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Classification of $\mathit{Aff}(2,\mathbb{R})$-orbit's dimensions for quadratic differential system

N. Gherstega, V. Orlov

Abstract. Affine invariant conditions for $\mathit{Aff}(2,\mathbb{R})-$orbit's dimensions are defined for two-dimensional autonomous quadratic differential system.

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Consider two-dimensional quadratic differential system

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta}x^\alpha x^\beta \quad (j,\alpha,\beta=\overline{1,2}), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta}$ is symmetrical in lower indices in which the complete convolution holds.

Consider also the group $\mathit{Aff}(2,\mathbb{R})$ of affine transformations given by the equalities:

$$\bar{x}^1 = \alpha x^1 + \beta x^2 + h^1, \quad \bar{x}^2 = \gamma x^1 + \delta x^2 + h^2, \quad \Delta = det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0,$$

where $\alpha,\beta,\gamma,\delta,h^1,h^2$ take real values.

Further will use the notations

$$\begin{gathered} a^1=a,\ \ a^2=b,\ \ a^1_1=c,\ \ a^1_2=d,\ \ a^2_1=e,\ \ a^2_2=f,\ \ a^1_{11}=g,\ \ a^1_{12}=h, \\ a^1_{22}=k,\ \ a^2_{11}=l,\ \ a^2_{12}=m,\ \ a^2_{22}=n,\ \ x^1=x,\ \ x^2=y. \end{gathered} \tag{2}$$

According to [1] and taking into consideration (2), the representation operators of the group $\mathit{Aff}(2,\mathbb{R})$ in the space of coefficients and variables of the system (1) will take the form

$$
\begin{aligned}
X_1 &= x\frac{\partial}{\partial x} + a\frac{\partial}{\partial a} + d\frac{\partial}{\partial d} - e\frac{\partial}{\partial e} - g\frac{\partial}{\partial g} + k\frac{\partial}{\partial k} - 2l\frac{\partial}{\partial l} - m\frac{\partial}{\partial m},\\
X_2 &= y\frac{\partial}{\partial x} + b\frac{\partial}{\partial a} + e\frac{\partial}{\partial c} + (f-c)\frac{\partial}{\partial d} - e\frac{\partial}{\partial f} + l\frac{\partial}{\partial g} +\\
&\quad +(m-g)\frac{\partial}{\partial h} + (n-2h)\frac{\partial}{\partial k} - l\frac{\partial}{\partial m} - 2m\frac{\partial}{\partial n},\\
X_3 &= x\frac{\partial}{\partial y} + a\frac{\partial}{\partial b} - d\frac{\partial}{\partial c} + (c-f)\frac{\partial}{\partial e} + d\frac{\partial}{\partial f} - 2h\frac{\partial}{\partial g} - k\frac{\partial}{\partial h} +\\
&\quad +(g-2m)\frac{\partial}{\partial l} + (h-n)\frac{\partial}{\partial m} + k\frac{\partial}{\partial n},\\
X_4 &= y\frac{\partial}{\partial y} + b\frac{\partial}{\partial b} - d\frac{\partial}{\partial d} + e\frac{\partial}{\partial e} - h\frac{\partial}{\partial h} - 2k\frac{\partial}{\partial k} + l\frac{\partial}{\partial l} - n\frac{\partial}{\partial n},\\
X_5 &= \frac{\partial}{\partial x} - c\frac{\partial}{\partial a} - e\frac{\partial}{\partial b} - 2g\frac{\partial}{\partial c} - 2h\frac{\partial}{\partial d} - 2l\frac{\partial}{\partial e} - 2m\frac{\partial}{\partial f},\\
X_6 &= \frac{\partial}{\partial y} - d\frac{\partial}{\partial a} - f\frac{\partial}{\partial b} - 2h\frac{\partial}{\partial c} - 2k\frac{\partial}{\partial d} - 2m\frac{\partial}{\partial e} - 2n\frac{\partial}{\partial f}.
\end{aligned}
\tag{3}
$$

The operators (3) form a six-dimensional Lie algebra [1]. Let $\tilde{a} = (a, b, ..., n) \in E^{12}(\tilde{a})$, where $E^{12}(\tilde{a})$ is the Euclidean space of the coefficients of the right-hand sides of the system (1). Denote by $\tilde{a}(q)$ the point from $E^{12}(\tilde{a})$ that corresponds to the system, obtained from the system (1) with coefficients $\tilde{a}$ by a transformation $q \in \mathit{Aff}(2, \mathbb{R})$.

