BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(57), 2008, Pages 106–109
ISSN 1024–7696

Groups with many hypercentral subgroups

O.D. Artemovych, L.V. Skaskiv

Abstract. We obtain a characterization of solvable groups with the minimal condition on non-hypercentral (respectively non-nilpotent) subgroups.

Mathematics subject classification: 20F16, 20F19, 20F22.
Keywords and phrases: Hypercentral group, nilpotent group, HM^*-group, minimal condition.

Introduction. We say that a group G satisfies the minimal condition on non-hypercentral subgroups Min-$\overline{ZA}$ (respectively, the minimal condition on non-nilpotent subgroups Min-$\overline{N}$) if for any properly descending chain $G_1 \geq G_2 \geq \cdots \geq G_n \geq \cdots$ of subgroups G_n in G there exists a number $m \in \mathbb{N}$ such that G_n is hypercentral (respectively, nilpotent) for each $n \geq m$. The minimal non-hypercentral (respectively, non-nilpotent) group satisfies Min-$\overline{ZA}$ (respectively, Min-$\overline{N}$). Recall if χ is a property pertaining to subgroups and an infinite group G is not a χ-group but all its proper subgroups have χ, then G is called a minimal non-χ group. Any Heineken-Mohamed type group (i.e. non-nilpotent groups with nilpotent and subnormal proper subgroups) is a minimal non-nilpotent group [1] (see e.g. [2]) and satisfies Min-$\overline{N}$ and Min-$\overline{ZA}$. In this paper we study the groups that satisfy Min-$\overline{ZA}$ (respectively Min-$\overline{N}$) and prove the following

Theorem. *Let G be a solvable group. Then G satisfies the minimal condition on non-hypercentral (respectively, non-nilpotent) subgroups if and only if one of the following holds:*

(1) G is a hypercentral (respectively, nilpotent) group;

(2) G is a Černikov group;

(3) $G = P \times Q$ is a group direct product of a hypercentral (respectively, nilpotent) Černikov p'-group Q and a non-hypercentral (respectively, non-nilpotent) p-group P which contains a normal HM^*-subgroup H of finite index (respectively, HM^*-subgroup H of finite index with the nilpotent commutator subgroup H') with the normalizer condition.

Note that earlier groups with the minimal condition on non-abelian subgroups have been studied by S.N. Černikov (see e.g. [3]) and V.P. Šunkov [4].

Throughout this paper p will always denote a prime. Most of the standard notations may by found in [2, 5] and [6]. Recall only that a group G is called an HM^*-group if its commutator subgroup G' is hypercentral and the quotient group

G/G' is a divisible Černikov p-group (see [7] and [8]). Every Heineken-Mohamed group is an HM^*-group. Other examples of HM^*-groups are contained in [8–10].

1. In the next we shall need some properfies of groups with Max-$\overline{N}$ (respectively, Max-$\overline{ZA}$).

Lemma 1. *Let G be a locally nilpotent group. If G satisfies Min-$\overline{N}$ (respectively, Min-$\overline{ZA}$), then G satisfies the normalizer condition and every minimal non-nilpotent (respectively, non-hypercentral) subgroup of G is subnormal.*

Proof. Let H be a proper subgroup of G. If H is either a non-nilpotent (respectively, non-hypercentral) or maximal nilpotent (respectively, hypercentral) subgroup of G, then the set $\{S|H < S \leq G\}$ has a minimal element, say M. Since H is a maximal subgroup of M, we conclude that H is normal in M. Moreover, every nilpotent (respectively, hypercentral) subgroup of G satisfies the normalizer condition and so G has also this property.

Let H be a minimal non-nilpotent (respectively, non-hypercentral) subgroup of G. Then the quotient group G/G' is quasicyclic (respectively, quasicyclic or trivial). If the derived subgroup H' is not normal in G, then $N_G(H')$ is a proper subgroup of $N_G(N_G(H'))$. Since any radicable abelian ascendant subgroup is subnormal [6, p.136], the subgroup H/H' is subnormal in M/H'. As a consequence, H is subnormal in M. The quotient group M/H' has a finite series whose quotients satisfy the minimal condition on subgroups and so it is a Černikov group.

