsigma_0\left(1-\eta_0^2\right)\left[C_1\int\frac{d\tau}{\varsigma_0} + C_3\right] + C_1,$$

and impose

$$\varsigma_0\frac{dC_3}{d\tau} + \varsigma_0\frac{dC_1}{d\tau}\int\frac{d\tau}{y_0} = 0, \tag{25}$$

$$\begin{aligned}\frac{d^2\eta_1}{d\tau^2} &= \frac{d}{d\tau}\left\{\varsigma_0\left(1-\eta_0^2\right)\left[C_1\int\frac{d\tau}{\varsigma_0} + C_3\right]\right\} + \frac{dC_1}{d\tau} = \\ &= \frac{d}{d\tau}\left[\eta_1\left(1-\eta_0^2\right)\right] + \frac{dC_1}{d\tau}\left[1 + \left(1-\eta_0^2\right)y_0\int\frac{d\tau}{y_0}\right] + y_0\left(1-\eta_0^2\right)\cdot\frac{dC_3}{d\tau} = \\ &= \frac{d}{d\tau}\left(1-\eta_0^2\right) + a - \eta_0 - bc.\end{aligned} \tag{26}$$

Hence from (25) and (26), we have successively $\frac{dC_1}{d\tau} = a - \eta_0 - bc$, i.e. $\frac{dC_1}{d\eta_0}\cdot\frac{d\eta_0}{d\tau} = a - \eta_0 - bc$, therefore $\frac{dC_1}{d\eta_0} = \frac{a-\eta_0-bc}{\varsigma_0}$, which implies $\frac{dC_1}{d\eta_0} = \frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}}$, whence $C_1(\eta_0) =$

$\psi(\eta_0) + C_{10}$, where $\psi(\eta_0)$ is an integral which can be computed immediately, for instance by using some formulae from [5]. Then the relation (25) reads

$$\frac{dC_3}{d\eta_0}\frac{d\eta_0}{d\tau} = -\frac{dC_1}{d\eta_0}\frac{d\eta_0}{d\tau}\int\left(c+\eta_0-\frac{\eta_0^3}{3}\right)^{-2} d\eta_0$$

and, by integration with respect to η_0, we have

$$C_3(\eta_0) - C_3 = -\int\left\{\frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}}\int\frac{d\eta_0}{\left(c+\eta_0-\frac{\eta_0^3}{3}\right)^2}\right\}d\eta_0 \ ,$$

i.e. $C_3(\eta_0) = \varphi(\eta_0) + C_{30}$, where $\varphi(\eta_0)$ is an integral which can be computed by using appropriate formulae from [5]. Finally, η_1 reads

$$\eta_1(\tau) = y_0\left[(\psi(\eta_0) + C_{10})\int\frac{d\tau}{y_0} + \varphi(\eta_0) + C_{30}\right], \qquad (27)$$

where C_{10} and C_{30} are constants related to y_0^0 and η_0^0. Thus η_1 and y_1 are completely determined.

An alternative procedure is: from (14) it follows

$$\frac{d\varsigma_1}{d\eta_0} = \frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}} + \frac{d}{d\eta_0}\left[\eta_1\left(1-\eta_0^2\right)\right],$$

hence

$$\frac{d}{d\eta_0}\left[\varsigma_1 - \eta_1\left(1-\eta_0^2\right)\right] = \frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}},$$

whence the prime integral

$$\varsigma_1 = \eta_1\left(1-\eta_0^2\right) + \int\frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}}d\eta_0 + K,$$

is a computation of η_1 and y_1 in terms of τ.

In order to determine ς_1 and η_1 as functions of η_0, we use (14) to obtain

$$\frac{d\eta_1}{d\eta_0}\cdot\frac{d\eta_0}{d\tau} = \eta_1\left(1-\eta_0^2\right) + \int\frac{a-\eta_0-bc}{c+\eta_0-\frac{\eta_0^3}{3}}d\eta_0 + K,$$

$$\frac{d\eta_1}{d\eta_0} = \eta_1\frac{\left(1-\eta_0^2\right)}{c+\eta_0-\frac{\eta_0^3}{3}} + \frac{1}{c-\eta_0-\frac{\eta_0^3}{3}}\cdot\int\frac{a-\eta_0-bC}{c+\eta_0-\frac{\eta_0^3}{3}}d\eta_0 + \frac{K}{c+\eta_0-\frac{\eta_0^3}{3}},$$

The last equation (affine in η_1) can be solved by the method of the variation of coefficients. This ends the determination of the closed-form of the solution of model (14), also by the alternative procedure.

References

[1] Rocşoreanu C., Georgescu A., Giurgiuţeanu N. *The FitzHugh-Nagumo model. Bifurcation and dynamics*. Mathematical Modelling: Theory and Applications, **10**, Kluwer, Dordrecht, 2000.

[2] Sterpu M. *Dinamică şi bifurcaţii pentru două modele Van der Pol generelizate*. Seria MatematicĂplicată şi Industrială **8**, Ed. Univ. Piteşti, 2001.

[3] Giurgiuţeanu N. *Contribuţii la studiul sistemelor de ecuaţii diferenţiale prin metode numerice. Aplicaţii la biologie*. Teză de Doctorat, Institutul de matematicăl Academiei Române, Bucureşti, 1997.

[4] Bazavan P. *Numerical algorithms in the study of dynamical systems*. Sitech, Craiova, 2005.

[5] Rîjîc I.M., Gradstein I.S. *Tabele de integrale, sume, serii şi produse*. Ed. Tehnică, Bucureşti, 1955.

Adelina Georgescu
University of Pitesti
E-mail: *adelinageorgescu@yahoo.com*

Received January 10, 2008