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Ideal Theory in Commutative Semirings

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Abstract. In this paper, we analyze some results on ideal theory of commutative semirings with non-zero identity analogues to commutative rings with non-zero identity. Here we will make an intensive examination of the notions of Noetherian semirings, Artinian semirings, local semirings and strongly irreducible ideals in commutative semirings. It is shown that this notion inherits most of essential properties of strongly irreducible ideals of a commutative rings with non-zero identity. Also, the relationship among the families of primary ideals, irreducible ideals and strongly irreducible ideals of a semiring R is considered.

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1 Introduction

The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many authors (see, for example, [1–3, 7, 12]). Semirings constitute a fairly natural generalization of rings, with broad applications in the mathematical foundations of computer science. The main part of this paper is devoted to stating and proving analogues to several well-known results in the theory of rings (see Sections 2 and 3).

For the sake of completeness, we state some definitions and notations used throughout. By a commutative semiring, we mean a commutative semigroup $(R, .)$ and a commutative monoid $(R, +, 0)$ in which 0 is the additive identity and $r.0 = 0.r = 0$ for all $r \in R$, both are connected by ring-like distributivity. In this paper, all semirings considered will be assumed to be commutative semirings with non-zero identity.

A subset I of a semiring R will be called an ideal if $a, b \in I$ and $r \in R$ implies $a + b \in I$ and $ra \in I$. A subtractive ideal ($= k$-ideal) I is an ideal such that if $x, x + y \in I$ then $y \in I$ (so $\{0\}$ is a k-ideal of R). The k-closure $\mathrm{cl}(I)$ of I is defined by $\mathrm{cl}(I) = \{a \in R : a + c = d \text{ for some } c, d \in I\}$ is an ideal of R satisfying $I \subseteq \mathrm{cl}(I)$ and $\mathrm{cl}(\mathrm{cl}(I)) = \mathrm{cl}(I)$. So an ideal I of R is a k-ideal if and only if $I = \mathrm{cl}(I)$. A prime ideal of R is a proper ideal P of R in which $x \in P$ or $y \in P$ whenever $xy \in P$. So P is prime if and only if A and B are ideals in R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$ where $AB = \{ab : a \in A \text{ and } b \in B\} \subseteq A \cap B$ (see [3, Theorem 5]). A primary ideal P of R is a proper ideal of R such that, if $xy \in P$ and $x \notin P$, then $y^n \in P$ for some positive integer n. If I is an ideal of R, then the radical of I,denoted by $\mathrm{rad}(I)$, is the set of all $x \in R$ for which $x^n \in I$ for some positive integer n. This is an ideal of R, containing I, and if $1 \in R$ is the intersection of all the prime ideals of R that contain I [1]. A proper ideal I of R is said to be maximal (resp. k-maximal) if J is an ideal (resp. k-ideal) in R such that $I \subsetneqq J$, then $J = R$. An ideal I of a semiring R is strongly irreducible if for ideals J and K of R, the inclusion $J \cap K \subseteq I$ implies that either $J \subseteq I$ or $K \subseteq I$. An ideal of R is said to be irreducible if I is not the intersection of two ideals of R that properly contain it. We say that R is a Noetherian (resp. Artinian) if any non-empty set of k-ideals of R has a maximal (resp. minimal) member with respect to set inclusion. This definition is equivalent to the ascending chain condition (resp. descending chain condition) on k-ideals of R.

2 Local semirings

Let R be a semiring with non-zero identity. A non-zero element a of R is said to be semi-unit in R if there exist $r, s \in R$ such that $1 + ra = sa$. R is said to be a local semiring if and only if R has a unique maximal k-ideal.

Lemma 1. *Let I be a k-ideal of a semiring R. Then the following hold:*
(i) If a is a semi-unit element of R with $a \in I$, then $I = R$.
(ii) If $x \in R$, then $\mathrm{cl}(Rx)$ is a k-ideal of R.

