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# Global Attractors of Quasi-Linear Non-Autonomous Difference Equations

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**Abstract.** The article is devoted to the study of global attractors of quasi-linear non-autonomous difference equations. We obtain the conditions for the existence of a compact global attractor. The obtained results are applied to the study of a special triangular map $T : R_+^2 \to R_+^2$ describing a growth model with logistic population growth rate.

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## 1 Introduction

The global attractors play a very important role in the qualitative study of difference equations (both autonomous and non-autonomous). The present work is dedicated to the study of global attractors of quasi-linear non-autonomous difference equations

$$u_{n+1} = A(\sigma(n,\omega))u_n + F(u_n, \sigma(n,\omega)), \tag{1}$$

where $\Omega$ is a metric space (generally speaking non-compact), $(\Omega, Z_+, \sigma)$ is a dynamical system with discrete time $Z_+$, $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness" (see condition (ii) in Theorem 4). An analogous problem was studied by Cheban D. and Mammana C. [6] when the space $\Omega$ is compact and Cheban D., Mammana C. and Michetti E. [8] in general case.

The obtained results are applied while studying a special class of triangular maps describing a discrete-time growth model of the Solow type where workers and shareholders have different but constant saving rates and the population growth rate dynamic is described by the logistic equation (see Brianzoni S., Mammana C. and Michetti E. [3]). The resulting system is given by $T = (T_2, T_1)$, where

$$T_2(u,\omega) = \frac{(1-\delta)u + (u^{\epsilon}+1)^{\frac{1-\epsilon}{\epsilon}}(s_w + s_r u^{\epsilon})}{1+\omega}$$

and

$$T_1(\omega) = \lambda\omega(1-\omega)$$

(for all $(u, \omega) \in R_+ \times [0,1]$), $\delta \in (0,1)$ is the depreciation rate of capital, $s_w \in (0,1)$ and $s_r \in (0,1)$ are the constant saving rates for workers and shareholders respectively, $\epsilon \in (-\infty, 1), \epsilon \neq 0$, is a parameter related to the elasticity of substitution between labor and capital.

This paper is organized as follows.

In Section 2 we establish the relation between triangular maps and non-autonomous dynamical systems with discrete time.

Section 3 is devoted to the study of the existence of compact global attractors of skew-product dynamical systems. The sufficient conditions of existence of compact global attractors for skew-product dynamical systems with non-compact base are given (Theorem 2).

In Section 4 we study the linear non-autonomous dynamical systems with discrete time and prove that they admit a unique compact invariant manifold and its description is given (Theorem 3).

In Section 5 we prove the existence of compact global attractors of quasi-linear dynamical systems (Theorem 5) and give the description of the structure of these attractors (Theorem 6).

In Section 6 we give some applications of general results from Sections 2–5 to the study of special class of the triangular maps $T : R^2_+ \to R^2_+$ describing a triangular growth model with logistic population growth rate as studied in Brianzoni S., Mammana C. and Michetti E. [3].

## 2 Triangular maps and non-autonomous dynamical systems

Let $W$ and $\Omega$ be two complete metric spaces and denote by $X := W \times \Omega$ their Cartesian product. Recall (see, for example, [16–18]) that a continuous map $F : X \to X$ is called triangular if there are two continuous maps $f : W \times \Omega \to W$ and $g : \Omega \to \Omega$ such that $F = (f, g)$, i.e. $F(x) = F(u, \omega) = (f(u, \omega), g(\omega))$ for all $x =: (u, \omega) \in X$.

Consider a system of difference equations

$$\begin{cases} u_{n+1} = f(u_n, \omega_n) \\ \omega_{n+1} = g(\omega_n), \end{cases} \tag{2}$$

for all $n \in Z_+$, where $Z_+$ is the set of all non-negative integer numbers.

Along with system (2) we consider the family of equations

$$u_{n+1} = f(u_n, g^n\omega) \ (\omega \in \Omega), \tag{3}$$

which is equivalent to system (2). Let $\varphi(n, u, \omega)$ be a solution of equation (3) passing through the point $u \in W$ for $n = 0$. It is easy to verify that the map $\varphi : Z_+ \times W \times \Omega \to W$ $((n, u, \omega) \mapsto \varphi(n, u, \omega)$ ) satisfies the following conditions:

1. $\varphi(0, u, \omega) = u$ for all $u \in W$ and $\omega \in \Omega$;
2. $\varphi(n+m, u, \omega) = \varphi(n, \varphi(m, u, \omega), \sigma(m, \omega))$ for all $n, m \in Z_+, u \in W$ and $\omega \in \Omega$, where $\sigma(n, \omega) := g^n\omega$;

3. the map $\varphi : Z_+ \times W \times \Omega \to W$ is continuous.

Denote by $(\Omega, Z_+, \sigma)$ the semi-group dynamical system generated by positive powers of the map $g : \Omega \to \Omega$, i.e. $\sigma(n, \omega) := g^n \omega$ for all $n \in Z_+$ and $\omega \in \Omega$.

Recall [5,19] that a triple $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$ (or briefly $\varphi$) is called a cocycle over the semi-group dynamical system $(\Omega, Z_+, \sigma)$ with fiber $W$.

Let $X := W \times \Omega$ and $(X, Z_+, \pi)$ be a semi-group dynamical system on $X$, where $\pi(n, (u, \omega)) := (\varphi(n, u, \omega), \sigma(n, \omega))$ for all $u \in W$ and $\omega \in \Omega$, then $(X, Z_+, \pi)$ is called [19] a skew-product dynamical system, generated by the cocycle $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$.

**Remark 1.** Thus, the reasoning above shows that every triangular map generates a cocycle and, obviously, vice versa, i.e. having a cocycle $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$ we can define a triangular map $F : W \times \Omega \to W \times \Omega$ by the equality

$$F(u, \omega) := (f(u, \omega), g(\omega)),$$

where $f(u, \omega) := \varphi(1, u, \omega)$ and $g(\omega) := \sigma(1, \omega)$ for all $u \in W$ and $\omega \in \Omega$. The semi-group dynamical system defined by the positive powers of the map $F : X \to X$ $(X := W \times \Omega)$ coincides with the skew-product dynamical system, generated by cocycle $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$

Taking into consideration this remark we can study triangular maps in the framework of cocycles with discrete time.

Let $(X, Z_+, \pi)$ (respectively, $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$) be a semi-group dynamical system (respectively, a cocycle).

A map $\gamma : Z \to X$ is called an entire trajectory of the semi-group dynamical system $(X, Z_+, \sigma)$ passing through the point $x \in X$ (respectively, $u \in W$) if $\gamma(0) = x$ and $\gamma(n + m) = \pi(m, \gamma(n))$ for all $n \in Z$ and $m \in Z_+$.

Denote by $\Phi_\omega(\sigma)$ the set of all the entire trajectories of the semi-group dynamical system $(\Omega, Z_+, \sigma)$ passing through the point $\omega \in \Omega$ at the initial moment $n = 0$ and $\Phi(\sigma) := \bigcup\{\Phi_\omega(\sigma) \mid \omega \in \Omega\}$.

A map $\mu : Z \to W$ is called an entire trajectory of the cocycle $\langle W, \varphi, (\Omega, Z_+, \sigma)\rangle$ passing through the point $(u, \omega) \in W \times \Omega$ if $\mu(0) = u$ and there exists $\alpha \in \Phi_\omega(\sigma)$ such that $\mu(n + m) = \varphi(m, \mu(n), \alpha(n))$ for all $n \in Z$ and $m \in Z_+$.

