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The transvectants and the integrals for Darboux systems of differential equations

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Abstract. We apply the algebraic theory of invariants of differential equations to integrate the polynomial differential systems $dx/dt = P_1(x,y) + x\,C(x,y)$, $dy/dt = Q_1(x,y) + y\,C(x,y)$, where real homogeneous polynomials P_1 and Q_1 have the first degree and $C(x,y)$ is a real homogeneous polynomial of degree $r \geq 1$. In generic cases the invariant algebraic curves and the first integrals for these systems are constructed. The constructed invariant algebraic curves are expressed by comitants and invariants of investigated systems.

Mathematics subject classification: 34C05, 58F14.
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1 Introduction

The problem of the integrability via invariant algebraic curves for planar polynomial differential systems has been investigated in many works. An ample survey on the Darboux integrability theory for planar complex and real polynomial systems can be found in [1]. In this book the authors mentioned that the detection of the integrable planar vector fields that are not Hamiltonian, in general, is a very difficult problem. In several works the problem of the integrability via invariant algebraic curves for some polynomial differential systems with degenerate infinity has been solved [2–11]. In works [12, 13] the invariant algebraic curve for Darboux differential systems with cubic nonlinearities has been expressed by invariants and comitants. In paper [14] the invariant algebraic curves, the integrating factors and some first integrals for Darboux differential systems with nonlinearities of degrees m $(2 \leq m \leq 7)$ has been constructed and expressed by invariants and comitants of investigated systems. In paper [15] a complete classification in the coefficient space $\mathbb{R}^{12}$ of quadratic systems with rational first integral of degree 2 has been obtained by using $Aff(2,\mathbb{R})$-invariants and comitants of these systems.

The main goal of this paper is to construct the invariant algebraic curves for integrable planar polynomial differential systems of Darboux type by using the $GL(2,\mathbb{R})$-invariants and the $GL(2,\mathbb{R})$-comitants of these systems [16] and classify the first integrals in generic cases. The generic cases include the systems with coprime right parts and exclude the linear systems.

In Section 2 we construct the main invariants and comitants for Darboux polynomial systems of differential equations. The definition of the transvectant of two polynomials and its properties are given. The construction part of the invariant algebraic curves of Darboux systems includes two subcases: the first one - for the polynomial $C(x,y)$ of odd degree and the second - for the polynomial $C(x,y)$ of even degree. Each subcase includes the results about forms of the invariant algebraic curves and the first integrals for investigated systems.

2 Darboux systems of differential equations

We consider the real systems of differential equations

$$\begin{aligned}\frac{dx}{dt} &= cx + dy + x\,C(x,y) = P(x,y),\\ \frac{dy}{dt} &= ex + fy + y\,C(x,y) = Q(x,y),\end{aligned} \tag{1}$$

where c, d, e, f are real coefficients and the polynomial $C(x,y)$ has real coefficients and degree $r \in \mathbb{N}^*$. This system can be written [17] in the following form

$$\begin{aligned}\frac{dx}{dt} &= \quad \frac{1}{2}\frac{\partial R}{\partial y} + \frac{1}{2}S\,x + x\,C(x,y) = P(x,y),\\ \frac{dy}{dt} &= -\frac{1}{2}\frac{\partial R}{\partial x} + \frac{1}{2}S\,y + y\,C(x,y) = Q(x,y),\end{aligned} \tag{2}$$

where the $GL(2,\mathbb{R})$-invariant S and the $GL(2,\mathbb{R})$-comitants $R(x,y)$ and $C(x,y)$ of the system (1) have the form

$$S = c + f, \quad R(x,y) = -ex^2 + (c-f)x + dy^2, \quad C(x,y) = \sum_{k=0}^{r} a_k \binom{r}{k} x^{r-k} y^k. \tag{3}$$

From the classical invariant theory [18] the definition of the transvectant of two polynomials is well known.

Definition 1. Let $f(x,y)$ and $\varphi(x,y)$ be homogeneous polynomials in x and y with real coefficients of the degrees $\rho \in \mathbb{N}^*$ and $\theta \in \mathbb{N}^*$, respectively, and $k \in \mathbb{N}^*$. The polynomial

$$(f,\varphi)^{(k)} = \frac{(\rho-k)!(\theta-k)!}{\rho!\theta!} \sum_{h=0}^{k} (-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h} \frac{\partial^k \varphi}{\partial x^h \partial y^{k-h}}$$

is called the *transvectant* of index k of polynomials f and φ.

Example 1. Hessian of the comitant $R(x,y)$ has the form

$$H = (R,R)^{(2)} = -\frac{1}{2}[4de + (c-f)^2] = -\frac{1}{2}\,\mathrm{Discr}\,R(x,y). \tag{4}$$

Remark 1. If the polynomials f and φ are $GL(2,\mathbb{R})$-comitants of the degrees $\rho \in \mathbb{N}^*$ and $\theta \in \mathbb{N}^*$, respectively, for the system (1), then the transvectant of the index $k \leq \min(\rho,\theta)$ is a $GL(2,\mathbb{R})$-comitant of the degree $\rho+\theta-2k$ for the system (1) [19]. If $k > \min(\rho,\theta)$, then $(f,\varphi)^{(k)} = 0$.