Let $t \in N_G(M) \setminus M$. Then $(H')^t$ is normal in M and therefore $M/(H' \cap (H')^t)$ is Černikov. Hence $H' = H' \cap (H')^t$. Now it is not difficult to prove that $H' = (H')^t$, a contrary with the choice of t. Thus a subgroup H' is normal in G. Since H/H' is ascendant in G/H', we conclude by the same argument as above that H/H' is subnormal in G/H' and consequently H is subnormal in G. □

Corollary 1. *Let G be a non-nilpotent (respectively, non-hypercentral) locally nilpotent group satisfying Max-$\overline{N}$ (respectively, Max-$\overline{ZA}$). If all proper normal subgroups of G are nilpotent (respectively, hypercentral), then G is minimal non-nilpotent (respectively, non-hypercentral) group.*

Proof. Let H be a proper subgroup of G and H be a minimal non-nilpotent (respectively, non-hypercentral) group. By Lemma 1 H is subnormal in G. Then the normal closure H^G of H in G is a proper normal subgroup of G and, moreover, H^G is non-nilpotent (respectively, non-hypercentral), a contradiction. Hence G is a minimal non-nilpotent (respectively, non-hypercentral) group. □

2. Proof of Theorem. ($\Leftarrow$) is obvious.

($\Rightarrow$) Let G be a solvable group satisfying Min-$\overline{N}$ (respectively, Min-$\overline{ZA}$). We assume that G is neither nilpotent (respectively, hypercentral) nor a Černikov group.

Then G contains a subnormal non-nilpotent (respectively, non-hypercentral) subgroup H in which any normal subgroup is nilpotent (respectively, hypercentral). Furthermore, if $H = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ is a finite subnormal series connecting H to G, then every quotient G_{i+1}/G_i satisfies the minimal condition on subgroups and so it is Černikov.

If the subgroup H is not locally nilpotent, then it contains a finitely generated non-nilpotent subgroup F. Then $H = H'F$ and the quotient H/H' is cyclic of prime power order, because in other case H is nilpotent (respectively, hypercentral) as product of two nilpotent (respectively, hypercentral) normal subgroups. So the intersection $H' \cap F$ is a nilpotent subgroup. Now it is easy to see that F is finite. Since the set of all subgroups containing F satisfies the minimal condition, G is a Černikov group. This is a contradiction. Hence H is a locally nilpotent group and therefore by Corollary 1 H is a minimal non-nilpotent (respectively, non-hypercentral) p-group for some prime p. As a consequence, G is a locally finite group. By the above argument G is a locally nilpotent group.

To complete the proof it is enough to suppose that G is a p-group and to prove that in this case it contains an HM^*-subgroup of finite index satisfying the normalizer condition. This is obvious if $n = 0$ because $G = H$ is a minimal non-nilpotent (respectively, non-hypercentral) group and so G/G' is quasicyclic.

By induction on n, we may suppose that G_{n-1} contains an HM^*-group T of finite index. Since G_{n-1} is normal in G, without loss of generality we can assume that T is normal in G. Then the quotient group G/T is Černikov. If D is a preimage of the finite residual D/T of G/T, then D is an HM^*-subgroup of finite index in G. In view of Lemma 1 D satisfies the normalizer condition. The proof is complete. $\square$

Corollary 2. *Let G be a non-hypercentral (respectively, non-nilpotent) group. Then G satisfies Min-$\overline{ZA}$ (respectively, Min-$\overline{N}$) if and only if G satisfies the normalizer condition (respectively, G satisfies the normalizer condition and G' is nilpotent).*

References

[1] Heineken H., Mohamed I.J. *A group with trivial centre satisfying the normalizer condition.* J.Algebra, 1968, **10**, 368–376.

[2] Lennox J.C., Stonehewer S.E. *Subnormal subgroups of groups.* Clarendon Press, Oxford, 1987.

[3] Černikov S.N. *Groups with prescribed systems of subgroups.* Moskow, Nauka, 1980 (in Russian).

[4] Šunkov V.P. *On abstract characterization of some linear groups.* Algebra. Matrices and groups of matrices, 1970, 3–54 (in Russian).

[5] Robinson D.J.S. *Finiteness Conditions and Generalized Soluble Groups, Vol. 1.* Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[6] Robinson D.J.S. *Finiteness Conditions and Generalized Soluble Groups, Vol. 2.* Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[7] Asar A.O. *On non-nilpotent p-group and the normalizer condition*, Turkish J. Math., 1994, **18**, 114–129.

[8] Artemovych O.D. *Groups with the minimal condition on non-"nilpotent-by-finite" subgroups*. Serdica Math. J., 2002, **28**, 153–162.

[9] Čarin V.S. *One note on the minimal condition on subgroups*. DAN USSR, 1949, **66**, 575–576 (in Russian).

[10] Belyaev V.V. *Locally finite groups with almost abelian proper subgroups*. Sibir. Mat. Zh., 1983, **24**, 11–17 (in Russian).

O.D. Artemovych
Institute of Mathematics
Cracow University of Technology
Poland
E-mail: *artemousk.pk.edu.pl*

Received November 19, 2007

L.V. Skaskiv
Department of Algebra and Logic
Faculty of Mechanics and Mathematics
Ivan Franko National University of Lviv
Ukraine
E-mail: *lila_yonykua.fm*