Let $Y$ be a complete metric space, $(X, Z_+, \pi)$ (respectively, $(Y, Z_+, \sigma)$) be a semi-group dynamical system on $X$ (respectively, $Y$), and $h : X \to Y$ be a homomorphism of $(X, Z_+, \pi)$ onto $(Y, Z_+, \sigma)$. Then the triple $\langle (X, Z_+, \pi), (Y, Z_+, \sigma), h\rangle$ is called a non-autonomous dynamical system.

Let $W$ and $Y$ be complete metric spaces, $(Y, Z_+, \sigma)$ be a semi-group dynamical system on $Y$ and $\langle W, \varphi, (Y, Z_+, \sigma)\rangle$ be a cocycle over $(Y, Z_+, \sigma)$ with the fiber $W$ (or, for short, $\varphi$), i.e. $\varphi$ is a continuous mapping of $Z_+ \times W \times Y$ into $W$ satisfying the following conditions: $\varphi(0, w, y) = w$ and $\varphi(t + \tau, w, y) = \varphi(t, \varphi(\tau, w, y), \sigma(\tau, y))$ for all $t, \tau \in Z_+, w \in W$ and $y \in Y$.

We denote $X := W \times Y$ and define on $X$ a skew product dynamical system $(X, Z_+, \pi)$ by the equality $\pi = (\varphi, \sigma)$, i.e. $\pi(t, (w, y)) = (\varphi(t, w, y), \sigma(t, y))$ for all

$t \in Z_+$ and $(w, y) \in W \times Y$. Then the triple $\langle (X, Z_+, \pi), ((Y, Z_+, \sigma), h\rangle$ is a non-autonomous dynamical system (generated by cocycle $\varphi$), where $h = pr_2 : X \mapsto Y$ is the projection on the second component.

## 3 Global attractors of dynamical systems

Let $\mathfrak{M}$ be a family of subsets from $X$.

A semi-group dynamical system $(X, Z_+, \pi)$ will be called $\mathfrak{M}$-dissipative if for every $\varepsilon > 0$ and $M \in \mathfrak{M}$ there exists $L(\varepsilon, M) > 0$ such that $\pi(n, M) \subseteq B(K, \varepsilon)$ for any $n \geq L(\varepsilon, M)$, where $K$ is a certain fixed subset from $X$ depending only on $\mathfrak{M}$. In this case we will call $K$ an attracting set for $\mathfrak{M}$.

For the applications the most important ones are the cases when $K$ is bounded or compact and $\mathfrak{M} := \{\{x\} \mid x \in X\}$ or $\mathfrak{M} := C(X)$, or $\mathfrak{M} := \{B(x, \delta_x) \mid x \in X,\ \delta_x > 0\}$, or $\mathfrak{M} := B(X)$ where $C(X)$ (respectively, $B(X)$) is the family of all compact (respectively, bounded) subsets from $X$.

The system $(X, Z_+, \pi)$ is called:

– point dissipative if there exists $K \subseteq X$ such that for every $x \in X$

$$\lim_{n \to +\infty} \rho(\pi(n, x), K) = 0; \tag{4}$$

– compactly dissipative if the equality (4) takes place uniformly w.r.t. $x$ on the compact subsets from $X$.

Let $(X, Z_+, \pi)$ be a compactly dissipative semi-group dynamical system and $K$ be an attracting set for $C(X)$. We denote by

$$J := \Omega(K) = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} \pi(m, K)},$$

then the set $J$ does not depend of the choice of $K$ and is characterized by the properties of the semi-group dynamical system $(X, Z_+, \pi)$. The set $J$ is called a Levinson center of the semi-group dynamical system $(X, Z_+, \pi)$.

**Theorem 1.** *[5] Let $(X, Z_+, \pi)$ be point dissipative. For $(X, Z_+, \pi)$ to be compactly dissipative it is necessary and sufficient that $\Sigma_K^+$ be relatively compact for any compact $K \subseteq X$.*

Let $E$ be a finite-dimensional Banach space and $\langle E, \varphi, (\Omega, Z_+, \sigma)\rangle$ be a cocycle over $(\Omega, Z_+, \sigma)$ with the fiber $E$ (or shortly $\varphi$).

A cocycle $\varphi$ is called:

- dissipative if there exists a number $r > 0$ such that

$$\limsup_{n \to +\infty} |\varphi(n, u, \omega)| \leq r \tag{5}$$

for all $\omega \in \Omega$ and $u \in E$;

- uniform dissipative on every compact subset from $\Omega$ if there exists a number $r > 0$ such that

$$\limsup_{n\to+\infty} \sup_{\omega\in\Omega', |u|\leq R} |\varphi(n,u,\omega)| \leq r$$

for all compact subset $\Omega^{'} \subseteq \Omega$ and $R > 0$.

Let $(X, Z_+, \pi)$ be a dynamical system and $x \in X$. Denote by $\omega_x$ the $\omega$-limit set of point $x$.

**Theorem 2.** *The following statements hold:*

1. *if the semi-group dynamical system $(\Omega, Z_+, \sigma)$ and the cocycle $\varphi$ are point dissipative, then the skew-product dynamical system $(X, Z_+, \pi)$ is point dissipative;*
2. *if the semi-group dynamical system $(\Omega, Z_+, \sigma)$ is compactly dissipative and the cocycle $\varphi$ is uniform dissipative on every compact subset from $\Omega$, then the skew-product system $(X, Z_+, \pi)$ is compactly dissipative.*

**Proof.** Let $x := (u, \omega) \in X := E \times \Omega$, then under the conditions of theorem the set $\Sigma_x^+ := \{\pi(n, x) : \ n \in Z_+\}$ is relatively compact and $\omega_x \subseteq B[0, r] \times K$, where $B[0, r] := \{u \in E : \ |u| \leq r\}$, $r$ is a number figuring in the inequality (5) and $K$ is the compact appearing in (4). Thus the semi-group dynamical system $(X, Z_+, \pi)$ is point dissipative.

According to first statement of theorem the skew-product dynamical system $(X, Z_+, \pi)$ is point dissipative. Let $M$ be an arbitrary compact subset from $X := E \times \Omega$, then there are $R > 0$ and a compact subset $\Omega^{'} \subseteq \Omega$ such that $M \subseteq B[0, R] \times \Omega^{'}$. Note that $\Sigma_M^+ := \{\pi(n, M) : \ n \in Z_+\} \subseteq \Sigma^+_{B[0,R]\times\Omega'} := \{(\varphi(n, u, \omega), \sigma(n, \omega)) : \ n \in Z_+, u \in B[0, R], \ \omega \in \Omega^{'}\}$. We will show that the set $\Sigma_M^+$ is relatively compact. In fact, let $\{x_k\} \subseteq \Sigma_M^+$, then there are $\{u_k\} \subseteq B[0, R]$, $\{\omega_k\} \subseteq \Omega^{'}$ and $\{n_k\} \subseteq Z_+$ such that $x_k = (\varphi(n_k, u_k, \omega_k), \sigma(n_k, \omega_k))$. By compact dissipativity of system $(\Omega, Z_+, \sigma)$ and uniform dissipativity of the cocycle $\varphi$ the sequences $\{\varphi(n_k, u_k, \omega_k)\}$ and $\sigma(n_k, \omega_k))$ are relatively compact and, consequently, the sequence $\{x_k\}$ is so. Now to finish the proof it is sufficient to refer to Theorem 1. $\square$

## 4 Linear non-autonomous dynamical systems

Let $\Omega$ be a complete metric space and $(\Omega, Z_+, \sigma)$ be a semi-group dynamical system on $\Omega$ with discrete time.

Recall that a subset $A \subseteq \Omega$ is called invariant (respectively, positively invariant, negatively invariant) if $\sigma(n, A) = A$ $(respectively, \sigma(n, A) \subseteq A, \ A \subseteq \sigma(n, A))$ for all $n \in Z_+$.