Definition 1. *Call the set* $O(\tilde{a}) = \{\tilde{a}(q) | q \in \mathit{Aff}(2, \mathbb{R})\}$ *the* $\mathit{Aff}(2, \mathbb{R})$ *– orbit of the point* $\tilde{a}$ *for the system* (1).

It is known from [1] that

$$
dim_{\mathbb{R}} O(\tilde{a}) = rank M_1, \tag{4}
$$

where M_1 is the following matrix

$$
M_1 = \begin{pmatrix}
a & 0 & 0 & d & -e & 0 & -g & 0 & k & -2l & -m & 0\\
b & 0 & e & -c+f & 0 & -e & l & -g+m & -2h+n & 0 & -l & -2m\\
0 & a & -d & 0 & c-f & d & -2h & -k & 0 & g-2m & h-n & k\\
0 & b & 0 & -d & e & 0 & 0 & -h & -2k & l & 0 & -n\\
-c & -e & 2g & -2h & -2l & -2m & 0 & 0 & 0 & 0 & 0 & 0\\
-d & -f & 2h & -2k & -2m & -2n & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{5}
$$

constructed on coordinate vectors of operators (3). Denote by $\Delta_{i,j,k,l,m,n}$ the minor of the 6th order of the matrix M_1, constructed on columns i, j, k, l, m, n $(i, j, k, l, m, n \in \{1, \ldots, 12\})$.

For the system (1) from [2], [4] are known the following center-affine comitants and invariants

$$
\begin{gathered}
I_1 = a^{\alpha}_{\alpha},\ I_2 = a^{\alpha}_{\beta}a^{\beta}_{\alpha},\ I_3 = a^{\alpha}_{p}a^{\beta}_{\alpha q}a^{\gamma}_{\beta\gamma}\varepsilon^{pq},\ I_4 = a^{\alpha}_{p}a^{\beta}_{\beta q}a^{\gamma}_{\alpha\gamma}\varepsilon^{pq},\\
I_5 = a^{\alpha}_{p}a^{\beta}_{\gamma q}a^{\gamma}_{\alpha\beta}\varepsilon^{pq}, I_7 = a^{\alpha}_{pr}a^{\beta}_{\alpha q}a^{\gamma}_{\beta s}a^{\delta}_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},\ I_8 = a^{\alpha}_{pr}a^{\beta}_{\alpha q}a^{\gamma}_{\delta s}a^{\delta}_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},\\
I_9 = a^{\alpha}_{pr}a^{\beta}_{\beta q}a^{\gamma}_{\gamma s}a^{\delta}_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs},\ K_3 = a^{\alpha}_{\beta}a^{\beta}_{\alpha\gamma}x^{\gamma},\ K_4 = a^{\alpha}_{\gamma}a^{\beta}_{\alpha\beta}x^{\gamma},\\
K_5 = a^{p}_{\alpha\beta}x^{\alpha}x^{\beta}x^{q}\varepsilon^{pq},\ K_6 = a^{\alpha}_{\alpha\beta}a^{\beta}_{\gamma\delta}x^{\gamma}x^{\delta},\ K_7 = a^{\alpha}_{\beta\gamma}a^{\beta}_{\alpha\delta}x^{\gamma}x^{\delta},\\
K_{11} = a^{p}_{\alpha}a^{\alpha}_{\beta\gamma}x^{\beta}x^{\gamma}x^{q}\varepsilon_{pq},\ K_{12} = a^{\alpha}_{\beta}a^{\beta}_{\alpha\gamma}a^{\gamma}_{\delta\mu}x^{\delta}x^{\mu},\ K_{13} = a^{\alpha}_{\gamma}a^{\beta}_{\alpha\beta}a^{\gamma}_{\delta\mu}x^{\delta}x^{\mu},\\
K_{17} = a^{\alpha}_{\beta\nu}a^{\beta}_{\alpha\gamma}a^{\gamma}_{\delta\mu}x^{\delta}x^{\mu}x^{\nu},\ K_{21} = a^{p}x^{q}\varepsilon_{pq},\ K_{23} = a^{p}a^{q}_{\alpha\beta}x^{\alpha}x^{\beta}\varepsilon_{pq}.
\end{gathered}
\tag{6}
$$

According to [3], write a transvectant of index k for binary forms f and φ as follows

$$
(f,\varphi)^{(k)} = \sum_{h=0}^{k}(-1)^h \begin{pmatrix} k \\ h \end{pmatrix} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\frac{\partial^k \varphi}{\partial x^h \partial y^{k-h}}. \tag{7}
$$

According to [4], the transvectant (7) on two comitants of the system (1) is a comitant (invariant) of this system too.