For every homogeneous GL-comitant $K(x,y)$ with degree $s \in \mathbb{N}^*$ of the system (1) from (2) we obtain the total derivative of $K(x,y)$ with respect to t:

$$\frac{dK}{dt} = \frac{\partial K}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial K}{\partial y}\cdot\frac{dy}{dt} = \frac{\partial K}{\partial x}\left(\frac{1}{2}\frac{\partial R}{\partial y} + \frac{1}{2}xS + xC\right) +$$
$$+\frac{\partial K}{\partial y}\left(-\frac{1}{2}\frac{\partial R}{\partial x} + \frac{1}{2}yS + yC\right) = s(K,R)^{(1)} + \frac{s}{2}KS + sKC, \tag{5}$$

where $(K,R)^{(1)}$ is a Jacobian (the transvectant of the first index) of GL-comitants K and R. The representation (5) shows that the derivative with respect to t of every homogeneous $GL(2,\mathbb{R})$-comitant with the degree $s \geq 1$ of the system (1) is a $GL(2,\mathbb{R})$-comitant too.

Remark 2. If the homogeneous polynomials f, g, φ and ψ have the degrees m, n, μ and 0 (m, n, $\mu \in \mathbb{N}^*$), respectively, with respect to x and y and $l, q \in \mathbb{N}$, $\alpha \in \mathbb{R}$, then

$$(\alpha f, g)^{(k)} = (f,\alpha g)^{(k)} = \alpha (f,g)^{(k)}, \qquad (f^q, f)^{(2l+1)} = 0,$$
$$(f+g,\varphi)^{(k)} = (f,\varphi)^{(k)} + (g,\varphi)^{(k)}, \qquad (\psi, f)^{(k)} = 0,$$
$$(f\cdot g,\varphi)^{(1)} = \frac{m}{m+n}(f,\varphi)^{(1)}g + \frac{n}{m+n}(g,\varphi)^{(1)}f.$$

Remark 3. If the homogeneous polynomials f and φ have the degrees $m \in \mathbb{N}^*$ and 2, respectively, with respect to x and y, then

$$((f,\varphi)^{(1)},\varphi)^{(1)} = \frac{m-1}{m}(f,\varphi)^{(2)}\varphi - \frac{1}{2}f(\varphi,\varphi)^{(2)}.$$

We shall suppose that the polynomials $P(x,y)$ and $Q(x,y)$ are coprime and the polynomial $C(x,y)$ has a nonzero degree, i.e.

$$R(x,y) \not\equiv 0, \qquad C(x,y) \not\equiv 0, \qquad \deg C(x,y) \geq 1. \tag{6}$$

Remark 4. From (5) for $K = R(x,y)$ we obtain

$$\frac{dR}{dt} = R(S + 2\,C), \tag{7}$$

which shows that $R(x,y) = 0$ is an invariant algebraic curve of the system (1).

Let the polynomial $C(x,y)$ have the degree r ($r \in \mathbb{N}^*$) with respect to x and y. We denote by p the integer part of the number $\frac{r}{2}$, i.e. $p = \left[\frac{r}{2}\right]$. Now we suppose the following assumptions: if the lower index in the symbol of the sum $\sum$ is greater than upper index then the sum is equal to zero; in repeated using of the transvectants a set of the parenthesis of the type $((\ldots($ will be replaced by a single parenthesis of the form $[\![$.

2.1 The polynomial $C(x,y)$ has odd degree

Let $r=\deg C(x,y)=2p+1$, where $p\in\mathbb{N}$.

Theorem 1. *The system (1) with the conditions (6) has real invariant algebraic curve $F_r(x,y)=0$ of the degree r, where the polynomial F_r is expressed by invariants and comitants of the system (1):*

$$F_r(x,y)=2^{2p+1}\cdot r!\cdot R^p\left(\frac{2}{r}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p},R)^{(1)}-[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}\cdot S\right)+$$
$$+\sum_{i=0}^{p-1}\left[\frac{2^{2i+1}\cdot r!}{(r-2i)!}\cdot R^i\left(\frac{2(r-2i)}{r}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}-[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot S\right)\times\right.$$
$$\left.\times\prod_{j=i+1}^{p}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\right]-\frac{1}{r^2}\prod_{j=0}^{p}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big),\qquad(8)$$

Proof. The polynomial F_r is a sum of two polynomials $F_r(x,y)=\widehat{F}_r(x,y)+\widetilde{F}_r$, where $\widehat{F}_r(x,y)$ is a comitant of the degree r with respect to x and y and $\widetilde{F}_r=-\frac{1}{r^2}\prod\limits_{j=0}^{p}\big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\big)$ is an invariant of the system (1).

