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# Symmetric random evolution in the space $\mathbb{R}^6$

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**Abstract.** A closed-form expression for the transition density of a symmetric Markovian random evolution in the Euclidean space $\mathbb{R}^6$ is presented.

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This note is motivated by the recent works on random motions at finite speed (also called the random flight, isotropic transport process or, in a more general sense, random evolution) in the Euclidean space $\mathbb{R}^m$. Such processes in the Euclidean spaces of different dimensions have thoroughly been examined in a series of works. In the study of such processes the most desirable goal is undoubtedly their explicit distributions in the cases (very few indeed) when such distributions can be obtained. The explicit form of the distribution of a two-dimensional symmetric random motion at finite speed was derived (by different methods) by Stadje [9], Masoliver *et al.* [7], Kolesnik and Orsingher [6], Kolesnik [2]. The distribution of a random flight in $\mathbb{R}^3$ was given by Tolubinsky [11] and Stadje [10] in fairly complicated integral forms. Finally, the explicit form of the distribution of a random flight in the space $\mathbb{R}^4$ was obtained by Kolesnik [4] and by Orsingher and De Gregorio [8]. The random flights in arbitrary higher dimensions were examined by Kolesnik [1, 3, 5] and by Orsingher and De Gregorio [8], however no new distributions were obtained in these works for higher dimensions $m \geq 5$.

Since the exact probability laws of random flights in lower dimensions were derived by fairly complicated and sometimes tricky methods, the possibility of obtaining the explicit form of the distributions seemed very doubtful in higher dimensions $m \geq 5$.

However, a general and unified method of studying the random flights in arbitrary dimension was suggested in the works by Kolesnik [1, 3, 5] based on the analysis of the integral transforms of their distributions. This method, applied to the six-dimensional random motion, enables us to obtain the explicit probability law of the process and this result is presented here. While this method works in any dimension, the derivation of the explicit probability law in such a fairly high dimension $m = 6$ looks like a "lucky accident" which, apparently, cannot be extended in higher dimensions.

The distribution derived has a considerably more complicated form in comparison with those obtained for the dimensions 2 and 4. It is presented as a series of the finite sums of the Gauss hypergeometric functions which seemingly cannot be reduced to a more elegant formula. Nevertheless, this formula is of a certain interest because it gives the *explicit* form of the distribution which can be directly used for practical calculations and, on the other hand, it is a new step toward the most desirable goal, namely, constructing a general theory of distributions for random flights in the Euclidean spaces $\mathbb{R}^m$ of arbitrary dimension $m \geq 2$.

We consider the stochastic motion performed by a particle starting its motion from the origin $\mathbf{0} = (0,0,0,0,0,0)$ of the six-dimensional Euclidean space $\mathbb{R}^6$ at time $t = 0$. The particle is endowed with constant, finite speed $c$ (note that $c$ is treated as the constant norm of the velocity). The initial direction is a six-dimensional random vector with uniform distribution (Lebesgue probability measure) on the unit sphere

$$S_1 = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 = 1 \right\}.$$

The particle changes direction at random instants which form a homogeneous Poisson process of rate $\lambda > 0$. At these moments it instantaneously takes on the new direction with uniform distribution on $S_1$, independently of its previous motion.

Let $\mathbf{X}(t) = (X_1(t), X_2(t), X_3(t), X_4(t), X_5(t), X_6(t))$ be the position of the particle at an arbitrary time $t > 0$ and denote by $d\mathbf{x}$ the infinitesimal element in the space $\mathbb{R}^6$.

At any time $t > 0$ the particle, with probability 1, is located in the six-dimensional ball of radius $ct$

$$\mathbf{B}_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 \leq c^2t^2 \right\}.$$

The distribution $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}, \ \mathbf{x} \in B_{ct}, \ t \geq 0$, consists of two components. The singular component corresponds to the case when no Poisson event occurs in the interval $(0,t)$ and is concentrated on the sphere

$$S_{ct} = \partial \mathbf{B}_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in R^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 = c^2t^2 \right\}.$$

In this case the particle is located on the sphere $S_{ct}$ and the probability of this event is

$$\Pr\{\mathbf{X}(t) \in S_{ct}\} = e^{-\lambda t}.$$

If one or more than one Poisson events occur, the particle is located strictly inside the ball $\mathbf{B}_{ct}$, and the probability of this event is

$$\Pr\{\mathbf{X}(t) \in \text{int}\,\mathbf{B}_{ct}\} = 1 - e^{-\lambda t}.$$

The part of the distribution $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$ corresponding to this case is concentrated in the interior

$$\text{int}\,\mathbf{B}_{ct} = \left\{ \mathbf{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 : \ \|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2 < c^2t^2 \right\},$$

and forms its absolutely continuous component.

Therefore there exists the density $p(\mathbf{x},t) = p(x_1, x_2, x_3, x_4, x_5, x_6; t)$, $\mathbf{x} \in \text{int}\, \mathbf{B}_{ct}$, $t > 0$, of the absolutely continuous component of the distribution function $\Pr\{\mathbf{X}(t) \in d\mathbf{x}\}$. Our principal result is given by the following theorem.

**Theorem.** *For any $t > 0$ the density $p(\mathbf{x},t)$ has the form*

$$p(\mathbf{x},t) = \frac{16\lambda t e^{-\lambda t}}{\pi^3 (ct)^6} \left(1 - \frac{5}{6}\, \frac{\|\mathbf{x}\|^2}{c^2t^2}\right) +$$

$$+\frac{e^{-\lambda t}}{2\pi^3 (ct)^6} \sum_{n=2}^{\infty} (\lambda t)^n (n+1)! \sum_{k=0}^{n+1} \frac{(k+1)(k+2)(n+2k+1)}{3^k (n-k+1)!(n+k-2)!} \times$$

$$\times F\left(-(n+k-2), k+3; 3; \frac{\|\mathbf{x}\|^2}{c^2t^2}\right) \tag{1}$$

*where* $\|\mathbf{x}\|^2 = \sum_{i=1}^{6} x_i^2$,

$$F(\xi,\eta;\zeta;z) = \ {}_2F_1(\xi,\eta;\zeta;z) = \sum_{k=0}^{\infty} \frac{(\xi)_k (\eta)_k}{(\zeta)_k} \frac{z^k}{k!}$$

*is the Gauss hypergeometric function and*

$$(a)_k = a(a+1)\dots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}$$

*is the standard Pochgammer symbol.*

It's interesting to note that, since the first coefficient of the hypergeometric function in formula (1) is always negative for arbitrary $n$ and $k$, the hypergeometric function itself represents, in fact, some polynomial. This is a characteristic feature of random flights in even-dimensional spaces.

The proof of the theorem is substantially based on the ideas and methods developed in the works by Kolesnik [1, 3, 5]. In particular, the applications of formulae (2.13) of [1] or (9) of [5] yield an explicit form of the Laplace transforms of the conditional characteristic functions corresponding to an arbitrary number of changes of directions, which then can be easily inverted. This gives the easy-treatable formulas for the conditional characteristic functions of the process from which the closed-form expressions for the conditional densities can be easily obtained by applying the classical Hankel inversion formula. By using then the total probability formula we immediately obtain our main result (1).

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