Below in this section we will suppose that the set $\Omega$ is invariant, i.e. $\sigma(n, \Omega) = \Omega$ for all $n \in Z_+$. Let $E$ be a finite-dimensional Banach space with the norm $|\cdot|$ and $W$ be a complete metric space. Denote by $[E]$ the space of all linear continuous operators on $E$ and by $C(\Omega, W)$ the space of all the continuous functions $f : \Omega \to W$

endowed with the compact-open topology, i.e. the uniform convergence on compact subsets in $\Omega$. The results of this section will be used in the next sections.

Consider a linear equation

$$u_{n+1} = A(\sigma(n,\omega))u_n \quad (\omega \in \Omega) \tag{6}$$

and an inhomogeneous equation

$$u_{n+1} = A(\sigma(n,\omega))u_n + f(\sigma(n,\omega)), \tag{7}$$

where $A \in C(\Omega, [E])$ and $f \in C(\Omega, E)$.

Recall that a linear bounded operator $P : E \to E$ is called a projection if $P^2 = P$, where $P^2 := P \circ P$.

Let $U(n,\omega)$ be the Cauchy operator of linear equation (6). Following [10] we will say that equation (6) has an exponential dichotomy on $\Omega$ if there exists a continuous projection valued function $P : \Omega \to [E]$ satisfying:

1. $P(\sigma(n,\omega))U(n,\omega) = U(n,\omega)P(\omega)$;
2. $U_Q(n,\omega)$ is invertible as an operator from $ImQ(\omega)$ to $ImQ(\sigma(n,\omega))$, where $Q(\omega) := I - P(\omega)$ and $U_Q(n,\omega) := U(n,\omega)Q(\omega)$;
3. there exist constants $0 < q < 1$ and $N > 0$ such that

   $$\|U_P(n,\omega)\| \leq Nq^n \text{ and } \|U_Q(n,\omega)^{-1}\| \leq Nq^n$$

   for all $\omega \in \Omega$ and $n \in Z_+$, where $U_P(n,\omega) := U(n,\omega)P(\omega)$.

Let $\omega \in \Omega$ and $\gamma_\omega \in \Phi_\omega(\sigma)$. Consider a difference equation

$$u_{n+1} = A(\gamma_\omega(n))u_n + f(\gamma_\omega(n)), \tag{8}$$

and the corresponding homogeneous linear equation

$$u_{n+1} = A(\gamma_\omega(n))u_n \quad (\omega \in \Omega). \tag{9}$$

Let $(X, \rho)$ be a metric space with distance $\rho$. Denote by $C(Z, X)$ the space of all the functions $f : Z \to X$ equipped with a product topology. This topology can be metricised. For example, by the equality

$$d(f_1, f_2) := \sum_{1}^{+\infty} \frac{1}{2^n} \frac{d_n(f_1, d_2)}{1 + d_n(f_1, d_2)},$$

where $d_n(f_1, d_2) := \max\{\rho(f_1(k), f_2(k)) \mid k \in [-n, n]\}$, a distance is defined on $C(Z, X)$ which generates the pointwise topology.

If $x \in X$ and $A, B \subseteq X$, then denote by $\rho(x, A) := \inf\{\rho(x, a) \mid a \in A\}$ and $\beta(A, B) := \sup\{\rho(a, B) \mid a \in A\}$ the semi-distance of Hausdorff.

Denote by $C(X)$ (respectively, $B(X)$) the family of all compact (respectively, bounded) subsets from $X$, $C(\Omega, E)$ the space of all the continuous functions $f : \Omega \to E$ , $C_b(\Omega, E) := \{f \in C(\Omega, E) : \; \|f\| := \sup_{\omega \in \Omega} |f(\omega)| < +\infty\}$ . Note that the space $C_b(\Omega, E)$ equipped with the norm $\| \cdot \|$ is a Banach space.

**Theorem 3.** *Suppose that the linear equation* (6) *has an exponential dichotomy on* $\Omega$*. Then for* $f \in C_b(\Omega, E)$ *the following statements hold:*

1. *the set* $I_\omega := \{u \in E \mid \exists \gamma_\omega \in \Phi_\omega$ *such that equation (8) admits a bounded solution* $\psi_\omega$ *defined on* $Z$ *with the initial condition* $\psi_\omega(0) = u\}$ *is nonempty and compact;*
2. $\varphi(n, I_\omega, \omega) = I_{\sigma(n,\omega)}$ *for all* $n \in Z_+$ *and* $\omega \in \Omega$*, where* $\varphi(n, u, \omega)$ *is a solution of equation (7) with the condition* $\varphi(0, u, \omega) = u$ *and* $\varphi(n, M, \omega) := \{\varphi(n, u, \omega) \mid u \in M\}$*;*
3. *the map* $\omega \to I_\omega$ *is upper-semicontinuous, i.e.*
$$\lim_{\omega \to \omega_0} \beta(I_\omega, I_{\omega_0}) = 0$$
*for every* $\omega_0 \in \Omega$*, where* $\beta$ *is the semi-distance of Hausdorff;*
4. *if* $\Omega$ *is compact, then the set* $I := \bigcup\{I_\omega \mid \omega \in \Omega\}$ *is also compact.*

**Proof.** Let $\omega \in \Omega$. Since $\Omega$ is invariant, the set $\Phi_\omega(\sigma) \neq \emptyset$. We fix $\gamma_\omega \in \Phi_\omega(\sigma)$. Under the conditions of Theorem 3 equation (9) has an exponential dichotomy on $\Omega$ with the same constants $N$ and $q$ that in equation (6). Then equation (8) admits the unique solution $\nu_{\gamma_\omega} : Z \to E$ with the condition

$$\|\nu_{\gamma_\omega}\|_\infty \le N\frac{1+q}{1-q}\|f(\nu_{\gamma_\omega}(\cdot))\|_\infty \le N\frac{1+q}{1-q}\|f\|, \tag{10}$$

where $\|b\| := \sup\{|f(\omega)| \mid \omega \in \Omega\}$ and $\|\nu_\omega\|_\infty := \sup\{|\nu_\omega(n)| \mid n \in Z\}$ (see, for example, [11, 15]). Thus, the set $I_\omega$ is not empty. From the continuity of the function $\varphi : Z_+ \times E \times \Omega \to E$ and inequality (10) it follows that the set $I_\omega$ is closed, bounded and
$$|u| \le N\frac{1+q}{1-q}\|f\|$$
for all $u \in I_\omega$ and $\omega \in \Omega$.

The second statement of the theorem follows from the equality $S_h(\Phi_\omega(\sigma)) = \Phi_{\sigma(h,\omega)}(\sigma)$ $(h \in Z)$, where $S_h\gamma_\omega$ is an $h$-translation of the trajectory $\gamma_\omega$, i.e. $S_h\gamma_\omega(n) := \gamma_\omega(n+h)$ for all $n \in Z$.