By using the relation (5), Remarks 2 and 3, for the polynomial (8) we obtain:

$$\frac{dF_r}{dt}=\frac{d(\widehat{F}_r+\widetilde{F}_r)}{dt}=r(\widehat{F}_r,R)^{(1)}+\frac{r}{2}\widehat{F}_rS+r\widehat{F}_rC=$$
$$=r2^{2p+1}\cdot r!\left(\frac{2}{r^2}R^p[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p},R)^{(1)},R)^{(1)}-\frac{1}{r}R^p[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p},R)^{(1)}\cdot S+\right.$$
$$\left.+\frac{1}{r}R^p[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p},R)^{(1)}\cdot S-\frac{1}{2}R^p[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}\cdot S^2\right)+$$
$$+r\sum_{i=0}^{p-1}\left[\frac{2^{2i+1}\cdot r!}{(r-2i)!}\left(\frac{2(r-2i)^2}{r^2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)},R)^{(1)}-\right.\right.$$
$$-\frac{(r-2i)}{r}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot S+\frac{(r-2i)}{r}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot S-$$
$$\left.\left.-\frac{1}{2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot S^2\right)\times\prod_{j=i+1}^{p}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\right]+r\widehat{F}_rC=$$
$$=r2^{2p+1}\cdot r!\left(\frac{2}{r^2}\cdot\frac{r-2p-1}{(r-2p)}R^{p+1}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p+1}-\right.$$

$$-\frac{1}{r^2}R^p[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}\cdot(R,R)^{(2)}-\frac{1}{2}R^p[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}\cdot S^2\Bigg)+$$

$$+r\sum_{i=0}^{p-1}\Bigg[\frac{2^{2i+1}\cdot r!}{(r-2i)!}\Bigg(\frac{2(r-2i)^2}{r^2}\cdot\frac{r-2i-1}{(r-2i)}R^{i+1}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i+1}-$$

$$-\frac{(r-2i)^2}{r^2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot(R,R)^{(2)}-\frac{1}{2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot S^2\Bigg)\times$$

$$\times\prod_{j=i+1}^{p}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]+r\widehat{F}_rC=$$

$$=-r2^{2p+1}\cdot r!\cdot\frac{1}{2r^2}R^p[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}\Big(2(R,R)^{(2)}+r^2S^2\Big)+$$

$$+r\sum_{i=0}^{p-1}\Bigg[\frac{2^{2i+1}\cdot r!}{(r-2i)!}\Bigg(\frac{2(r-2i)(r-2i-1)}{r^2}R^{i+1}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i+1}-$$

$$-\frac{1}{2r^2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\Big(2(r-2i)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg)\times$$

$$\times\prod_{j=i+1}^{p}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]+r\widehat{F}_rC=$$

$$=-r2^{2p}\cdot r!\cdot\frac{1}{r^2}R^p[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}\Big(2(R,R)^{(2)}+r^2S^2\Big)+$$

$$+r\sum_{i=0}^{p-1}\Bigg[\frac{2^{2(i+1)}\cdot r!}{(r-2(i+1))!}\cdot\frac{1}{r^2}R^{i+1}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i+1}\times\prod_{j=i+1}^{p}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]-$$

$$-r\sum_{i=0}^{p-1}\Bigg[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\times\prod_{j=i}^{p}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]+r\widehat{F}_rC=$$

$$=r\sum_{i=1}^{p}\Bigg[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\times\prod_{j=i}^{p}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]-$$

$$-r\sum_{i=0}^{p}\Bigg[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\times\prod_{j=i}^{p}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\Bigg]+r\widehat{F}_rC=$$

$$=-rC\frac{1}{r^2}\prod_{j=0}^{p}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)+r\widehat{F}_rC=rC\widetilde{F}_r+r\widehat{F}_rC=rC(\widehat{F}_r+\widetilde{F}_r)=rCF_r.$$

So,
$$\frac{dF_r}{dt}=rCF_r \tag{9}$$
and $F_r(x,y)=0$ is a real invariant algebraic curve for (1). Theorem 1 is proved.

Example 2. For $r\in\{1,3\}$ we obtain the invariant algebraic curves:
$$F_1(x,\,y)=4(C,R)^{(1)}-2C\cdot S-2(R,R)^{(2)}-S^2=0,$$
$$F_3(x,\,y)=32R((C,R)^{(2)},R)^{(1)}-48R(C,R)^{(2)}\cdot S+$$
$$+\left(4[\![C,R)^{(1)}-2C\cdot S\right)\left(2(R,R)^{(2)}+9S^2\right)-\left(2(R,R)^{(2)}+9S^2\right)\left(2(R,R)^{(2)}+S^2\right)=0,$$

The next theorem classifies first integrals of the system (1) in this subcase.

