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The Euler Tour of n-Dimensional Manifold with Positive Genus

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Abstract. In the paper [1] it is proved that abstract cubic n-dimensional torus [2] possesses a directed Euler tour of the same dimension. The result prompts to a new (virtual) device for transmission and reception of information. In the present paper it is shown that every abstract cubic n-dimensional manifold without borders, of positive genus possesses a n-dimensional directed Euler tour. This result has practical application.

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Via a complex of multi-ary relations K^n and their groups of the direct homologies on the groups of integer numbers Z [5] we had defined an abstract cube and a cubic oriented manifold, without borders [3, 4].

Definition 1 [2]. *A cubic complex $\mathcal{I}^n$ is called an* ***abstract cubic*** n***-dimensional manifold without borders*** *if the following proprieties are satisfied:*

A. *any $I^{n-1} \subset \mathcal{I}^n$ is a joint face exactly of two n-dimensional cubes from $\mathcal{I}^n$;*
B. *for $\forall I_i^n, I_j^n \in \mathcal{I}^n$, $i \neq j$, there exists a sequence of n-dimensional cubes $I_{i_1}^n = I_i^n, I_{i_2}^n \dots, I_{i_q}^n = I_j^n$, where $I_r^n \cap I_{r+1}^n = I_{r,r+1}^{n-1}$, $r \in \{i_1, i_2, \dots, i_{q-1}\}$;*
C. *for $\forall I^p \in \mathcal{I}^n$, $0 \leq p \leq n-1$, $\exists I^n \in \mathcal{I}^n$, where I^p is a face of I^n;*
D. *for disjoint cubes $\forall I_i^n, I_j^n \in \mathcal{I}^n$, $I_i^n \cap I_j^n = I^p$, $2 \leq p < n$, there exists a sequence of abstract cubes, $I_{i_1}^n = I_i^n, I_{i_2}^n, \dots, I_{i_q}^n = I_j^n$, such that $\bigcap\limits_{j=1}^{q} I_{i_j}^n = I^p$.*

Definition 2 [3]. *The property of n-dimensional abstract cubic manifold without borders $V_p^n = K^n$, that* ***every*** *m-dimensional cube $I^m \subset K^n$, $0 \leq m \leq n$, belongs to 2^{n-m} n-dimensional cubes, is called a* ***normal cubiliaj****.*

The set of abstract multidimensional oriented manifolds without borders defined by abstract cubes can be classified in the same way as the classification of the abstract multidimensional orientated manifolds without borders defined by abstract simplexes is done [4]. By the groups of direct homologies on the integer numbers Z this is done for the complexes of multi-ary relations [5] and for the complexes of abstract cubes similarly [3]. As a result, manifolds of the sequence from Figure 1, denoted by $V_0^n(\Box)$, $V_1^n(\Box)$, $\dots$ $V_p^n(\Box)$, $\dots$, represents as an element of every class of manifolds defined by abstract cubes.

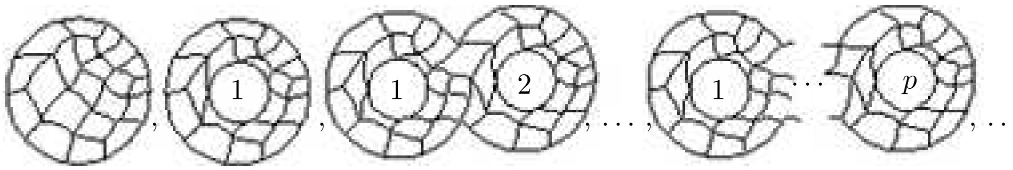

Figure 1

Let $K^n = \{\mathcal{I}^0, \mathcal{I}^1, \ldots, \mathcal{I}^n\}$ be an abstract cubic complex, where $\mathcal{I}^m$ is the set of all abstract m-dimensional cubes of K^n, $0 \le m \le n$ [2].