We will prove now the third statement. Let $\omega_0 \in \Omega$, $\omega_k \to \omega_0$, $u_k \in I_{\omega_k}$ and $u_k \to u$. To prove our statement it is sufficient to show that $u \in I_{\omega_0}$. Since $u_k \in I_{\omega_k}$, there is a trajectory $\gamma_{\omega_k} \in \Phi_{\omega_k}(\sigma)$ such that $\gamma_{\omega_k}$ converges to $\gamma_{\omega_0} \in \Phi_{\omega_0}(\sigma)$ in $C(Z, \Omega)$ and the equation

$$u_{n+1} = A(\gamma_{\omega_k}(n))u_n + f(\gamma_{\omega_k}(n)) \tag{11}$$

has a solution $\nu_{\gamma_{\omega_k}}$ with the initial condition $\nu_{\gamma_{\omega_k}}(0) = u_k$ and satisfying inequality (10), i.e.
$$|\nu_{\gamma_{\omega_k}}(n)| \le N\frac{1+q}{1-q}\|f(\nu_{\gamma_{\omega_k}})\|_\infty \le N\frac{1+q}{1-q}\|f\| \tag{12}$$

for all $n \in Z$ and $k = 1, 2, \dots$ . We will show that the sequence $\{\nu_{\gamma_{\omega_k}}(n)\}$ converges for every $n \in Z$. In fact, by Tihonoff theorem the sequence $\{\nu_{\omega_k}\} \subset C(Z, E)$ is relatively compact. From equality (11) and inequality (12) it follows that every limit point of the sequence $\{\nu_{\gamma_{\omega_k}}\}$ is a (bounded on $Z$) solution of the equation

$$u_{n+1} = A(\gamma_{\omega_0}(n))u_n + f(\gamma_{\omega_0}(n)). \tag{13}$$

Taking into account that equation (13) admits a unique solution bounded on $Z$, we obtain the convergence of the sequence $\{\nu_{\gamma_{\omega_k}}\}$ in the space $C(Z, E)$. We put $\nu_0 := \lim_{k \to +\infty} \nu_{\gamma_{\omega_k}}$. It is easy to see that $\nu_0(0) = u$ and, consequently, $u \in I_{\omega_0}$.

To prove the fourth assertion it is sufficient to remark that for every $\omega \in \Omega$ the set $I_\omega$ is compact, the map $\omega \to I_\omega$ is upper-semicontinuous and, consequently, the set $I := \bigcup\{I_\omega \mid \omega \in \Omega\}$ is compact. The theorem is completely proved. □

## 5 Global attractors of quasi-linear triangular systems

Consider a difference equation

$$u_{n+1} = \mathcal{F}(u_n, \sigma(n, \omega)) \ (\omega \in \Omega). \tag{14}$$

Denote by $\varphi(n, u, \omega)$ a unique solution of equation (14) with the initial condition $\varphi(0, u, \omega) = u$.

Equation (14) is said to be dissipative (respectively, uniform dissipative on every compact subset from $\Omega$) if there exists a positive number $r$ such that

$$\limsup_{n \to +\infty} |\varphi(n, u, \omega)| \le r \quad (\text{respectively, } \limsup_{n \to +\infty} \sup_{\omega \in \Omega', |u| \le R} |\varphi(n, u, \omega)| \le r)$$

for all $u \in E$ and $\omega \in \Omega$ (respectively, for all $R > 0$ and $\Omega' \in C(\Omega)$).

Consider a quasi-linear equation

$$u_{n+1} = A(\sigma(n, \omega))u_n + F(u_n, \sigma(n, \omega)), \tag{15}$$

where $A \in C(\Omega, [E])$ and the function $F \in C(E \times \Omega, E)$ satisfies "the condition of smallness" (condition (ii) in Theorem 4).

Denote by $U(k, \omega)$ the Cauchy matrix for the linear equation

$$u_{n+1} = A(\sigma(n, \omega))u_n.$$

**Theorem 4.** *Suppose that the following conditions hold:*

1. *there are positive numbers $N$ and $q < 1$ such that*

$$\|U(n, \omega)\| \le Nq^n \qquad (n \in Z_+); \tag{16}$$

2. *$|F(u, \omega)| \le C + D|u|$ $(C \ge 0,\ 0 \le D < (1 - q)N^{-1})$ for all $u \in E$ and $\omega \in \Omega$.*

*Then equation (15) is uniform dissipative on every compact subset from* $\Omega$.

**Proof.** Let $\varphi(\cdot,u,\omega)$ be the solution of equation (14) passing through the point $u\in E$ for $n=0$. According to the formula of the variation of constants (see, for example,[14] and [15]) we have

$$\varphi(n,u,\omega)=U(k,\omega)u+\sum_{m=0}^{n-1}U(n-m-1,\omega)F(\varphi(m,u,\omega),\sigma(m,\omega)),$$

and, consequently,

$$|\varphi(n,u,\omega)|\le Nq^n|u|+\sum_{m=0}^{n-1}q^{n-m-1}(C+D|\varphi(m,u,\omega)|). \tag{17}$$

We set $u(n):=q^{-n}|\varphi(n,u,\omega)|$ and, taking into account (17), obtain

$$u(n)\le N|u|+CNq^{-1}\sum_{m=0}^{n-1}q^{-m}+DNq^{-1}\sum_{m=0}^{n-1}u(m). \tag{18}$$

Denote the right hand side of inequality (18) by $v(n)$. Note that

$$v(n+1)-v(n)=q^{-n}\frac{CN}{q}+\frac{DN}{q}u(n)\le\frac{DN}{q}v(n)+\frac{CN}{q}q^{-n},$$

and, hence,

$$v(n+1)\le\Big(1+\frac{DN}{q}\Big)v(n)+\frac{CN}{q}q^{-n}.$$

From this inequality we obtain

$$v(n)\le\Big(1+\frac{DN}{q}\Big)^{n-1}v(1)+\frac{CN}{q}\frac{1-q^{n-1}}{1-q}.$$

Therefore,

$$|\varphi(n,u,\omega)|\le(q+DN)^{n-1}qN|u|+\frac{CN}{q-1}(q^{n-1}-1), \tag{19}$$

because $v(1)=N|u|$. From (19) it follows that

$$\limsup_{n\to+\infty}\sup_{\omega\in\Omega',|u|\le R}|\varphi(n,u,\omega)|\le\frac{CN}{1-q}$$

for all $R>0$ and $\Omega'\in C(\Omega)$. The theorem is proved. $\square$

Let $\langle E,\varphi,(\Omega,Z_+,\sigma)\rangle$ be a cocycle over $(\Omega,Z_+,\sigma)$ with the fiber $E$.

A family $\{I_\omega\mid\omega\in J_\Omega\}$ of nonempty compact subsets $I_\omega\subset E$ is called a compact global attractor of the cocycle $\varphi$ if the following conditions are fulfilled:

1. the semi-group dynamical system $(\Omega,Z_+,\sigma)$ is compactly dissipative;

2. the set $I := \bigcup\{I_\omega \mid \omega \in J_\Omega\}$ is relatively compact, where $J_\Omega$ is the Levinson center of $(\Omega, Z_+, \sigma)$;

3. the family $I := \{I_\omega \mid \omega \in J_\Omega\}$ is invariant with respect to the cocycle $\varphi$, i.e. $\cup\{\varphi(n, I_q, q) \mid q \in (\sigma^n)^{-1}(\sigma(n,\omega))\} = I_{\sigma(n,\omega)}$ for all $n \in Z_+$ and $\omega \in J_\Omega$, where $\sigma^n := \sigma(n, \cdot)$;

4. the equality
$$\lim_{n\to+\infty} \sup_{\omega\in\Omega'} \beta(\varphi(n, K, \omega), I) = 0$$
takes place for every $K \in C(E)$ and $\Omega' \in C(\Omega)$, where $C(E)$ (respectively, $C(\Omega)$) is a family of compact subsets from $E$ (respectively, $\Omega$).

**Lemma 1.** *The cocycle $\varphi$ is compactly dissipative if and only if the skew-product system $(X, Z_+, \pi)$ ($X := E \times \Omega$ and $\pi := (\varphi, \sigma)$) is so.*

**Proof.** This statement follows directly from the correspondig definitions. □

**Theorem 5.** *Let $(\Omega, Z_+, \sigma)$ be a compactly dissipative system and $\varphi$ be a cocycle generated by equation (15). Under the conditions of Theorem 4 the skew-product system $(X, Z_+, \pi)$ ($X := E \times \Omega$ and $\pi := (\varphi, \sigma)$), generated by cocycle $\varphi$ admits a compact global attractor.*

**Proof.** This statement follows directly from Theorems 4, 2 and Lemma 1. □

**Remark 2.** Simple examples show that under the conditions of Theorem 5 the compact global attractor $\{I_\omega \mid \omega \in \Omega\}$, generally speaking, is not trivial, i.e. the component set $I_\omega$ contains more than one point. This statement can be illustrated by the following example: $u_{n+1} = \dfrac{1}{2}u_n + \dfrac{2u_n}{1+u_n^2}$.