Theorem 2. *The system (1) with the conditions (6) has the following real first integrals:*

a) for $S\neq 0,\ H>0:$
$$\Big|F_r\Big|^{\frac{2}{r}}\cdot\Big|R\Big|^{-1}\cdot G_1=c_1,\quad G_1=\exp\left[\frac{2S}{\sqrt{2H}}\arctan\frac{\dfrac{\partial R}{\partial x}-y\cdot\sqrt{2H}}{\dfrac{\partial R}{\partial x}+y\cdot\sqrt{2H}}\right]; \tag{10}$$

b) for $S\neq 0,\ H<0:$
$$\Big|F_r\Big|^{\frac{2}{r}}\cdot\Big|R\Big|^{-1}\cdot G_2=c_2,\quad G_2=\left|\frac{\dfrac{\partial R}{\partial x}-y\cdot\sqrt{-2H}}{\dfrac{\partial R}{\partial x}+y\cdot\sqrt{-2H}}\right|^{\frac{S}{\sqrt{-2H}}}; \tag{11}$$

c) for $S\neq 0,\ H=0:$
$$\Big|F_r\Big|^{\frac{2}{r}}\cdot\Big|R\Big|^{-1}\cdot G_3=c_3,\quad G_3=\exp\left[\frac{S[(c-f)x^2+2(d+e)xy-(c-f)y^2]}{4(d-e)\,R}\right]; \tag{12}$$

d) for $S=0:$
$$\Big|F_r\Big|^{\frac{2}{r}}\cdot\Big|R\Big|^{-1}=c_4, \tag{13}$$
where c_1,c_2,c_3 *and* c_4 *are real constants.*

Proof. If $S\neq 0$, then from (7) and (9) and after calculation of the derivatives with respect to t of the functions: G_1 for $H>0$, G_2 for $H<0$ and G_3 for $H=0$, we obtain
$$\frac{dR}{dt}=R(S+2\,C),\ \frac{dF_r}{dt}=rCF_r,\ \frac{dG_1}{dt}=SG_1,\ \frac{d\ln G_2}{dt}=S,\ \frac{dG_3}{dt}=SG_3. \tag{14}$$

From (14) we easy obtain first integrals (10), (11) and (12).

If $S=0$, the relations (7) and (9) have the forms
$$\frac{dR}{dt}=2\,R\,C,\quad \frac{dF_r}{dt}=rCF_r. \tag{15}$$

The relations (15) determine the first integral (13). Theorem 2 is proved.

2.2 The polynomial $C(x,y)$ has even degree

Let $r = \deg C(x,y) = 2p$, where $p \in \mathbb{N}^*$.

Theorem 3. *The system (1) with the conditions (6) has real invariant algebraic curve $F_r(x,y)=0$ of the degree r, where the polynomial F_r is expressed by invariants and comitants of the system (1):*

$$F_r(x,y) = -2R^p I_r + rS\cdot\Phi_r(x,\,y), \tag{16}$$

where

$$I_r = 2^{2p}\cdot(r-1)!\llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}, \tag{17}$$

$$\Phi_r(x,y) = \frac{2^{2p-1}\cdot r!}{2!}\cdot R^{p-1}\left(\frac{4}{r}\llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-1},R)^{(1)} - \llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-1}\cdot S\right)+$$
$$+\sum_{i=0}^{p-2}\left[\frac{2^{2i+1}\cdot r!}{(r-2i)!}\cdot R^i\left(\frac{2(r-2i)}{r}\llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)} - \llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot S\right)\times\right.$$
$$\left.\times\prod_{j=i+1}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\right] - \frac{1}{r^2}\prod_{j=0}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big). \tag{18}$$

Proof. The first step is to calculate the derivative $\dfrac{d\Phi_r(x,y)}{dt}$. The polynomial $\Phi_r(x,y)$ is a sum of two terms $\Phi_r(x,y) = \widehat{\Phi}_r(x,y)+\widetilde{\Phi}_r$, where $\widehat{\Phi}_r(x,y)$ is a comitant of the degree r with respect to x and y and $\widetilde{\Phi}_r = -\dfrac{1}{r^2}\prod_{j=0}^{p-1}\big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\big)$ is an invariant of the system (1). By using the relation (5), Remarks 2 and 3, we obtain:

$$\frac{d\Phi}{dt} = \frac{d(\widehat{\Phi}_r+\widetilde{\Phi}_r)}{dt} = r(\widehat{\Phi}_r,R)^{(1)} + \frac{r}{2}\widehat{\Phi}_r S + r\widehat{\Phi}_r C =$$
$$= r\frac{2^{2p-1}\cdot r!}{2!}\left(\frac{8}{r^2}R^{p-1}\llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-1},R)^{(1)},R)^{(1)} - \frac{2}{r}R^{p-1}\llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-1},R)^{(1)}\cdot S+\right.$$
$$\left.+\frac{2}{r}R^{p-1}\llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-1},R)^{(1)}\cdot S - \frac{1}{2}R^{p-1}\llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-1}\cdot S^2\right)+$$
$$+r\sum_{i=0}^{p-2}\left[\frac{2^{2i+1}\cdot r!}{(r-2i)!}\left(\frac{2(r-2i)^2}{r^2}R^i\llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)},R)^{(1)} -\right.\right.$$
$$-\frac{(r-2i)}{r}R^i\llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot S + \frac{(r-2i)}{r}R^i\llbracket C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot S-$$