Definition 3 [2]. *A* ***linear*** ***n-dimensional chain*** *of the complex K^n is a sequence of oriented cubes, $I_1^n, I_2^n, \ldots, I_i^n, I_{i+1}^n, \ldots, I_k^n$, $k > 1$, where $I_i^n \cap I_{i+1}^n \in \in \mathcal{I}^{n-1}$, $I_i^n \cap I_{i+2}^n = \emptyset, \forall i \in \{1, 2, \ldots, k-1\}$. If I_i^n, I_{i+1}^n are coherences cubs for all $i = 1, 2, .., k-1$, then this chain is called a* ***linear n-dimensional oriented chain*** *of the complex K^n.*

Any abstract cubic oriented manifold without borders V^n can be transformed into a manifold in which all n-dimensional cubes are coherences [2, 5].

Definition 4 [2]. *If an abstract cubic oriented manifold without borders V^n possesses a linear directed n-dimensional chain $I_1^n, I_2^n, \ldots, I_i^n, I_{i+1}^n, \ldots, I_{\alpha_n}^n$ that contains every cube $I^n \in V^n$ exactly once, where $I_{i-1}^n \cap I_i^n = I_{i-1,i}^{n-1}$ and $I_i^n \cap I_{i+1}^n = I_{i,i+1}^n$ are common faces of the cube I_i^n, $2 \le i \le \alpha_{n-1}$, then this directed chain is called a* ***linear directed Euler chain*** *of dimension n of the manifold V^n, where I_i^n and I_{i+1}^n are coherences cubs for all $i = 1, 2, .., k-1$. If additionally it is verified that I_1^n and I_k^n are the same, then it is said that this directed chain is a* ***linear directed Euler tour*** *of dimension n of the manifold V^n.*

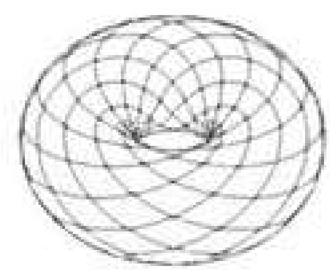

Figure 2

In the paper [1] is proved the theorem which affirms that the *unique* abstract oriented manifold without borders V_1^n that satisfies the property of normal cubiliaj is torus. In the paper [1] it is defined as an abstract oriented cartesian product. It is indicated that an abstract n-dimensional torus V_1^n is represented as a cartesian product of n abstract circumferences (abstract manifolds of dimension 1). This product and the property of normal cubiliaj, which is proper for torus, permitted us to indicate the conditions of existence of directed Euler tour of dimension n for an abstract n-dimensional torus.

Was proved

Theorem 1 [1]. *The abstract cubic n-dimensional concordant oriented torus V_1^n possesses a directed Euler tour of dimension n (see Figure 2).*

This theorem permits us to indicate a (virtual) device which is more efficient in solving Posthumus' problem of reception and transmission of information and in other applications [1].

In this paper we indicate the conditions of existence of n-dimensional directed Euler tour on any abstract cubic oriented n-dimensional manifold without borders, with positive genus (more holes than on the torus), and which is defined by abstract cubes.

Theorem 2. *The abstract cubic oriented manifold without borders V_p^n, where $p \geq 1$ is its genus, possesses a directed Euler tour of dimension n.*

Proof. To prove the theorem we use the induction on manifold's genus q.

For $q = 1$ the main theorem results by Theorem 1.

Let $q = p - 1$. We admit that the theorem is true for the manifold V_{p-1}^n.

Let $q = p$. We consider the abstract manifolds V_{p-1}^n and V_1^n, which possess normal cubiliaj and V_1^n has the same orientation as V_{p-1}^n. By induction, V_{p-1}^n and V_1^n possess directed Euler tours of dimension n, $C_{(p-1)}^n$ and $C_{(p)}^n$ respectively. Let $I_{i_1}^n$ be an arbitrary abstract cube of the torus V_{p-1}^n and $\overset{\circ}{I}{}_{i_{p-1}}^n$ its vacuum [2]. We take out this vacuum from the manifold V_{p-1}^n. Similarly we do with the torus V_1^n, taking out the vacuum $\overset{\circ}{I}{}_{j_1}^n$, which corresponds to the concordance given on an arbitrary n-dimensional cube $I_{j_1}^n$ of this torus. Therefore we obtain the manifolds $V_{p-1}^n \backslash \overset{\circ}{I}{}_{i_{p-1}}^n$ and $V_1^n \backslash \overset{\circ}{I}{}_{j_1}^n$, which possess the borders $I_{i_{p-1}}^n \backslash \overset{\circ}{I}{}_{i_{p-1}}^n$ and $I_{j_1}^n \backslash \overset{\circ}{I}{}_{j_1}^n$ respectively. We glue together these borders on the same concordant orientation, and we obtain the abstract oriented manifold V_p^n.