**Theorem 6.** *Let $A \in C(\Omega, [E])$ and $F \in C(E \times \Omega, E)$ and the following conditions be fulfilled:*

1. *the semi-group dynamical system $(\Omega, Z_+, \sigma)$ is compactly dissipative and $J_\Omega$ is its Levinson center;*

2. *there exist positive numbers $N$ and $q < 1$ such that inequality (16) holds;*

3. *there exists $C > 0$ such that $|F(0, \omega)| \le C$ for all $\omega \in \Omega$;*

4. *$|F(u_1, \omega) - F(u_2, \omega)| \le L|u_1 - u_2|$ $(0 \le L < N^{-1}(1 - q))$ for all $\omega \in \Omega$ and $u_1, u_2 \in E$.*

*Then*

1. *the equation (15) (the cocycle $\varphi$ generated by this equation) admits a compact global attractor;*

*2. there are two positive constants $\mathcal{N}$ and $\nu < 1$ such that*

$$|\varphi(n,u_1,\omega) - \varphi(n,u_2,\omega)| \le \mathcal{N}\nu^n |u_1 - u_2| \tag{20}$$

*for all $u_1, u_2 \in E,\ \omega \in \Omega$ and $n \in Z_+$.*

**Proof.** First step. We will prove that under the conditions of Theorem 6 equation (15) admits a compact global attractor $I = \{I_\omega \mid \omega \in J_\Omega\}$. In fact,

$$|F(u,\omega)| \le |F(0,\omega)| + L|u| \le C + L|u|$$

for all $u \in E$, where $C := \sup\{|F(0,\omega)| \mid \omega \in \Omega\}$. According to Theorems 2 and 4, equation (15) admits a compact global attractor $I = \{I_\omega \mid \omega \in J_\Omega\}$.

Second step. Let $\varphi$ be the cocycle generated by equation (15). In virtue of the formula of the variation of constants, we have

$$\varphi(n,u,\omega) = U(n,\omega)u + \sum_{m=0}^{n-1} U(n-m-1,\omega)F(\varphi(m,u,\omega),\sigma(m,\omega)).$$

Consequently,

$$\varphi(n,u_1,\omega) - \varphi(n,u_2,\omega) = U(n,\omega)(u_1 - u_2) +$$

$$\sum_{m=1}^{n-1} U(n-m-1,A)[F(\varphi(m,u,\omega),\sigma(m,\omega)) - F(\varphi(m,u_2,\omega),\sigma(m,\omega))].$$

Thus,

$$\begin{aligned} &|\varphi(n,u_1,\omega) - \varphi(n,u_2,\omega)| \le Nq^n(|u_1 - u_2| \\ &+Lq^{-1}\sum_{m=0}^{n-1} q^{-m}|\varphi(m,u_1,\omega) - \varphi(m,u_2,\omega)|). \end{aligned} \tag{21}$$

Let $u(n) := |\varphi(n,u_1,\omega) - \varphi(n,u_2,\omega)|q^{-n}$. From (21) it follows that

$$u(n) \le N\Big(|u_1 - u_2| + Lq^{-1}\sum_{m=0}^{n-1} u(m)\Big). \tag{22}$$

Denote by $v(n)$ the right hand side of (22). The following inequality

$$v(n+1) - v(n) = LNq^{-1}u(n) \le LNq^{-1}v(n). \tag{23}$$

holds. From (23) we obtain

$$v(n) \le (1 + LNq^{-1})^{n-1}v(1)$$

and, since $v(1) = N|u_1 - u_2|$, we get

$$u(n) \le (1 + LNq^{-1})N|u_1 - u_2|. \tag{24}$$

From (24) we have

$$|\varphi(n,u_1,\omega)-\varphi(n,u_2,\omega)| \le (q+LN)^{n-1}qN|u_1-u_2| \tag{25}$$

for all $u_1,u_2 \in E$ and $\omega \in \Omega$.

To finish the proof of Theorem it is sufficient to put $\nu := q+LN$ and $\mathcal{N} := qN(q+LN)^{-1}$. The theorem is proved. □

**Remark 3.** It is possible to show that under the conditions of Theorems 3 and 6 the set $I_\omega$ contains a single point (for all $\omega \in J_\Omega$) if the mapping $\sigma(1,\cdot) : \Omega \to \Omega$ is invertible. If the mapping $\sigma(1,\cdot)$ is not invertible, then the set $I_\omega$ may be very complicated (for example $I_\omega$ may be a Cantor set). Below we give an example which confirms this statement.

**Example 1.** Let $Y := [-1,1]$ and $(Y,Z_+,\sigma)$ be a cascade generated by positive powers of the odd function $g$, defined on $[0,1]$ in the following way:

$$g(y)=\begin{cases} -2y & , \quad 0\le y\le \frac{1}{2} \\ 2(y-1) & , \quad \frac{1}{2}< y\le 1. \end{cases}$$

It is easy to check that $g(Y)=Y$. Let us put $X := R\times Y$ and denote by $(X,Z_+,\pi)$ a semi-group dynamical system generated by the positive powers of the mapping $P : X \to X$

$$P\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} f(u,y) \\ g(y) \end{pmatrix}, \tag{26}$$

where $f(u,y) := \frac{1}{10}u + \frac{1}{2}y$. Finally, let $h = pr_2 : X \to Y$. From (26), it follows that $h$ is a homomorphism of $(X,Z_+,\pi)$ onto $(Y,Z_+,\sigma)$ and, consequently, $\langle (X,Z_+,\pi),(Y,Z_+,\sigma),h\rangle$ is a non-autonomous dynamical system. Note that

$$|(u_1,y)-(u_2,y)| = |u_1-u_2| = 10|P(u_1,y)-P(u_2,y)|. \tag{27}$$

From (27), it follows that

$$|P^n(u_1,y)-P^n(u_2,y)| \le \mathcal{N}e^{-\nu n}|(u_1,y)-(u_2,y)| \tag{28}$$

for all $n \in Z_+$, where $\mathcal{N} = 1$ and $\nu = \ln 10$. By Theorem 6 the cocycle $\langle R,\varphi,(Y,Z_+,\sigma)\rangle$ admits a compact global attaror $I := \{I_y : \ y \in Y\}$ and $\varphi$ is exponentially convergent, i.e. the inequality (20) takes place. According to [18, p.43] $I_y$ is homeomorphic to the Cantor set for all $y \in [-1,1]$.

**Remark 4.** 1. If $\Omega$ is a compact metric space the close results (Sections 2–5) were established in [6].

2. The results of Sections 2–5 are true also in the case we replace the finite-dimensional Banach space $E$ by its closed subset.

# 6 Applications

## 6.1 The model

The model we consider is a particular case of the growth model by Solow; it has been obtained while considering the standard, neoclassical one-sector growth model where the two types of agents, workers and shareholders, have different but constant saving rates as in Bohm V. and Kaas L. [4] and where the production function $F : R_+ \to R_+$, mapping capital per worker $k$ into output per worker $y$, is of the CES type (as in Brianzoni S., Mammana C. and Michetti E. [1] and [2]), that is given by

$$F(u) = (1 + u^\epsilon)^{\frac{1}{\epsilon}}. \tag{29}$$

However in the present work we add a further assumption, that is the population growth rate evolves according to the logistic law, as also considered in Brianzoni S., Mammana C. and Michetti E. [3].