Proof. (i) Is clear. To see (ii), let $y, x + y \in \mathrm{cl}(Rx)$; we show that $x \in \mathrm{cl}(Rx)$. There are elements a, b, c and d of Rx such that $x + y + a = b$ and $y + c = d$. It follows that $x + a + d = b + c$, as needed. □

Lemma 2. *Let R be a semiring with $1 \neq 0$. Then R has at least one k-maximal ideal.*

Proof. Since $\{0\}$ is a proper k-ideal of R, the set Δ of all proper k-ideals of R is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on Δ, and by using Zorn's Lemma to this partially ordered set, a maximal k-ideal of R is just a maximal member of the partially ordered set $(\Delta, \subseteq)$. □

An ideal I of a semiring R is called a partitioning ideal ($= Q$-ideal) if there exists a subset Q of R such that $R = \cup\{q + I : q \in Q\}$ and if $q_1, q_2 \in Q$ then $(q_1 + I) \cap (q_2 + I) \neq \emptyset$ if and only if $q_1 = q_2$. Let I be a Q-ideal of a semiring R and let $R/I = \{q + I : q \in Q\}$. Then R/I forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 + I) \oplus (q_2 + I) = q_3 + I$, where $q_3 \in Q$ is the unique element such that $q_1 + q_2 + I \subseteq q_3 + I$. $(q_1 + I) \odot (q_2 + I) = q_4 + I$, where $q_4 \in Q$ is the unique element such that $q_1q_2 + I \subseteq q_4 + I$. This semiring R/I is called the quotient semiring of R by I. By definition of Q-ideal, there exists a unique $q_0 \in Q$ such that $0 + I \subseteq q_0 + I$. Then $q_0 + I$ is a zero element of R/I. Clearly, if R is commutative , then so is R/I (see [7, 8]). The next result should be compared with [12, Corollary 2.2].

Theorem 1. *Let I be a proper Q-ideal of a semiring R. Then there exists a maximal k-ideal M of R with $I \subseteq M$.*

Proof. Since R/I is non-trivial, and so, by Lemma 2, has a k-maximal ideal L, which, by [5, Theorem 2.3], will have to have the form M/I for some k-ideal M of R with $I \subseteq M$. It now follows from [5, Theorem 2.14] that M is a k-maximal ideal of R. □

Lemma 3. *Let R be a semiring and let $a \in R$. Then a is a semi-unit of R if and only if a lies outside each k-maximal ideal of R.*

Proof. By Lemma 1 (i), a is a semi-unit of R if and only if $R = \mathrm{cl}Ra$. First, Suppose that a is a semi-unite of R and let $a \in M$ for some maximal k-ideal ideal M of R. Then we should have $Ra \subseteq M \subsetneqq R$, so that a could not be a semi-unit of R. Conversely, if a were not a semi-unit of R, then $1 + ra = sa$ holds for no $r, s \in R$. Hence, $1 \notin \mathrm{cl}(Ra)$ yields that $\mathrm{cl}(Ra)$ is a proper k-ideal of R by Lemma 1 (ii). By [12, Corollary 2.2], $\mathrm{cl}(Ra) \subseteq J$ for some maximal k-ideal J of R; but this would contradict the fact that a lies outside each maximal k-ideal of R. □

Theorem 2. *Let R be a semiring. Then R is a local semiring if and only if the set of non-semi-unit elements of R is a k-ideal.*

Proof. Assume that R is a local semiring with unique maximal k-ideal P. By Lemma 3, P is precisely the set of non-semi-unit elements of R. Conversely, assume that the set of non-semi-units of R is a k-ideal I of R (so $I \neq R$ since 1 is a semi-unit of R). Since R is not trivial, it has at least one maximal k-ideal: let J be one such. By Lemma 3, J consists of non-semi-units of R, and so $J \subseteq I \subsetneqq R$. Thus $I = J$ since J is k-maximal. We have thus shown that R has at least one maximal k-ideal, and for any maximal k-ideal of R must be equal to I. □

Let R be a given semiring, and let S be the set of all multiplicatively cancellable elements of R (so $1 \in S$). Define a relation $\sim$ on $R \times S$ as follows: for $(a, s), (b, t) \in R \times S$, we write $(a, s) \sim (b, t)$ if and only if $ad = bc$. Then $\sim$ is an equivalence relation on $R \times S$. For $(a, s) \in R \times S$, denote the equivalence class of $\sim$ which contains (a, s) by a/s, and denote the set of all equivalence classes of $\sim$ by R_S. Then R_S can be given the structure of a commutative semiring under oprerations for which $a/s + b/t = (ta + sb)/st$, $(a/s)(b/t) = (ab)/st$ for all $a, b \in R$ and $s, t \in S$. This new semiring R_S is called the semiring of fractions of R with respect to S; its zero element is $0/1$, its multiplicative identity element is $1/1$ and each element of S has a multiplicative inverse in R_S (see [10, 11]).