A directed n-dimensional torus is constructed on V_{p-1}^n beginning with an arbitrary n-dimensional cube, for example with $I_{i_{p-1}}^n$. By the $q = p - 1$ step of induction, V_{p-1}^n possesses a directed n-dimensional Euler tour, $C_{(p-1)}^n$, constructed of $I_{i_{p-1}}^n = I_{i_1}^n, I_{i_2}^n, \ldots, I_{i_{\beta_{p-1}^n}}^n$, where β_{p-1}^n is the number of all n-dimensional cubes on V_{p-1}^n. The last cube of the directed Euler tour is $I_{i_1}^n$, because this tour begins with $I_{i_{\beta_{p-1}^n}}^n$. This is why we take out the vacuum of the cube $I_{i_{\beta_{p-1}^n}}^n$, and so we have the subcomplex $C_{(p-1)}^n \backslash \overset{\circ}{I}{}_{i_{\beta_{p-1}^n}}^n$ of V_{p-1}^n. Similarly by $q = 1$ step of induction, the torus V_1^n possesses the directed Euler tour of dimension n, $C_{(1)}^n$, constructed of cubes $I_{j_1}^n, I_{j_2}^n, \ldots, I_{j_{\beta_1^n}}^n$. Taking out the vacuums $\overset{\circ}{I}{}_{i_{p-1}}^n$ and $\overset{\circ}{I}{}_{i_1}^n$ from $C_{(p-1)}^n$ and $C_{(1)}^n$ respectively, we glue together the borders $I_{i_{p-1}}^n \backslash \overset{\circ}{I}{}_{i_{p-1}}^n$ and $I_{i_1}^n \backslash \overset{\circ}{I}{}_{i_1}^n$ in the same concordance of orientation. This is possible because of the same orientation of V_{p-1}^n and V_1^n. Therefore the directed Euler tour on the manifold of genus p, V_p^n, is determined by the sequence of abstract n-dimensional cubes $I_{i_1}^n, I_{i_2}^n, \ldots, I_{i_{\beta_{p-1}^n}-1}^n, I_{j_1}^n, I_{j_2}^n, \ldots, I_{j_{\beta_1^n}-1}^n$. This sequence determines the directed Euler tour $C_{(p)}^n$ (see Figure 3, where $n = 2$ and the genus is p).

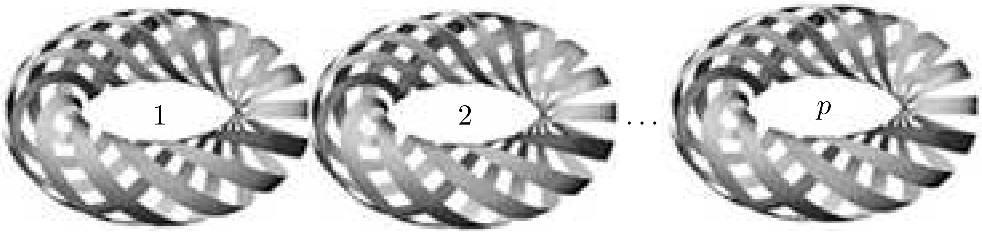

Figure 3

The theorem is proved.

Corollary 1. *Every manifold $V_p^n(\Box)$, $p \geq 2$, possesses a cubiliaj, but this one is not a normal cubiliaj as it results from the proof of Theorem 2.*

Hypothesis. For every abstract spherical manifold V_0^n, defined by cubes, there exists a cube $I^m \subset V_0^n$, $0 \leq m \leq n-1$, which is incident to less than 2^{n-m} number of n-dimensional cubes.

The hypothesis is verified for $n = 2$ [6].

Remark 1. It is possible that the device from paper [1] can be done more efficiently using any manifold $V_p^n(\Box)$, $p \geq 2$, instead of torus.

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