The resulting system, $T = (\omega', u')$, describing capital per worker ($u$) and population growth rate ($\omega$) dynamics, is given by:

$$T := \begin{cases} u' = \frac{1}{1+\omega}\left[(1-\delta)u + (u^\epsilon + 1)^{\frac{1-\epsilon}{\epsilon}}(s_w + s_r u^\epsilon)\right] \\ \\ \omega' = \lambda\omega(1-\omega) \end{cases} \tag{30}$$

where $\delta \in (0, 1)$ is the depreciation rate of capital, $s_w \in (0, 1)$ and $s_r \in (0, 1)$ are the constant saving rates for workers and shareholders respectively, $\epsilon \in (-\infty, 1), \epsilon \neq 0$ is a parameter related to the elasticity of substitution between labor and capital (the elasticity of substitution between the two production factors is given by $\dfrac{1}{1-\epsilon}$) and, finally, $\lambda \in (0, 4]$ for the dynamics generated by the logistic map not being explosive.

We get a dicrete-time dynamical system described by the iteration of a map of the plane of triangular type. In fact the second component of the previous system does not depend on $k$, therefore the map is characterized by the triangular structure:

$$T := \begin{cases} u' = g(u, \omega) \\ \\ \omega' = f(\omega) \end{cases} . \tag{31}$$

As a consequence, the dynamics of the map $T$ are influenced by the dynamics of the one–dimensional map $f$, that is the well-known logistic map.

## 6.2 Dynamics of the logistic map $f_\lambda(x) = \lambda x(1-x)$

We recall some general results for map $f_\lambda$ (see, for example, [20]). For $\lambda \in (0, 4]$ the map $f_\lambda$ acts from interval $[0, 1]$ into itself and, consequently, it admits a compact global attractor $I_\lambda \subseteq [0, 1]$. Since $I_\lambda$ is connected (see, for example, Theorem 1.33 [5]) and $0 \in I_\lambda$, then $I_\lambda = [0, a_\lambda]$ ($a_\lambda \leq 1$).

1. If $0 < \lambda \leq \lambda_0 := 1$, then $I_\lambda = \{0\}$.

2. If $\lambda_0 < \lambda < \lambda_1 := 3$, then the map $f_\lambda$ has two fixed points: $x = 0$ is a repelling fixed point and $p_0 = 1 - 1/\lambda$ is an attracting fixed point. If $x \in I_\lambda \setminus \{0, p_0\}$, then $\alpha_x = 0$ and $\omega_x = p_0$.

3. If $\lambda_1 < \lambda \leq \lambda_2 := 1 + \sqrt{6}$, then the map $f_\lambda$ has one repelling fixed point $x = 0$ and there is an attracting 2-periodic point $p_1$.

4. There exists a increasing sequence $\{\lambda_k\}_{k=0}^{\infty}$ such that

   (a) $\lambda_k \to \lambda_\infty$ as $k \to \infty$, where $\lambda_\infty \approx 3,569\dots$.

   (b) If $\lambda_k < \lambda < \lambda_{k+1}$ $(k = 2, 3, \dots)$, then the map $f_\lambda$ has one repelling fixed point $x = 0$ and there is an attracting $2^k$-periodic point $p_k$.

5. For all $0 < \lambda < \lambda_\infty$ the structure of the attractor $I_\lambda$ is sufficiently simple. Every trajectory is asymptotically periodic. There exists a unique attracting $2^m-$periodic point $p$ (the number $m$ depends on $\lambda$) which attracts all trajectory from $[0, 1]$, except for a countable set of points. For $\lambda \geq \lambda_\infty$ the attractor $I_\lambda$ is more complicated, in particularly, it may be a strange attractor (see [20]).

Let $(X, Z_+, \pi)$ be a semi-group dynamical system with discrete time.

A number $m$ is called an $\varepsilon$-almost period of the point $x$ if $\rho(\pi(m+n, x), \pi(n, x)) < \varepsilon$ for all $n \in Z_+$.

The point $x$ is called almost periodic if for any $\varepsilon > 0$ there exists a positive number $l \in Z_+$ such that on every segment (in $Z_+$) of length $l$ there may be found an $\varepsilon$-almost period of the point $x$.

(vi) Denote by $Per(f_\lambda)$ the set of all periodic points of $f_\lambda$. If $\lambda = \lambda_\infty$, then the map $f_\lambda$ has the $2^i$-periodic point $p_i$ for all $i \in Z_+$ (all the points $p_i$ are repelling). The boundary $K = \partial Per(f_\lambda)$ of set $P(f_\lambda)$ is a Cantor set. The set $K$ is an almost periodic minimal and it does not contain periodic points. The set $K$ attracts all trajectory from $[0, 1]$, except for a countable set of points $P = \cup_{i=0}^{\infty} f_\lambda^{-i}(Per(f_\lambda))$. If $x \in [0, 1] \setminus P$, then $\omega_x = K$ (see [20]).

### 6.3 Existence of an attractor for $\epsilon \in (-\infty, 0)$

**Lemma 2.** *Let $(R_+ \times [0, 1], T)$ be a triangular map admitting a compact global attractor $J \subset R_+ \times [0, 1]$. If $p \in [0, 1]$ is a $m$-periodic point of the map $T_1 : [0, 1] \mapsto [0, 1]$ ($T = (T_2, T_1)$), then*

1. *$J_p = I_p \times \{p\}$, where $I_p = [a_p, b_p]$ ($a_p, b_p \in R_+$ and $a_p \leq b_p$);*

2. *there exists $q \in I_p = [a_p, b_p]$ such that $(q, p)$ is an $m$-periodic point of the map $T$.*

**Proof.** Let $p \in [0,1]$ be an $m$-periodic point of $T_1$, i.e. $T_1^m(p) = p$. Denote by $S := T^m$ the mapping from $X_p := R_+ \times \{p\}$ into itself. Then, the semi-group dynamical system $(X_p, S)$ is compactly dissipative and its Levinson center coincides with $J_p = I_p \times \{p\}$. By Theorem 1.33 from [5] the compact set $I_p \subset R_+$ is connected and, consequently, there are $a_p, b_p \in R_+$ such that $a_p \leq b_p$, $I_p = [a_p, b_p]$ and

$$U(m,p)[a_p, b_p] = [a_p, b_p], \tag{32}$$

where $T^m(q,p) = (U(m,p)q, T_1^m(p))$ for all $(q,p) \in R_+ \times [0,1]$. Since $U(m,p)$ is a continuous mapping from $[a_p, b_p]$ onto itself, then there exists at least one $q \in [a_p, b_q]$ such that $U(m,p)q = q$. It is evident that $(q,p)$ is an $m$-periodic point of the mapping $T = (T_2, T_1)$. □

**Theorem 7.** *For all $\epsilon < 0$ the dynamical system $(R_+ \times [0,1], T)$ admits a compact global attractor $J \subset R_+ \times [0,1]$. If $p \in [0,1]$ is an $m$-periodic point of the map $T_1 : [0,1] \mapsto [0,1]$ ($T = (T_2, T_1)$), then*

1. *$J_p = I_p \times \{p\}$, where $I_p = [a_p, b_p]$ ($a_p, b_p \in R_+$ and $a_p \leq b_p$);*
2. *there exists $q \in I_p = [a_p, b_p]$ such that $(q,p)$ is an $m$-periodic point of the map $T$.*