Thoughout this paper we shall assume unless otherwise stated, that **S** is the set of all multiplicatively cancellable elements of a semiring R. Now suppose that I be an ideal of a semiring R. The ideal generated by I in R_S, that is, the set of all finite sums $s_1 a_1 + ..., s_n a_n$ where $r_i \in R_S$ and $s_i \in I$, is called the extention of I to R_S, and it is denoted by IR_S. Again, if J is an ideal of R_S then by the contraction of J in R we mean $J \cap R = \{r \in R : r/1 \in J\}$, which is clearly an ideal of R.

We need the following lemma proved in [6, Lemma 2.3].

Lemma 4. *Assume that I, J and K are ideals of a semiring R and let L be an ideal of semiring R_S. Then the following hold:*

(i) $x \in IR_S$ if and only if it can be written in the form $x = a/c$ for some $a \in I$ and $c \in S$.

(ii) $(L \cap R)R_S = L$.

(iii) $(I \cap J)R_S = (IR_S) \cap (JR_S)$.

Lemma 5. *Let I be a k-ideal of a semiring R. Then IR_S is a k-ideal of R_S.*

Proof. Suppose that $a/s, a/s + b/t \in IR_S$; we show that $b/t \in IR_S$. By Lemma 4, there are elements $c, d \in I$ and $u, v \in S$ such that $(at+bs)/st = c/u$ and $a/t = d/w$, so $atuw + bsuw = t^2ud + sbuw = cstw \in I$; hence $bsuw \in I$ since I is a k-ideal. It follows that $b/t = (bsuw)/(tsuw) \in IR_S$, as required. □

Theorem 3. *Let R be a local semiring with unique maximal k-ideal P such that $S \cap P = \emptyset$. Then R_S is a local semiring with unique maximal k-ideal of PR_S.*

Proof. By Lemma 5 and Theorem 2, it is enough to show that PR_S is exactly the set of non-semi-units of R_S. Let $z \in R_S - PR_S$, and take any representation $z = a/s$ with $a \in R$, $s \in S$. We must have $a \notin P$, so $1 + ra = sa$ for some $r, s \in R$ by Lemma 3. It then follows from $1/1 + ((rs)/1)(a/s) = (s^2/1)(a/s)$ that a/s is a semi-unit of R_S. On the other hand, if y is a semi-unit of R_S, and $y = b/t$ for some $b \in R$, $t \in S$, then there exist $c, d \in R$ and $u, w \in S$ such that $1/1 + (bc)/(tu) = (bd)/(tw)$. It follows that $t^2uw + bctw = tubc$; hence $b \notin P$ since P is a k-ideal, and since this reasoning applies to every representation $y = b/t$ with $b \in R$, $t \in S$, of y as a formal fraction, it follows that $y \notin PR_S$, and so the proof is complete. □

For the remainder of this section we turn our attention to study some essential properties of Noetherian and Artinian semirings.

Lemma 6. *If R is a Noetherian semiring, then every proper k-ideal is a finite intersection of irreducible k-ideals.*

Proof. The proof is trivial. □

Proposition 1. *Let R be a Noetherian semiring and let I be an irreducible k-ideal of R. Then I is primary.*

Proof. The proof is straightforward (see [13, Proposition 4.34]). □

Theorem 4. *If R is a Noetherian semiring, then every proper k-ideal is a finite intersection of primary k-ideals.*

Proof. This follows from Lemma 6 and Proposition 1. $\square$

Let R be a semiring. R is called cancellative if whenever $ac = ab$ for some elements a, b and c of R with $a \neq 0$, then $b = c$. Also, we define the Jacobson radical of R, denoted by $\mathrm{Jac}(R)$, to be the intersection of all the maximal k-ideals of R. Then by [12, Corollary 2.2], the Jacobson radical of R always exists and by [5, Lemma 2.12], it is a k-ideal of R.

Theorem 5. *Let R be an Artinian cancellative semiring. Then the following hold:*

(i) Every prime k-ideal of R is k-maximal.

