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On numerical algorithms for solving multidimensional analogs of the Kendall functional equation*

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Abstract. The numerical algorithms for solution of multidimensional analogs of the Kendall (Kendall-Takacs) equation, including effective fast algorithm are discussed.

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The mathematical models of Queueing Systems, in particular Priority Systems, play an important role in the analysis, modelling and design of various modern networks and their components (Vishnevskii 2003 [1]). Thus, Priority models with switchover time represent a special class of Polling Systems and are widely used in broadband WLAN (Wireless Local Area Networks) (Vishnevskii and Semenova 2007 [2]). However, the obtained theoretical results for mentioned systems have a rather complicated mathematical structure. For example, the presented below multidimensional analogs of the famous Kendall equation represent a system of r recurrent functional equations expressed in terms of Laplace-Stieltjes transform (LST) which has no exact analytical solution. Besides that, this system is involved in the core of many important characteristics of the evolution of priority system, such as traffic intensity, probabilities of states, etc. In what follows we discuss some numerical algorithms of the solution of mentioned recurrent functional equation.

Let us consider the queueing systems $M|G|1$ with an exhaustive service. Denote by λ the parameter of input Poisson flow, by $B(x) = P\{B < x\}$ the distribution function of service, by $\beta(s) = \int_0^\infty e^{-sx} dB(x)$ and by $\beta_1 = \int_0^\infty x dB(x)$ its LST and first moment, respectively. Denote by $\Pi(x)$ the distribution function of the busy period, by $\pi(s)$ and π_1 the LST of $\Pi(x)$ and its first moment, respectively.

The following result is known as Kendall (Kendall-Takacs) functional equation.

The LST of the busy period is determined in the unique way from the functional equation

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)). \tag{1}$$

If $\lambda\beta_1 < 1$, *then*

$$\pi_1 = \frac{\beta_1}{1 - \lambda\beta_1}. \tag{2}$$

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Formulae (1) and (2) are referred to in most standard textbooks on Queueing Theory (see, for example Gnedenko 2005 [3], Takagi 1991 [4]).

Let us consider the queueing system $M_r|G_r|1$ with r priority classes of messages. The moments of i-messages' appearance represent a Poisson flow with parameter λ_i and the service times are random variables with the distribution functions $B_i(x)$, $i = 1, \ldots, r$. The priority classes are numbered in the decreasing order of priorities, namely, it is assumed that i-messages have a higher priority than j-messages if $1 \leq i, j \leq r$. It is also assumed that the server needs some time C_{ij} to switch the service process from the queue i to queue j. The length of the ij – switching C_{ij} is considered to be a random variable with the distribution function $C_{ij}(t)$, $1 \leq i, j \leq r$, $i \neq j$.

Also suppose that the switching C_{ij} depends only on index j, $C_{ij} = C_j$. The strategy in the free state is considered "reset". Denote by $\Pi_k(x)$ the distribution function of the busy period with the messages of the priority not less than k, $\sigma_k = \lambda_1 + \ldots + \lambda_k$, $\sigma = \sigma_r$, $\sigma_0 = 0$, $\beta_i(s)$, $c_j(s)$, $\pi(s)$, $\ldots$, $\pi_k(s)$ are the LST of the distribution functions $B_i(x)$, $C_j(x)$, $\Pi(x)$,..., $\Pi_k(x)$, respectively. More details regarding the priority queueing systems with switchover times are presented in Mishkoy 2007 [5], Mishkoy et al. 2008 [6].

In what follows we suppose that the interrupted switching and interrupted service of message will be continued from the time point it was interrupted at. As it is shown in previous studies (see for example Mishkoy et. al 2008 [6]) busy periods distribution and traffic intensity can be determined solving the following recurrent system of functional equations:

$$\pi_k(s) = \frac{\sigma_{k-1}}{\sigma_k}\pi_{k-1}(s+\lambda_k) + \frac{\sigma_{k-1}}{\sigma_k}(\pi_{k-1}(s+\lambda_k[1-\pi_{kk}(s)])-$$

$$-\pi_{k-1}(s+\lambda_k))\nu_k(s+\lambda_k[1-\pi_{kk}(s)])+$$

$$+\frac{\lambda_k}{\sigma_k}\nu_k(s+\lambda_k[1-\pi_{kk}(s)])\pi_{kk}(s), \tag{3}$$

$$\pi_{kk}(s) = h_k(s+\lambda_k[1-\pi_{kk}(s)]), \tag{4}$$

$$\nu_k(s) = c_k(s+\sigma_{k-1}[1-\pi_{k-1}(s)]), \tag{5}$$

$$h_k(s) = \beta_k(s+\sigma_{k-1}[1-\pi_{k-1}(s)\nu_k(s)]). \tag{6}$$

The system of functional equations presented above can be viewed as the generalization of the Kendall-Takacs equation (1). Namely, in Mishkoy 2007 [5] it is shown that for $C_j = 0$ and the number of priority classes $r = 1$, equation (1) follows from the mentioned system.