$$- \frac{1}{2} R^i [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i} \cdot S^2 \Bigg) \times \prod_{j=i+1}^{p-1} \Big(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \Big) \Bigg] + r\widehat{\Phi}_r C =$$

$$= r \frac{2^{2p-1} \cdot r!}{2!} \Bigg(\frac{8}{r^2} \cdot \frac{r-2p+1}{(r-2p+2)} R^p [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p} -$$

$$- \frac{4}{r^2} R^{p-1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p-1} \cdot (R,R)^{(2)} - \frac{1}{2} R^{p-1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p-1} \cdot S^2 \Bigg) +$$

$$+ r \sum_{i=0}^{p-2} \Bigg[\frac{2^{2i+1} \cdot r!}{(r-2i)!} \Bigg(\frac{2(r-2i)^2}{r^2} \cdot \frac{r-2i-1}{(r-2i)} R^{i+1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i+1} -$$

$$- \frac{(r-2i)^2}{r^2} R^i [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i} \cdot (R,R)^{(2)} - \frac{1}{2} R^i [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i} \cdot S^2 \Bigg) \times$$

$$\times \prod_{j=i+1}^{p-1} \Big(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \Big) \Bigg] + r\widehat{\Phi}_r C =$$

$$= 2^{2p} \cdot (r-1)! \cdot R^p [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p} -$$

$$- r \frac{2^{2p-1} \cdot r!}{2!} \cdot \frac{1}{2r^2} R^{p-1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p-1} \Big(8(R,R)^{(2)} + r^2 S^2 \Big) +$$

$$+ r \sum_{i=0}^{p-2} \Bigg[\frac{2^{2i+1} \cdot r!}{(r-2i)!} \Bigg(\frac{2(r-2i)(r-2i-1)}{r^2} R^{i+1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i+1} -$$

$$- \frac{1}{2r^2} R^i [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i} \Big(2(r-2i)^2 (R,R)^{(2)} + r^2 S^2 \Big) \Bigg) \times$$

$$\times \prod_{j=i+1}^{p-1} \Big(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \Big) \Bigg] + r\widehat{\Phi}_r C =$$

$$= R^p I_r - r \frac{2^{2(p-1)} \cdot r!}{2!} \cdot \frac{1}{r^2} R^{p-1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{p-1} \Big(8(R,R)^{(2)} + r^2 S^2 \Big) +$$

$$+ r \sum_{i=0}^{p-2} \Bigg[\frac{2^{2(i+1)} \cdot r!}{(r-2(i+1))!} \cdot \frac{1}{r^2} R^{i+1} [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i+1} \times \prod_{j=i+1}^{p-1} \Big(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \Big) \Bigg] -$$

$$- r \sum_{i=0}^{p-2} \Bigg[\frac{2^{2i} \cdot r!}{(r-2i)!} \cdot \frac{1}{r^2} R^i [\![C, \overbrace{R)^{(2)}, \ldots, R)^{(2)}}^{i} \times \prod_{j=i}^{p-1} \Big(2(r-2j)^2 (R,R)^{(2)} + r^2 S^2 \Big) \Bigg] + r\widehat{\Phi}_r C =$$

$$= R^p I_r + r\sum_{i=1}^{p-1}\left[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\times\prod_{j=i}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\right]-$$

$$-r\sum_{i=0}^{p-1}\left[\frac{2^{2i}\cdot r!}{(r-2i)!}\cdot\frac{1}{r^2}R^i[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\times\prod_{j=i}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)\right]+r\widehat{\Phi}_rC=$$

$$= R^pI_r - rC\frac{1}{r^2}\prod_{j=0}^{p-1}\Big(2(r-2j)^2(R,R)^{(2)}+r^2S^2\Big)+r\widehat{\Phi}_rC=$$

$$= R^pI_r + rC\widetilde{\Phi}_r + r\widehat{\Phi}_rC = R^pI_r + rC(\widehat{\Phi}_r+\widetilde{\Phi}_r) = R^pI_r + rC\Phi_r.$$

Thus,
$$\frac{d\Phi_r}{dt} = R^pI_r + rC\Phi_r. \tag{19}$$

By using the relation (5) and Remark 2 for polynomial R^p we have:

$$\frac{dR^p}{dt} = r(R^p,R)^{(1)} + \frac{r}{2}R^pS + rR^pC = \frac{r}{2}R^p(S+2C). \tag{20}$$

By virtue of the relations (19) and (20) for polynomial (16) we obtain:

$$\frac{dF_r}{dt} = \frac{d(-2R^pI_r + rS\Phi_r)}{dt} = -2I_r\frac{dR^p}{dt} + rS\frac{d\Phi_r}{dt} =$$
$$= -rI_rR^p(S+2C) + rS(R^pI_r + rC\Phi_r) = rC(-2R^pI_r + rS\Phi_r) = rCF_r.$$

So, $F_r(x,y)=0$ is an invariant algebraic curve for (1). Theorem 3 is proved.