(ii) $\mathrm{Jac}(R) = \mathrm{rad}(0)$ *(the nilradical of R).*

Proof. (i) Assume that I is a prime k-ideal of R and let $I \subsetneqq J$ for some k-ideal J of R; we show that $J = R$. There is an element $x \in J$ with $x \notin I$. Then by Lemma 1, $\mathrm{cl}(Rx) \supseteq \mathrm{cl}(Rx^2) \supseteq ...$ is a descending chain of k-ideals of R, so $\mathrm{cl}(Rx^n) = \mathrm{cl}(Rx^{n+1})$ for some n; hence $x^n + rx^{n+1} = sx^{n+1}$ for some $r, s \in R$. Since R is cancellative and $x \neq 0$, it follows that we may cancel x^n, hence $1 + rx = sx$. Hence x is a semi-unit in J, and therefore $J = R$ by Lemma 1. (ii) follows from (i). $\square$

Lemma 7. *Let R be a semiring. Then the following hold:*

(i) Let $P_1, ..., P_n$ be prime k-ideals and let I be an ideal of R contained in $\bigcup_{i=1}^n P_i$. Then $I \subseteq P_i$ for some i.

(ii) Let $I_1, ..., I_n$ be ideals and let P be a prime ideal containing $\bigcap_{i=1}^n I_i$. Then $I_i \subseteq P$ for some i. If $P = \bigcap_{i=1}^n$, then $P = I_i$ for some i.

Proof. (i) The proof is straightforward by induction on n (see [13, Theorem 3.61]).

(ii) Suppose $I_i \nsubseteq P$ for all i. Then there exist $x_i \in I_i$, $x_i \notin P$ $(1 \leq i \leq n)$, and therefore $x_1x_2...x_n \in I_1I_2...I_n \subseteq \bigcap_{i=1}^n I_i$; but $x_1x_2...x_n \notin P$; hence $\bigcap_{i=1}^n I_i \nsubseteq P$ which is a contradiction. Finally, if $P = \bigcap_{i=1}^n I_i$, then $I_i \subseteq P$ and hence $P = I_i$ for some i. $\square$

Theorem 6. *An Artinian cancellative semiring has only a finite number of maximal k-ideals.*

Proof. Consider the set of all finite intersections $P_1 \cap ... \cap P_n$, where the P_i are maximal k-ideals (note that an intersection of a family of k-ideals of R is a k-ideal by [5, Lemma 2.12]). This set has a minimal element, say $Q_1 \cap ... \cap Q_s$; hence for any maximal k-ideal Q we have $Q \cap Q_1 \cap ... \cap Q_s = Q_1 \cap ... \cap Q_s$, and therefore $Q_1 \cap ... \cap Q_s \subseteq Q$. By Lemma 7, $Q_i \subseteq Q$ for some i, hence $Q = Q_i$ since Q_i is k-maximal, as required. $\square$

3 Strongly irreducible ideals

In this section we list some basic properties concerning strongly irreducible ideals. The results of this section should be compared with [9].

Theorem 7. *Let I be an ideal of a semiring R. Then the following hold:*

(i) If I is strongly irreducible, then I is irreducible.

(ii) If R is Noetherian and I is a strongly irreducible k-ideal of R, then I is primary.

Proof. (i) Assume that I is strongly irreducible and let J and K be ideals of R such that $J \cap K = I$. Then $J \cap K \subseteq I$, so either $J \subseteq I$ or $K \subseteq I$, and it then follows that either $I = J$ or $I = K$, so I is irreducible.

(ii) This follows from (i) and Proposition 1. □

Proposition 2. *Let I be an ideal of a semiring R. Then the following hold:*

(i) To show that I is strongly irreducible, it suffices to show that if Ra and Rb are cyclic ideals of R such that $Ra \cap Rb \subseteq I$, then either $a \in I$ or $b \in I$.

(ii) If I is a prime ideal of R, then I is strongly irreducible.

Proof. (i) Let J and K be ideals of R such that $J \cap K \subseteq I$; we show that either $J \subseteq I$ or $k \subseteq I$. Suppose $J \nsubseteq I$. Then there exists $a \in J$ such that $a \notin I$. Then for all $b \in K$ it holds $Ra \cap Rb \subseteq J \cap K \subseteq I$, so $b \in I$, as required.