**Proof.** Assume $\epsilon \in (-\infty, 0)$ and let $\lambda = -\epsilon$, then $\lambda \in (0, +\infty)$. We write $T_1$ in terms of $\lambda$

$$\begin{aligned}
T_1(u,\omega) &= \frac{1}{1+\omega}\left[(1-\delta)u + (u^{-\lambda}+1)^{\frac{1+\lambda}{-\lambda}}(s_w + s_r u^{-\lambda})\right] = \\
&= \frac{1}{1+\omega}\left[(1-\delta)u + \left(\frac{1+u^\lambda}{u^\lambda}\right)^{-\frac{1+\lambda}{\lambda}}\left(\frac{s_r + s_w u^\lambda}{u^\lambda}\right)\right] = \\
&= \frac{1}{1+\omega}\left[(1-\delta)u + \left(\frac{u^\lambda}{1+u^\lambda}\right)^{\frac{1+\lambda}{\lambda}}\left(\frac{s_r + s_w u^\lambda}{u^\lambda}\right)\right] = \\
&= \frac{1}{1+\omega}\left[(1-\delta)u + \frac{u}{(1+u^\lambda)^{\frac{1+\lambda}{\lambda}}}(s_r + s_w u^\lambda)\right] = \\
&= \frac{1}{1+\omega}\left[(1-\delta)u + \frac{u}{(1+u^\lambda)^{\frac{1}{\lambda}}}\frac{s_r + s_w u^\lambda}{1+u^\lambda}\right].
\end{aligned} \tag{33}$$

Note that $\dfrac{u}{(1+u^\lambda)^{\frac{1}{\lambda}}} \longrightarrow 1$ as $u \longrightarrow +\infty$, $\dfrac{s_r + s_w u^\lambda}{1+k^\lambda} \longrightarrow s_w$ as $u \longrightarrow +\infty$ and, consequently, there exists $M > 0$ such that

$$\left|\frac{u}{(1+u^\lambda)^{\frac{1}{\lambda}}}\frac{s_r + s_w u^\lambda}{1+u^\lambda}\right| \leq M, \tag{34}$$

for all $u \in [0, +\infty)$.

Since $0 \leq \frac{1}{1+\omega} \leq 1$ for all $\omega \in [0, 1]$, then from (33) and (34) we obtain

$$0 \leq T_1(u, \omega) \leq \alpha u + M \tag{35}$$

for all $(u, \omega) \in R_+ \times [0, 1]$, where $\alpha := 1 - \delta > 0$.

Since the map $T$ is triangular, to prove the first statement of Theorem it is sufficient to apply Theorem 5. The second statement follows from Lemma 2. $\square$

**Remark 5.** 1. It is easy to see that the previous theorem is true also for $\delta = 1$ because in this case $\alpha = 1 - \delta = 0$ and from (35) we have $T_1(u, \omega) \leq M$, $\forall (u, \omega) \in R_+ \times [0, 1]$. Now it is sufficient to refer to Theorem 2.

2. If $\delta = 0$ the problem is open.

### 6.4 Existence of an attractor for $\epsilon \in (0, 1)$ and $s_r < \delta$

The semi-group dynamical system $(X, Z_+, \pi)$ is said to be:

- locally completely continuous if for every point $p \in X$ there exist $\delta = \delta(p) > 0$ and $l = l(p) > 0$ such that $\pi^l B(p, \delta)$ is relatively compact;
- weakly dissipative if there exists a nonempty compact $K \subseteq X$ such that for every $\varepsilon > 0$ and $x \in X$ there is $\tau = \tau(\varepsilon, x) > 0$ for which $\pi(\tau, x) \in B(K, \varepsilon)$. In this case we will call $K$ a weak attractor.

Note that every semi-group dynamical system $(X, Z_+, \pi)$ defined on the locally compact metric space $X$ is locally completely continuous.

**Theorem 8.** *[5] For the locally completely continuous dynamical systems the weak, point and compact dissipativity are equivalent.*

**Theorem 9.** *For all $\epsilon \in (0, 1)$ and $s_r < \delta$ the dynamical system $(R_+ \times [0, 1], T)$ admits a compact global attractor $J \subset R_+ \times [0, 1]$. If $p \in [0, 1]$ is an $m$-periodic point of the map $T_1 : [0, 1] \mapsto [0, 1]$ $(T = (T_2, T_1))$, then*

1. *$J_p = I_p \times \{p\}$, where $I_p = [a_p, b_p]$ ($a_p, b_p \in R_+$ and $a_p \leq b_p$);*
2. *there exists $q \in I_p = [a_p, b_p]$ such that $(q, p)$ is an $m$-periodic point of the map $T$.*

**Proof.** If $\epsilon \in (0, 1)$ we have

$$T_1(u, \omega) = \frac{1}{1+\omega}\left[(1-\delta)u + (u^\epsilon + 1)^{\frac{1-\epsilon}{\epsilon}}(s_w + s_r u^\epsilon)\right] =$$

$$= \frac{1}{1+\omega}\left[(1-\delta)u + \frac{(u^\epsilon + 1)^{\frac{1}{\epsilon}}}{1+u^\epsilon}(s_w + s_r u^\epsilon)\right] =$$

$$= \frac{1}{1+\omega}\left[(1-\delta)u + s_r u + \theta(u)u\right] \tag{36}$$

where $\theta(u) \to 0$ as $u \to +\infty$. In fact $\dfrac{(u^\epsilon+1)^{\frac{1}{\epsilon}}}{u} \to 1$ as $u \to +\infty$ while $\dfrac{(s_w + s_r u^\epsilon)}{1+u^\epsilon} \to s_r$ as $u \to +\infty$ and, consequently,

$$\frac{\frac{(u^\epsilon+1)^{\frac{1}{\epsilon}}}{1+u^\epsilon}(s_w+s_ru^\epsilon)}{s_ru} = \frac{(u^\epsilon+1)^{\frac{1}{\epsilon}}}{u}\frac{(s_w+s_ru^\epsilon)}{s_r(u^\epsilon+1)} \to 1$$

as $u \to +\infty$, i.e. $\dfrac{(u^\epsilon+1)^{\frac{1}{\epsilon}}}{1+u^\epsilon}(s_w+s_ru^\epsilon) = s_ru + \theta(u)u$. From (36) we have

$$T_1(u,\omega) = \frac{1}{1+\omega}\left[(1-\delta+s_r)u + \theta(u)u\right]$$

for all $(u,\omega) \in R^2_+$.

Since $s_r < \delta$ then $\alpha := 1-\delta+s_r < 1$. Let $R_0 > 0$ be a positive number such that

$$|\theta(u)| < \frac{1-\alpha}{2}, \tag{37}$$

for all $u > R_0$. Note that for every $(u_0,\omega_0) \in R_+ \times [0,1]$, with $u_0 > R_0$, the trajectory $\{T^n(u,\omega) \mid n \in Z_+\}$ starting from point $(u_0,\omega_0)$ at the initial moment $n = 0$, at least one time intersects the compact $K_0 := [0,h_0] \times [0,R_0]$, $(h_0 > h)$. In fact, if we suppose that this statement is false, then there exists a point $(u_0,\omega_0) \in R_+ \times [0,1] \setminus K_0$ such that

$$(u_n,\omega_n) := T^n(u_0,\omega_0) \in R_+ \times [0,1] \setminus K_0 \tag{38}$$

for all $n \in Z_+$. Taking into consideration that $\omega_n \to h$ (or 0) as $n \to +\infty$, we obtain from (38) that $u_n > R_0$ for all $n \geq n_0$, where $n_0$ is a sufficiently large number from $Z_+$. Without loss of generality, we may suppose that $n_0 = 0$ (if $n_0 > 0$ then we start from the initial point $(u_{n_0},\omega_{n_0}) := T^{n_0}(u_0,\omega_0)$, where $T^{n_0} := T \circ T^{n_0-1}$ for all $n_0 \geq 2$). Thus we have

$$u_n > R_0 \tag{39}$$

for all $n \geq 0$ and

$$u_{n+1} = \frac{1}{1+\omega}\left[\alpha u_n + \theta(u_n)u_n\right] \tag{40}$$

From (37) and (40) we obtain

$$u_{n+1} \leq \alpha u_n + \frac{1-\alpha}{2}u_n = \frac{1+\alpha}{2}u_n \tag{41}$$

since $\dfrac{1}{1+\omega} \leq 1$ for all $\omega \geq 0$. From (41) we have

$$u_n \leq \left(\frac{1+\alpha}{2}\right)^n u_0 \to 0 \text{ as } n \to +\infty, \tag{42}$$

but (39) and (42) are contradictory. The obtained contradiction proves the statement. Let now $(u_0,\omega_0) \in R_+ \times [0,1]$ be an arbitrary point.