Example 3. For $r\in\{2,4\}$ we obtain the invariant algebraic curves:

$$F_2(x,\,y) = -8R(C,R)^{(2)} + 2S\Big(4(C,R)^{(1)} - 2C\cdot S - 2(R,R)^{(2)} - S^2\Big) = 0,$$

$$F_4(x,\,y) = -192R^2((C,R)^{(2)},R)^{(2)} + 4S\Big[96R((C,R)^{(2)},R)^{(1)} - 96R(C,R)^{(2)}\cdot S +$$
$$+\Big(4(C,R)^{(1)} - 2C\cdot S\Big)\Big(8(R,R)^{(2)}+16S^2\Big) - \Big(8(R,R)^{(2)}+16S^2\Big)\Big(2(R,R)^{(2)}+S^2\Big)\Big] = 0.$$

The next theorem is similar to Theorem 2 and classifies first integrals of the system (1) in this subcase for $S\neq 0$.

Theorem 4. *The system (1) with the conditions (6) and $S\neq 0$ has the following real first integrals:*

a) for $H>0$:

$$\Big|F_r\Big|^{\frac{1}{p}}\cdot\Big|R\Big|^{-1}\cdot G_1 = c_5,\quad G_1 = \exp\left[\frac{2S}{\sqrt{2H}}\arctan\frac{\dfrac{\partial R}{\partial x} - y\cdot\sqrt{2H}}{\dfrac{\partial R}{\partial x} + y\cdot\sqrt{2H}}\right]; \tag{21}$$

b) for $H<0$:

$$\Big|F_r\Big|^{\frac{1}{p}}\cdot\Big|R\Big|^{-1}\cdot G_2 = c_6,\quad G_2 = \left|\frac{\dfrac{\partial R}{\partial x} - y\cdot\sqrt{-2H}}{\dfrac{\partial R}{\partial x} + y\cdot\sqrt{-2H}}\right|^{\frac{S}{\sqrt{-2H}}}; \tag{22}$$

c) for $H=0$:

$$\Big|F_r\Big|^{\frac{1}{p}}\cdot\Big|R\Big|^{-1}\cdot G_3=c_7,\quad G_3=\exp\left[\frac{S[(c-f)x^2+2(d+e)xy-(c-f)y^2]}{4(d-e)\,R}\right],\qquad (23)$$

where c_5, c_6 *and* c_7 *are real constants.*

The proof of Theorem 4 is similar to the proof of Theorem 2.

Let $S=0$. The first result in this subcase for the system (1) with $S=0$ is the following theorem.

Theorem 5. *The system* (1) *with the conditions (6) and* $S=0,\ H=(R,R)^{(2)}=0$ *has the invariant algebraic curve* $\Psi_r(x,y)=0$ *of the form*

$$\Psi_r(x,y)=J_rV_rR+W_rQ_r,\qquad (24)$$

where

$$J_r=\frac{I_r}{2^{2p}\cdot(2p-1)!}=[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p},\quad Q_r(x,y)=[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-1},R)^{(1)},$$

$$V_r(x,y)=\frac{r+1}{r}R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}+$$
$$+\sum_{i=0}^{p-1}\left(\binom{r}{2i+1}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1},R)^{(1)}-\right.\qquad (25)$$
$$\left.-\binom{r}{2i+2}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1}\right),$$

$$W_r(x,y)=\sum_{i=0}^{p}\binom{r}{2i}R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}-$$
$$-\sum_{i=0}^{p-1}\binom{r}{2i+1}[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1},R)^{(1)}.\qquad (26)$$

Proof. From the first we calculate the derivative $\dfrac{dV_r(x,y)}{dt}$. The polynomial $V_r(x,y)$ is a sum of two terms $V_r(x,y)=\widetilde{V}_r(x,y)+\widehat{V}_r(x,y)$, where $\widehat{V}_r(x,y)$ is homogeneous polynomial of the degree $r+2$ with respect to x and y and the comitant $\widetilde{V}_r(x,y)=\dfrac{r+1}{r}R\cdot[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}$ has the second degree with respect to x and y. By using the relation (5), Remarks 2 and 3, we obtain:

$$\frac{dV_r}{dt}=\frac{d(\widetilde{V}_r+\widehat{V}_r)}{dt}=2(\widetilde{V}_r,R)^{(1)}+2\widetilde{V}_rC+(r+2)(\widehat{V}_r,R)^{(1)}+(r+2)\widehat{V}_rC=$$