(ii) Assume that I is prime and let J and K be ideals of R such that $J \cap K \subseteq I$. Since $IJ \subseteq I \cap J \subseteq I$, I prime gives either $J \subseteq I$ or $K \subseteq I$, as needed. □

Lemma 8. *Let I be a Q-ideal of a semiring R. If J, K and L are k-ideals of R containing I, then $(J/I) \cap (K/I) = L/I$ if and only if $J \cap K = L$.*

Proof. Suppose that $(J/I) \cap (K/I) = L/I$; we show that $J \cap K = L$. Let $x \in J \cap K$. Then there exist $q_1 \in Q$ and $a \in I$ such that $x = q_1 + a$, so $q_1 \in Q \cap J$ and $q_1 \in Q \cap K$ since J, K are k-ideals; hence $q_1 + I \in (J/I) \cap (K/I) = L/I$ by [5, Proposition 2.2]. Therefore, $q_1 \in L$; thus $x \in L$ since L is a k-ideal. So, $J \cap K \subseteq L$. Now suppose that $z \in L$. Then $z = q_2 + b$ for some $q_2 \in Q$ and $b \in I$. It follows that $q_2 + I \in L/I = (J/I) \cap (K/I)$, so $q_2 \in K \cap J$; hence $z \in K \cap J$. Thus $L = J \cap K$. The other implication is similar. □

Theorem 8. *Let R be a semiring, I a Q-ideal of R and J a strongly irreducible k-ideal of R with $I \subseteq J$. Then J/I is a strongly irreducible ideal of R/I.*

Proof. Let N and M be ideals of R/I such that $N \cap M \subseteq J/I$. Then there are k-ideals K, H of R such that $N = K/I$ and $M = H/I$ by [5, Theorem 2.3]; hence Lemma 8 gives $K \cap H \subseteq J$. Since J is strongly irreducible it follows that either $K \subseteq J$ or $H \subseteq J$; hence either $N = K/I \subseteq J/I$ or $M = H/I \subseteq J/I$ by [5, Lemma 2.13 (ii)]. So J/I is strongly irreducible. □

Lemma 9. *Let I be a primary ideal of a semiring R with $\mathrm{rad}(I) = P$. Then the following hold:*

(i) IR_S is a primary ideal of R_S.

(ii) If $P \cap S = \emptyset$, then $IR_S \cap R = I$.

(iii) If $P \cap S = \emptyset$, then $(IR_S :_{R_S} PR_S) = (I :_R P)R_S$.

(iv) If $P \cap S = \emptyset$ and J is an ideal of R such that $JR_S \subseteq IR_S$, then $J \subseteq I$.

Proof. (i) Let $a/s, b/t \in R_S$, where $a, b \in R$, $s, t \in S$, be such that $(ab)/(st) \in IR_S$ but $a/s \notin IR_S$. Then there exist $e \in I$ and $z \in S$ such that $abz = ste \in I$ but $az \notin I$ (otherwise, $a/s = (az)/(zs) \in IR_S$); hence I primary gives $b^n \in I$ for some positive integer n. It follows that $b^n/t^n = (b/t)^n \in IR_s$, as needed.

(ii) Clearly, $I \subseteq IR_S \cap R$. Let $a \in IR_S \cap R$. Then there are elements $b \in I$ and $s \in S$ such that $a/1 = b/s$, so $as = b \in I$; hence I primary gives $a \in I$, as required.

(iii) Clearly, $(I :_R P)R_S \subseteq (IR_S :_{R_S} PR_S)$. For the other direction, let $a/s \in (IR_S :_{R_S} PR_S)$, where $a \in R$, $s \in S$. It suffices to show that $ab \in I$ for every $b \in P$. There are elements $c \in I$ and $t \in S$ such that $(a/s)(b/1) = (ab)/s = c/t$, so $abt = sc \in I$ but $t \notin P$; hence $ab \in I$, and so the proof is complete.

(iv) Let $a \in J$. As $a/1 \in JR_S$, there are elements $b \in I$ and $t \in S$ such that $a/1 = b/t$, so $at = b \in I$; hence I primary gives $a \in I$. Thus $J \subseteq I$. □

Proposition 3. *Let I be an ideal of a semiring R. If IR_S is a strongly irreducible ideal of R_S, then $IR_S \cap R$ is an irreducible ideal of R.*

Proof. Assume that IR_S is strongly irreducible and let J and K be ideals of R such that $J \cap K \subseteq IR_S \cap R$. Then $JR_S \cap KR_S \subseteq IR_S$ by Lemma 4. It follows that either $JR_S \subseteq IR_S$ or $KR_S \subseteq IR_S$, so either $J \subseteq IR_S \cap R$ or $K \subseteq IR_S \cap R$, as required. □