(a) If $u_0 < R_0$ and $u_n \le R_0$ for all $n \in N$, then $\limsup_{n\to+\infty} u_n \le R_0$;

(b) If there exists $n_0 \in N$ such that $u_{n_0} > R_0$, then there exists $m_0 \in N$ ($m_0 > n_0$) such that $(u_{m_0}, \omega_{m_0}) \in K_0$ (see the proof above).

Thus we proved that for all $(u_0, \omega_0) \in R_+^2$ there exists $m_0 \in N$ such that $(u_{m_0}, \omega_{m_0}) \in K_0$. According to Theorem 8 the semi-group dynamical system $(R_+ \times [0,1], T)$ admits a compact global attractor.

The second statement follows from Lemma 2. The theorem is proved. $\square$

### 6.5 Structure of the attractor

**Lemma 3.** *Suppose that the following conditions are fulfilled:*

1. $(R_+ \times [0,1], T)$ *is a triangular map admitting a compact global attractor* $J \subset R_+ \times [0,1]$;
2. $p \in [0,1]$ *is a periodic point of the map* $T_1 : [0,1] \mapsto [0,1]$ $(T = (T_2, T_1))$;
3. *there are two positive numbers* $\mathcal{N}$ *and* $q < 1$ *such that*

$$\rho(T^n(u_1,\omega), T^n(u_2,\omega)) \le \mathcal{N} q^n \rho(u_1,u_2) \tag{43}$$

*for all* $(u_i,\omega) \in R_+ \times [0,1]$ $(i = 1,2)$ *and* $n \in N$.

*Then* $J_p = I_p \times \{p\}$, *where* $I_p = [a_p, b_p]$ ($a_p, b_p \in R_+$ *and* $a_p = b_p$, *i.e.* $I_p$ *consists of a single point.*

**Proof.** To prove this statement we note that from the conditions (43) and (32) we have

$$diam(J_p) = diam(T^{mk}(J_p)) \le \mathcal{N} q^k diam(J_p) \tag{44}$$

for all $k \in N$. From the inequality (44) we obtain $diam(J_p) = 0$. Taking into consideration the equalities $J_p = I_p \times \{p\}$ and (32) we obtain $a_p = b_p$. $\square$

**Theorem 10.** *[9] Let* $X$ *be a compact metric space and* $\langle (X, Z_+, \pi), (\Omega, Z_+, \sigma), h\rangle$ *be a non-autonomous dynamical system. Suppose that the following conditions are fulfilled:*

1. *The point* $\omega \in \Omega$ *is almost periodic;*
2. $\lim_{t\to+\infty} \rho(\pi(t,x_1), \pi(t,x_2)) = 0$ *for all* $x_1, x_2 \in X$ *such that* $h(x_1) = h(x_2)$.

*Then there exists a unique almost periodic point* $x_\omega \in X_\omega$ *such that*

$$\lim_{t\to+\infty} \rho(\pi(t,x), \pi(t,x_\omega)) = 0$$

*for all* $x \in X_\omega$.

**Theorem 11.** *Suppose that $\epsilon < 0$ and one of the following conditions holds:*

1. $s_w < \min\{\delta, s_r\}$ *and* $0 < \lambda < \lambda_0$, *where* $\lambda_0$ *is a positive root of the quadratic equation* $(s_r - s_w)\lambda^2 + (s_r - 2\delta)\lambda - \delta = 0$;

2. $s_r < s_w < \delta$.

*Then*

1. *the semi-group dynamical system* $(R_+ \times [0,1], T)$ *admits a compact global attractor* $J \subset R_+ \times [0,1]$;

2. *if* $p \in [0,1]$ *is an* $m$*-periodic (respectively, almost periodic) point of the map* $T_1 : [0,1] \mapsto [0,1]$ $(T = (T_2, T_1))$, *then* $J_p = I_p \times \{p\}$, *where* $I_p = [a_p, b_p]$ $(a_p, b_p \in R_+$ *and* $a_p = b_p$, *i.e.* $I_p$ *consists of a single* $m$*-periodic (respectively, almost periodic) point .*

**Proof.** Assume $\epsilon \in (-\infty, 0)$ and let $\lambda = -\epsilon$, then $\lambda \in (0, +\infty)$. We write $T_1$ in terms of $\lambda$ (see the proof of Theorem 9)

$$T_1(u,\omega) = \frac{1}{1+\omega}\left[(1-\delta)u + \frac{u}{(1+u^\lambda)^{\frac{1}{\lambda}}}\frac{s_w + s_r u^\lambda}{1+u^\lambda}\right].$$

Denote by

$$f(u) := \frac{u}{(1+u^\lambda)^{\frac{1}{\lambda}}}\frac{s_w + s_r u^\lambda}{1+u^\lambda},$$

then

$$f'(u) = \frac{s_w + (-s_w\lambda + (\lambda+1)s_r)u^\lambda}{(1+u^\lambda)^{2+1/\lambda}}.$$

It is easy to verify that under the conditions of theorem $f'(u) < s_w$ for all $u \geq 0$. Consider the non-autonomous difference equation

$$u_{n+1} = A(\sigma(n,\omega))u_n + F(u_n, \sigma(n,\omega)) \tag{45}$$

corresponding to triangular map $T = (T_1, T_2)$, where $A(\omega) := \frac{1}{\omega+1}$, $F(u,\omega) := \frac{1}{\omega+1}f(u)$ and $\sigma(n,\omega) := T_2^n(\omega)$ for all $n \in Z_+$ and $\omega \in [0,1]$. Under the conditions of theorem we can apply Theorem 6. By this theorem the semi-group dynamical system $(R_+ \times [0,1], T)$ is compactly dissipative with Levinson center $J$ and there are two positive numbers $\mathcal{N}$ and $q < 1$ such that

$$\rho(T^n(u_1,\omega), T^n(u_2,\omega)) \leq \mathcal{N} q^n \rho(u_1, u_2) \tag{46}$$

for all $(u_i, \omega) \in R_+ \times [0,1]$ $(i = 1,2)$. To finish the proof of theorem it is sufficient to apply Lemma 3 and Theorem 10. $\square$

## 6.6 Conclusion

Under the conditions of Theorem 7 or 9 the mapping $T = (T_2, T_1)$ $(T_1 = f_\lambda)$ admits a compact global attractor $J_\lambda \subset R_+ \times [0,1]$. There exists an increasing sequence $\{\lambda_k\}_{k=0}^{\infty}$ such that

1. $\lambda_k \to \lambda_\infty$ as $k \to \infty$, where $\lambda_\infty \approx 3,569\ldots$.
2. If $\lambda_k < \lambda < \lambda_{k+1}$ $(k = 2, 3, \ldots)$, then the map $T = (T_2, T_1)$ has at least one fixed point $(q_0, 0) \in J_\lambda$ and there is a $2^k$-periodic point $(q_k, p_k) \in J_\lambda$.
3. For $\lambda \geq \lambda_\infty$ the set $J_\lambda$ may be a strange attractor. For example, under the conditions of Theorem 11, for $\lambda = \lambda_\infty$ the attractor $J_\lambda$ contains an almost periodic (but not periodic) minimal set.

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