Note that equation (4) appear as a key equation of mentioned system of recurrent functional equations. Namely, to obtain $\pi_k(s)$, $h_k(s)$, $\nu_k(s)$ it is necessary to solve the functional equation (4) first. The numerical algorithms for solving (4) can be elaborated using the classical scheme of successive approximation (see for example Gnedenko et al. 1973 [7]):

...
$\pi_{kk}^{(0)}(s^*) := 0$; $n := 1$;
Repeat
$\pi_{kk}^{(n)}(s^*) := h_k(s^* + \lambda_k - \lambda_k \pi_{kk}^{(n-1)})$;
inc(n);
Until $|\pi_{kk}^{(n)}(s^*) - \pi_{kk}^{(n-1)}(s^*)| < \varepsilon$;
$\pi_{kk}(s^*) := \pi_{kk}^{(n)}(s^*)$
...
or using the improved algorithms elaborated in Bejan 2006 [8]:
...
$\underline{\pi}_{kk}^{(n)}(0) := 0$; $\overline{\pi}_{kk}^{(n)}(0) = 1$;
Repeat
$\overline{\pi}_{kk}^{(n)}(s^*) = h_k(s^* + \lambda_k - \lambda_k \overline{\pi}_{kk}^{(n-1)}(s^*))$;
$\underline{\pi}_{kk}^{(n)}(s^*) = h_k(s^* + \lambda_k - \lambda_k \underline{\pi}_{kk}^{(n-1)}(s^*))$;
inc(n);
Until $\frac{|\overline{\pi}_{kk}^{(n)}(s^*) - \underline{\pi}_{kk}^{(n-1)}(s^*)|}{2} < \varepsilon$;

$\pi_{kk}(s^*) := \frac{\overline{\pi}_{kk}^{(n)}(s^*) + \underline{\pi}_{kk}^{(n-1)}(s^*)}{2}$;
...

Unfortunately the procedure based on such type of algorithms is consuming very much time especially for large r ($r \geq 10$). A new effective algorithm for evaluation the characteristics can be elaborated using the method elaborated in Mishkoy, Grama 1994 [9] based on the binary tree data structure. The main idea of this procedure is to solve the system of equations for fixed r and s with respect to the collection of the unknown values $\pi_{kk}(s(k,j))$, were $k = r, \ldots, 1, j = 1, \ldots, 2^{r-k}$. To this end we define an iteration process each iteration m of which represents a recurrent procedure constructed by means of the binary tree in the following manner:

$$\sigma_k \pi_k(s(k,j)^{(m)}) = \sigma_{k-1}\pi_{k-1}(s(k-1,2j-1)^{(m)}) +$$
$$+\sigma_{k-1}\{\pi_{k-1}(s(k-1,2j)^{(m)}) - \pi_{k-1}(s(k-1,2j-1)^{(m)})\}\times$$
$$\times \nu_k(s(k-1,2j)^{(m)}) + \lambda_k \nu_k(s(k-1,2j)^{(m)})\pi_{kk}^{(m)}(s(k,j)^{(m)}),$$
$$\pi_{kk}^{(m)} s(k,j)^{(m)} = h_{k-1}(s(k-1,2j)^{(m)}), s(r,1)^{(m)} = s,$$
$$s(k-1,2j-1)^{(m)} = s_{k1}(s(k,j)^{(m)}), s(k-1,2j)^{(m)} = s_{k2}(s(k,j)^{(m)}),$$

where $k = r, \ldots, 1, j = 1, \ldots, 2^{r-k}$, $s_{ki}(s(k,j)^{(m-1)})$, $i = 1, 2$, are determined by $s_{k1}(s) = s + \lambda_k$, $s_{k2}(s) = s + \lambda_k(1 - \pi_{kk}(s))$ with $\pi_{kk}^{(m-1)}(s(k,j)^{(m-1)})$ instead of $\pi_{kk}(s)$ and $\nu_k(s)$, $h_k(s)$, are determined by (5), (6), $m = 1, 2, \ldots$, the initial values for $\pi_{kk}^{(0)}(s(k,j)^{(0)})$, $k = r, \ldots, 1, j = 1, \ldots, 2^j$, may be taken arbitrary in the interval [0,1].

The iteration process may be easily performed by the computer and turns out to provide very rapid calculations for the above mentioned characteristics even for enough large value of k.

References

[1] Vishnevskii V.M. *Theoretical bases of design of the computer communication networks.* Moscow, 2003 (in Russian).

[2] Vishnevskii V.M., Semenova O.V. *Polling Systems: The theory and applications in the brodband wireless networks.* Moscow, 2007 (in Rissian).

[3] Gnedenko B.V., Kovalenko I.N. *Introduction to Queueing Theory.* Moscow, 2005 (in Russian).

[4] Takagi H. *Queueing Analysis. Volume 1: Vacation and Priority Systems.* North-Hodland, Elsevier Science Publ., 1991.

[5] Mishkoy Gh. *An analog of Kendall-Takacs equation for generalized queueing systems.* Proceedings of the 31th Annual Congress of the American Roumanian Academy of Arts and Sciences, Brasov, Romania, 2007, p. 483–486.

[6] Mishkoy Gh.K., Rykov V.V., Giordano S., Bejan A.Iu. *Multidimensional analogs of the Kendall equation for priority queueing systems: computational aspects.* Automatics and Remote Control No. 5, Moscow, 2008, p. 82–95.

[7] Gnedenko B.V., Danielyan E.A, Dimitrov B.N, Klimov G.P., Matveev V.F. *Priority Queueing Systems.* Moscow State University Press, Moscow, 1973 (in Russian).

[8] Bejan A.Iu. *Numerical treatment of the Kendall equation in the analysis of priority queueing systems.* Bul. Acad. Ştiinţe Repub. Mold., Math. 2006, No. 2(51), 17–28.

[9] Mishkoy G.K., Grama I. G. *Fast Numerical Algorithms for Evaluating Characteristics of Priority Queueing Systems.* Proceedings of the IMACS Symposium on MATHEMATICAL MODELLING. Austria, Vienna, Volume 5, 1994, p. 893–896.

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