Theorem 9. *Let I be a strongly irreducible primary ideal of a semiring R such that $\mathrm{Rad}(I) \cap S = \emptyset$. Then IR_S is strongly irreducible.*

Proof. Assume that I is a strongly irreducible primary ideal of R and let H and G be ideals of R_S such that $H \cap G \subseteq IR_S$. Then $(H \cap R) \cap (G \cap R) \subseteq IR_S \cap R = I$ by Lemma 9. So either $H \cap R \subseteq I$ or $G \cap R \subseteq I$ since I is strongly irreducible. Therefore it follows that either $G = (G \cap R)R_S \subseteq IR_S$ or $H = (H \cap R)R_S \subseteq IR_S$ by Lemma 4, and hence IR_S is strongly irreducible. □

Theorem 10. *Assume that I is a primary ideal of a semiring R with $\mathrm{Rad}(I) \cap S = \emptyset$ and let IR_S be strongly irreducible ideal of R_S. Then I is strongly irreducible.*

Proof. By Lemma 9 and Proposition 3, $IR_S \cap R = I$ is a strongly irreducible ideal of R. □

Lemma 10. *Assume that R is a semiring and let $a, b \in R$. Then the following hold:*

(i) $Ra \cap Rb = (Ra : Rb)b = (Rb : Ra)a$. Moreover, if I is an ideal of R such that $I \subseteq Ra$, then $I = (I : Ra)a$.

(ii) $\mathrm{cl}(Ra) \cap Rb = (\mathrm{cl}(Ra) : Rb)b$.

(iii) If I is a k-ideal of R such that $I \subseteq Ra$, then $I = (I : \mathrm{cl}(Ra))a$.

Proof. (i) Clearly, $(Ra : Rb)b \subseteq Ra \cap Rb$. For the other direction, if $z \in Ra \cap Rb$, then $z = ra = sb$ for some $r, s \in R$. It is clear that $s \in (Ra : Rb)$; hence $z \in (Ra : Rb)b$. By symmetry it follows that $Ra \cap Rb = (Rb : Ra)a$. For the last statement, assume that I is an ideal of R such that $I \subseteq Ra$. Then it is clear that $(I : Ra)a \subseteq I$, and if $x \in I \subseteq Ra$, then $x = ta$ for some $t \in R$, so $t \in (I : Ra)$; thus $x = ta \in (I : Ra)a$, as required.

(ii) Since the inclusion $(\mathrm{cl}(Ra) : Rb)b \subseteq (\mathrm{cl}(Ra) \cap Rb$ is clear, we will prove the reverse inclusion. Let $y \in \mathrm{cl}(Ra) \cap Rb$. Then there are elements r, s and t of R such that $y + ra = sa$ and $y = tb$, so $tb \in \mathrm{cl}(Ra)$ gives $t \in (\mathrm{cl}(Ra) : Rb)$. Therefore, we must have $y \in (\mathrm{cl}(Ra) : Rb)b$.

(iii) It is clear that $(I : \mathrm{cl}(Ra))a \subseteq I$. For the other containment, assume that $z \in I \subseteq Ra$, so $z = ra$ for some $r \in R$. Let $c \in \mathrm{cl}(Ra)$. Then $c + ta = sa$, so $rc + rta = rsa$; hence $rc \in I$ since I is a k-ideal. It follows that $r \in (I : \mathrm{cl}(Ra))$; hence $z \in (I : \mathrm{cl}(Ra))a$, as needed. □

Theorem 11. *Let R be a local semiring with unique maximal k-ideal P and let I be a strongly irreducible P-primary k-ideal in R. Assume that $I \subsetneqq (I : P)$. Then the following hold:*

(i) $(I : P) = \mathrm{cl}(Rx)$ for some $x \in R$.

(ii) For each k-ideal J in R either $J \subseteq I$ or $(I : P) \subseteq J$.