For the system (1) from [4] will use the following transvectants

$$
T_5 = (C,K)^{(2)},\ T_{16} = (C,D)^{(1)}, \tag{8}
$$

where $C = K_5,\ D = I_1K_1K_2 + K_2K_3 - K_2K_4 + \frac{1}{2}I_1^2K_5 - \frac{1}{2}I_2K_5 - I_1K_{11} - 2K_1^2K_{21} + 4K_6K_{21} - 2K_7K_{21},\ K = \frac{1}{2}(K_1^2 - K_7)$.

Denote by $\tilde{a}(\tau)$ the point from the space $E^{12}(\tilde{a})$, that corresponds to the system , obtained from the system (1) with coefficients $\tilde{a}$ by the transformation of translation $\tau : x = \bar{x} + x_0,\ y = \bar{y} + y_0$. It is evident $\tilde{a}(\tau) = \tilde{a}(x_0, y_0)$. According to [6], if $I(\tilde{a})$ is the center-affine invariant of the system (1), then the polynomial

$$
K(\tilde{a}, x, y) = I(\tilde{a}(x_0, y_0))|_{\{x_0=x, y_0=y\}}
$$

is a affine comitant of this system. Then considering (6), we construct the following affine comitants:

$$
Af_i(\tilde{a}, x, y) = I_i(\tilde{a}(x_0, y_0))|_{\{x_0=x, y_0=y\}}\quad (i = 3, 4, 5). \tag{9}
$$

Lemma 1. *For $\beta \neq 0$, the rang of matrix M_1 is equal to six, where*

$$
\beta = 27I_8 - I_9 - 18I_7 \tag{10}
$$

and $I_8,\ I_9,\ I_7$ are from (6).

Proof. We suppose the contrary $\beta \neq 0$, and all minors of the sixth order of the matrix M_1 are zero. We observe that β is the discriminant of the cubic form K_5. It is known [2] that for $\beta \neq 0$ there exists a linear transformation of the system (1). Such that the comitant K_5 from (6) evaluated for the transformed system takes the form:

$$K_5 = x(x^2+\delta y^2),\ (\delta=\pm1,\ l=-1, k=0,\ g=2m,\ n=2h-\delta). \qquad (11)$$

In case of (10) we obtain $\Delta_{3,4,7,8,9,10} = -16\delta h^4 = 0$, $\delta \neq 0$, this implies $h=0$. Taking into consideration $h=0$, we obtain $\Delta_{3,6,7,9,11,12} = -16\delta^4 m^2 = 0$, this implies $m=0$. Taking into consideration $m=0$, we obtain $\Delta_{5,6,9,10,11,12} = 8\delta^4 \neq 0$. We obtain a contradiction. Lemma 1 is proved. □

Lemma 2. *For $K_5 \not\equiv 0$, $\beta = 0$, the rang of matrix M_1 is equal to six if and only if*

$$Af_5^2 + T_5^2 + T_{16}^2 + Kom^2 \not\equiv 0, \qquad (12)$$

where

$$\begin{aligned} Kom \equiv\ & I_1K_1^2(2K_1^2K_2 - 2K_2K_6 - K_1K_{11}) - 2K_1^3(K_2K_4 + 2K_7K_{21}) + \\ & +4K_1K_6^2K_{21} + K_1^2[2K_4K_{11} + K_2K_{13} + 2K_{23}(K_6+K_7)] - 4K_6^2K_{23}, \end{aligned} \qquad (13)$$

and $I_1, K_1, K_2, K_4, K_5, K_6, K_7, K_{11}, K_{12}, K_{13}, K_{16}, K_{17}, K_{21}, K_{23}$ are from (6), T_5, T_{16} are from (8), β is from (9), Af_5 is affine comitant of the system (1).