$$= (r+2)V_rC - r\widetilde{V}_rC + (r+2)(\widehat{V}_r, R)^{(1)} = (r+2)V_rC - r\widetilde{V}_rC +$$

$$+(r+2)\sum_{i=0}^{p-1}\Bigg(\binom{r}{2i+1}\frac{r-2i}{r+2}\llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}, R)^{(1)} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1}, R)^{(1)} +$$

$$+\binom{r}{2i+1}\frac{2i+2}{r+2}\llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1}, R)^{(1)}, R)^{(1)} -$$

$$-\binom{r}{2i+2}\frac{r-2i}{r+2}\llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}, R)^{(1)}, R)^{(1)} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1} -$$

$$-\binom{r}{2i+2}\frac{2i+2}{r+2}\llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}, R)^{(1)} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1}, R)^{(1)}\Bigg) =$$

$$= (r+2)V_rC - r\widetilde{V}_rC +$$

$$+\sum_{i=0}^{p-1}\Bigg(\binom{r}{2i+1}(r-2i)\llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}, R)^{(1)} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1}, R)^{(1)} +$$

$$+\binom{r}{2i+1}(2i+1)R \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i} -$$

$$-\binom{r}{2i+2}(r-2i-1)R \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i+1} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1} -$$

$$-\binom{r}{2i+1}(r-2i-1)\llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}, R)^{(1)} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1}, R)^{(1)}\Bigg) =$$

$$= (r+2)V_rC - r\widetilde{V}_rC +$$

$$+\sum_{i=0}^{p-1}\binom{r}{2i+1}\llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}, R)^{(1)} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1}, R)^{(1)} +$$

$$+\sum_{i=0}^{p-1}\binom{r}{2i}(r-2i)R \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i} -$$

$$-\sum_{i=1}^{p}\binom{r}{2i}(r-2i+1)R \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i} =$$

$$= (r+2)V_rC - (r+1)RC \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p} +$$

$$+\sum_{i=0}^{p-1}\binom{r}{2i+1}\llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}, R)^{(1)} \cdot \llbracket C, \overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i-1}, R)^{(1)} +$$

$$+rR\cdot C\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p}-R\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p}\cdot C-$$

$$-\sum_{i=1}^{p-1}\binom{r}{2i}R\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i}=$$

$$=(r+2)V_rC-\sum_{i=0}^{p}\binom{r}{2i}R\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i}+$$

$$+\sum_{i=0}^{p-1}\binom{r}{2i+1}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)}=$$

$$=(r+2)V_rC-W_r.$$

Thus,
$$\frac{dV_r}{dt}=(r+2)V_rC-W_r. \tag{27}$$

Now we calculate the derivative $\dfrac{dW_r(x,y)}{dt}$, where $W_r(x,y)$ is a homogeneous comitant of the degree $r+2$ with respect to x and y.

$$\frac{dW_r}{dt}=(r+2)(W_r,R)^{(1)}+(r+2)W_rC=(r+2)W_rC+$$

$$+(r+2)\sum_{i=0}^{p}\binom{r}{2i}\frac{r-2i}{r+2}R\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i}+$$

$$+(r+2)\sum_{i=0}^{p}\binom{r}{2i}\frac{2i}{r+2}R\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i},R)^{(1)}-$$

$$-(r+2)\sum_{i=0}^{p-1}\binom{r}{2i+1}\frac{r-2i}{r+2}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)},R)^{(1)}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)}-$$

$$-(r+2)\sum_{i=0}^{p-1}\binom{r}{2i+1}\frac{2i+2}{r+2}[\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)},R)^{(1)}=$$

$$=(r+2)W_rC+\sum_{i=0}^{p}\binom{r}{2i}(r-2i)R\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i}+$$

$$+\sum_{i=0}^{p}\binom{r}{2i}2iR\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i},R)^{(1)}-$$

$$-\sum_{i=0}^{p-1}\binom{r}{2i+1}(r-2i-1)R\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i+1}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i-1},R)^{(1)}-$$

$$-\sum_{i=0}^{p-1}\binom{r}{2i+1}(2i+1)R\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{i},R)^{(1)}\cdot [\![C,\overbrace{R)^{(2)},\ldots,R)^{(2)}}^{p-i}=$$

$$= (r+2)W_rC + \sum_{i=0}^{p-1}\binom{r}{2i}(r-2i)R\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}+$$

$$+\sum_{i=1}^{p}\binom{r}{2i}2iR\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i},R)^{(1)}-$$

$$-\sum_{i=0}^{p-1}\binom{r}{2(i+1)}2(i+1)R\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i+1}\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-(i+1)},R)^{(1)}-$$

$$-\sum_{i=0}^{p-1}\binom{r}{2i}(r-2i)R\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i},R)^{(1)}\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i}=$$

$$=(r+2)W_rC+\sum_{i=1}^{p}\binom{r}{2i}2iR\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i},R)^{(1)}-$$

$$-\sum_{i=1}^{p}\binom{r}{2i}2iR\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{i}\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-i},R)^{(1)}=(r+2)W_rC.$$