Proof. (i) By hypothesis, $I \subsetneqq (I : P)$, so there exists $x \in (I : P) - I$; we show that $(I : P) = \mathrm{cl}(Rx)$. Suppose not. Let $y \in (I : P) - \mathrm{cl}(Rx)$. Since $(\mathrm{cl}(Rx) : Ry) \neq R$, [12, Corollary 2.2], [4, Lemma 2.1], Lemma 1 and Lemma 10 gives $\mathrm{cl}(Rx) \cap Ry = (\mathrm{cl}(Rx) : Ry)y \subseteq yP \subseteq I$. However, I strongly irreducible implies that either $\mathrm{cl}(Rx) \subseteq I$ or $Ry \subseteq I$, hence $y \in I$. Therefore, $(I : P) \subseteq I \cup \mathrm{cl}(Rx)$. For the other direction, it suffices to show that $\mathrm{cl}(Rx) \subseteq (I : P)$. Let $d \in \mathrm{cl}(Rx)$. Then $d + tx = ux$ for some $u, t \in R$, so $d \in (I : P)$ since $x \in (I : P)$ and $(I : P)$ is a k-ideals by [4, Lemma 2.1]. Thus, $(I : P) = I \cup \mathrm{cl}(Rx)$. Since by [4, Lemma 2.2], if an ideal is the union of two k-ideals, then it is equal to one of them, we must have $(I : P) \subseteq \mathrm{cl}(Rx)$ or $(I : P) \subseteq I$ which is a contradiction. Therefore, $(I : P) = \mathrm{cl}(Rx)$, so (i) holds.

(ii) It may clearly be assumed that $J \nsubseteq I$, so it remains to show that $\mathrm{cl}(Rx) = (I : P) \subseteq J$; that is, that $x \in J$ since J is a k-ideal. For this, if $x \notin J$, then let $b \in J$, so $x \notin Rb$ and $(Rb : Rx) \neq R$. Therefore [12, Corollary 2.2] and Lemma 10 gives $Rx \cap Rb = (Rb : Rx)x \subseteq xP \subseteq I$, hence $Rb \subseteq I$ since I is a strongly irreducible ideal and $Rx \nsubseteq I$. Since this holds for each $b \in J$, it follows that $J \subseteq I$, and this is a contradiction. Therefore $x \in J$, hence (ii) holds. □

Corollary 1. *Let I be a strongly irreducible k-ideal in a local Noetherian semiring R with the unique maximal k-ideal of $\mathrm{rad}(I) = P$, and assume that $I \neq P$ and $P \cap S = \emptyset$. Then the following hold:*

(i) $(IR_S :_R PR_S)R_S = \mathrm{cl}(R_S X)$ for some $X \in R_S$.

(ii) For each k-ideal L in R_S either $L \subseteq IR_S$ or $(IR_S :_{R_S} PR_S) \subseteq L$.

Proof. By Theorem 3, R_S is a local semiring with unique maximal k-ideal PR_S and by Lemma 5, Lemma 9 and Theorem 9, IR_S is a PR_S-primariy strongly irreducible k-ideal of R_S. So (i) and (ii) follows immediately from Theorem 11. □

Proposition 4. *Let I be a strongly irreducible k-ideal in a local Noetherian semiring R with the unique maximal k-ideal of* $\mathrm{rad}(I) = P$*, and assume that $I \neq P$ and $P \cap S = \emptyset$. Then the following hold:*

(i) $(I :_R P)R_S = \mathrm{cl}(R_S X)$ for some $X \in R_S$.

(ii) For each k-ideal J in R either $J \subseteq I$ or $(I : P)R_S \subseteq JR_S$.

Proof. This follows from Lemma 9 and Corollary 1. □

Proposition 5. *Let R be a local semiring with unique maximal k-ideal P and let I be a strongly irreducible P-primary k-ideal of R with $P \neq I$. Then I and $(I : P)$ are comparable (under containment) to all ideals in R; in fact, $I = \bigcup\{J : J \text{ is a } k\text{-ideal in } R \text{ and } J \subsetneqq (I : P)\}$ and $(I : P) = \bigcap\{J : J \text{ is a } k\text{-ideal in } R \text{ and } I \subsetneqq J\}$.*

Proof. As $(I : P)$ is a k-ideal of R, we must have $(I : P) = \bigcap\{J : J \text{ is a k-ideal in R and } I \subsetneqq J\}$ by Theorem 11 (ii). Also, if J is a k-ideal in R such that $J \subsetneqq (I : P)$, then $(I : P) \nsubseteq J$, so $J \subseteq I$ by Proposition 4 (ii); hence $I = \bigcup\{J : J \text{ is a k-ideal in R and } J \subsetneqq (I : P)\}$ since I is a k-ideal. □