Proof. It is known from [2] that for $K_5 \not\equiv 0$ and $\beta = 0$ there exists such affine transformation exists that the comitant K_5 will take the form:

$$K_5 = x^2(x+\delta y),\ (\delta = 0,1;\ l=-1, k=0,\ g=2m+\delta,\ n=2h). \qquad (14)$$

Let (12) be not true. It is easy to verify that in case (14) $T_5 \equiv 8h^2\delta y + 8h(3h - 2m\delta - \delta^2)x$, from $T_5 \equiv 0$ we have $h=0$. Taking into consideration $h=0$ in the case (14), we obtain $Af_5 \equiv -d(\delta^2 + 4m\delta + 5m^2)$. From $Af_5 \equiv 0$ it follows

$$d[\delta + (1-\delta)m] = 0,\ (\delta = 0,1) \qquad (15)$$

In the case $\delta = 1$ from (15) we obtain $d=0$, $T_{16} \equiv 6m(cf+ef-f^2-2am-2bm)x^4$, and Kom is the CT comitant [5], $Kom \equiv (1+3m)^3(cf+ef-f^2-2am-2bm)x^6$. It is evident if $T_{16} \equiv Kom \equiv 0$, we obtain:

$$h = d = cf + ef - f^2 - 2am - 2bm = 0. \qquad (16)$$

In the case when $\delta = 0$ from (15) we obtain: $m=0$, $Kom \equiv 0$, $T_{16} \equiv 6dfx^4 - 6d^2x^3y$ or $d=0$, $Kom \equiv 27m^3(cf - f^2 - 2am)x^6$, $T_{16} \equiv 6m(cf-f^2-2am)x^4$. It is evident if $T_{16} \equiv Kom \equiv 0$ it follows:

$$h = d = m(cf - f^2 - 2am) = 0. \qquad (17)$$

It is easy to verify that in cases (16), (17) all 6^{th} order minors of the matrix M_1 are equal to zero. The necessity of Lemma 2 is proved.

Prove the sufficiency. In the case when (14) holds for $\delta = 1$ we obtain: $\Delta_{3,4,7,8,11,12} = -8h^4$, for $h=0$ $\Delta_{1,3,4,6,7,11} = 2d^3(1+3m)^2$, $\Delta_{4,5,6,10,11,12} = -8dm^4$;

for $h = d = 0$ $\Delta_{2,5,6,10,11,12} = 4m^3(cf + ef - f^2 - 2am - 2bm)$, $\Delta_{2,3,5,7,10,11} = 2(1 + 2m)(1 + 3m)(cf + ef - f^2 - 2am - 2bm)$.

In the case when (14) holds for $\delta = 0$ we obtain: $\Delta_{3,4,7,8,10,11} = -24h^4$, for $h = 0$ $\Delta_{1,3,4,6,7,11} = 18d^3m^2$; for $h = m = 0$, $\Delta_{1,3,4,5,7,10} = 2d^3$. If $h = d = 0$, we obtain $\Delta_{2,3,5,7,10,11} = 12m^2(cf - f^2 - 2am)$. Lemma 2 is proved. □

Lemma 3. *For $K_5 \equiv 0$, the rang of matrix M_1 is equal to six if and only if $Af_4(Af_4 - Af_3) \not\equiv 0$, where Af_3, Af_4 are affine comitants of the system* (1).

Proof. For $K_5 \equiv 0$, according to (2), (6) $(l = k = 0,\ g = 2m,\ n = 2h)$ we obtain that $Af_4(Af_4 - Af_3) \equiv 27(eh^2 + chm + fhm - dm^2)^2$, and all 6^{th} order minors of the matrix M_1 consist of the factor expession $eh^2 + chm - fhm - dm^2$. Taking into consideration the above mentioned proof of the Lemma 3 follows. □

From Lemmas 1,2,3 follows:

Theorem. *The dimension of $Aff(2, \mathbb{R})$-orbit of the system (1) is equal to six if and only if*

$$\beta \neq 0, \textit{ or}$$
$$\beta = 0,\ K_5(Af_5^2 + T_5^2 + T_{16}^2 + Kom^2) \not\equiv 0, \textit{ or}$$
$$\beta = 0,\ K_5 \equiv 0,\ Af_4(Af_4 - Af_3) \not\equiv 0,$$

where K_5 from (6)*, T_5, T_{16} from* (8)*, β from* (9)*, Kom from* (13)*, Af_3, Af_4, Af_5 from* (9) *are affine comitants of the system* (1).

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Institute of Mathematics and Computer Sciences
Academy of Sciences of Moldova
str. Academiei 5, MD-2028 Chisinau
Moldova

E-mail: *gherstega@gmail.com*, *orlovictor@gmail.com*

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