Thus,

$$\frac{dW_r}{dt}=(r+2)W_rC. \tag{28}$$

In analogous way for polynomial $Q_r(x,y)$ we have:

$$\frac{dQ_r}{dt}=2(Q_r,R)^{(1)}+2Q_rC=2Q_rC+2[\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p-1},R)^{(1)},R)^{(1)}=$$
$$=2Q_rC+R\cdot [\![C,\overbrace{R)^{(2)},\dots,R)^{(2)}}^{p}=2Q_rC+RJ_r. \tag{29}$$

By using (24), (27), (28) and (29), we obtain:

$$\frac{d\Psi_r}{dt}=J_r\frac{dV_r}{dt}R+J_rV_r\frac{dR}{dt}+\frac{dW_r}{dt}Q_r+W_r\frac{dQ_r}{dt}=$$
$$=J_r\left((r+2)V_rC-W_r\right)R+2J_rV_rRC+(r+2)W_rCQ_r+W_r\left(2Q_rC+RJ_r\right)=$$
$$=(r+4)(J_rV_rR+W_rQ_r)C=(r+4)\Psi_rC.$$

So,

$$\frac{d\Psi_r}{dt}=(r+4)\Psi_rC. \tag{30}$$

and $\Psi_r(x,y)=0$ is an invariant algebraic curve for (1). Theorem 5 is proved.

Remark 5. If $H=(R,R)^{(2)}=0$, then the following identity $W_r\equiv R^{p+1}\cdot(C,C)^{(r)}$ holds. By virtue of this identity the invariant algebraic curve $\Psi_r(x,y)=0$ can be written in the form $R\cdot\Psi_r^*(x,y)=0$, where $\Psi_r^*(x,y)=J_r\cdot V_r+Q_r\cdot R^p\cdot(C,C)^{(r)}$. So, $\Psi_r^*(x,y)=0$ is an invariant algebraic curve for the system (1) with $S=0,\ H=0$.

The next theorem classifies in this subcase first integrals of (1) for $S=0$.

Theorem 6. *The system (1) with the conditions (6) and $S=0$ has the following real first integrals:*

a) for $H>0,\ I_r\neq 0:$

$$\sqrt{\frac{H}{2}}\cdot\frac{1}{I_r}\cdot\frac{\Phi_r}{R^p}+G_4=c_8,\quad G_4=\arctan\frac{ex+fy}{y\cdot\sqrt{\dfrac{H}{2}}};\tag{31}$$

b) for $H<0,\ I_r\neq 0:$

$$G_5\cdot\exp\left(-\frac{\sqrt{-2H}}{I_r}\cdot\frac{\Phi_r}{R^p}\right)=c_9,\quad G_5=\left|\frac{\dfrac{\partial R}{\partial x}-y\cdot\sqrt{-2H}}{\dfrac{\partial R}{\partial x}+y\cdot\sqrt{-2H}}\right|;\tag{32}$$

c) for $H=0,\ I_r\neq 0:$

$$\Psi_r\cdot R^{-(p+2)}=c_{10};\tag{33}$$

d) for $I_r=0:$

$$\Phi_r\cdot R^{-p}=c_{11},\tag{34}$$

where c_8, c_9, c_{10} *and* c_{11} *are real constants.*

Proof. For $I_r\neq 0$ we have

$$\frac{d}{dt}\left(\frac{\Phi_r}{R^p}\right)=\frac{1}{R^{2p}}\cdot\Big[R^p\cdot(R^p\cdot I_r+rC\Phi_r)-\Phi_r\cdot(p\cdot R^{p-1}\cdot 2CR)\Big]=$$
$$\frac{1}{R^{2p}}\cdot\Big[I_r\cdot R^{2p}+2pC\Phi_rR^p-2pC\Phi_rR^p\Big]=I_r.\tag{35}$$

If $H>0$, then $\dfrac{dG_4}{dt}=-\sqrt{\dfrac{H}{2}}$ and we easy obtain first integral (31).

If $H<0$, then $\dfrac{d\ln G_5}{dt}=\sqrt{-2H}$ and we have first integral (32).

Let $H=0,\ I_r\neq 0$. In this case by virtue of (5) and (30) we have

$$\frac{d\left(\Psi_r/R^{p+2}\right)}{dt}=\frac{1}{R^{2(p+2)}}\left(\frac{d\Psi_r}{dt}R^{p+2}-\Psi_r\frac{dR^{p+2}}{dt}\right)=$$
$$=\frac{1}{R^{2(p+2)}}\big((r+4)\Psi_rCR^{p+2}-2(p+2)\Psi_rR^{p+2}C\big)=0.$$

So, the system (1) has first integral (33).

The first integral (34) for $I_r=0$ is given by (35). Theorem 6 is proved.

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