Theorem 12. *Let I be an ideal of a local Noetherian semiring R with* $\mathrm{rad}(I) \cap S = \emptyset$. *Then I is a non-prime strongly irreducible k-ideal if and only if there exist k-ideals J and P of R such that $I \subsetneqq J \subseteq P$ and: (1) P is prime; (2) I is P-primary; and, (3) for all k-ideals L in R either $L \subseteq I$ or $JR_S \subseteq LR_S$. Also if this holds, then $JR_S = (IR_S :_{R_S} PR_S)$. In particular, a local Noetherian semiring R contains a non-prime strongly irreducible k-ideal if and only if there exists a k-ideal of R satisfying these conditions.*

Proof. Proposition 4 and Proposition 5 gives a non-prime strongly irreducible k-ideal in a local Notherian semiring satisfies the stated conditions. For the converse, assume that I is a P-primary k-ideal of R. By Theorem 10, it suffices to show that IR_S is strongly irreducible. Let L and T be ideals in R_S such that $L \cap T \subseteq IR_S$. If $L \nsubseteq IR_S$ and $T \nsubseteq IR_S$, then it follows from Lemma 4 and Lemma 9 that $(L \cap R) \nsubseteq I$ and $(T \cap R) \nsubseteq I$. By assumption, $IR_S \subseteq JR_S \subseteq T \cap L$, and this is a contradiction. Therefore, either $L \subseteq IR_S$ or $T \subseteq IR_S$, hence IR_S is strongly irreducible.

Finally, the ideal JR_S is clearly uniquely determined by the properties (1) $IR_S \subsetneqq JR_S \subseteq PR_S$ and (2) for all k-ideals L in R_S either $L \subseteq IR_S$ or $JR_S \subseteq L$. Since $(IR_S : PR_S)$ also has these properties by Corollary 1 (ii), $JR_S = (IR_S : PR_S)$. □

Theorem 13. *Let I be an irreducible P-primary k-ideal over a local Noetherian semiring R with the unique maximal k-ideal of P, and assume that $I \neq P$ and $P \cap S = \emptyset$. Then I is strongly irreducible if and only if I comparable to all k-ideals of R.*

Proof. By Corollary 1 and Proposition 4, it is enough to show that an irreducible ideal that is comparable to all k-ideals in R is strongly irreducible. To see that, by Theorem 12, it suffices to show that $(I : P)$ is comparable to all k-ideals in R. For this, if J is a k-ideal of R that is not contained in I, then $I \subsetneqq J$, by hypothesis. Since I is irreducible, it follows that $(I : P) \subseteq J$. □

References

[1] Allen P. *Ideal theory in semirings.* Dissertation, Texas Christian University, 1967.

[2] Allen P.J. *A fundamental theorem of homomorphisms for simirings.* Proc. Amer. Math. Soc., 1969, **21**, 412–416.

[3] Allen P.J., Neggers J. *Ideal theory in commutative A-semirings.* Kyungpook Math. J., 2006, **46**, 261–271.

[4] Ebrahimi Atani S. *On k-weakly primary ideals over semirings.* Sarajevo J. of Math., 2007, **15**, 9–13.

[5] Ebrahimi Atani S. *The ideal theory in quotients of commutative semirings.* Glasnik Matematicki, 2007, **42**, 301–308.

[6] Ebrahimi Atani S. *On primal and weakly primal ideals over commutative semirings.* Glasnik Matematicki, to appear.

[7] Gupta V., Chaudhari J.N. *Some remarks on semirings.* Radovi Matematicki, 2003, **12**, 13–18.

[8] Gupta V., Chaudhari J.N. *Some remarks on Right π-regular semirings.* Sarajevo J. of Math., 2006, **14**, 3–9.

[9] Heinzer W.J., Ratlif L.J., Rush D.E. *Strongly Irreducible Ideals of a Commutative Ring.* J. Pure Appl. Algebra, 2002, **166**, 267–275.

[10] Mosher J.R. *Generalized quotients of semirings.* Compositio Math., 1970, **22**, 275–281.

[11] Murta K. *On the quotient semigroup of non-commutative semigroup.* Osaka Math. J., 1950, **2**, 1–5.

[12] Sen M.K., Adhikari M.R. *On maximal k-ideals of simirings.* Proc. Amer. Math. Soc., 1993, **118**, 699–703.

[13] Sharp R.Y. *Steps in Commutative Algebra.* Cambridge University Press, Cambriddge, 1990.

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