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Open problems on the algebraic limit cycles of planar polynomial vector fields

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Abstract. We collect several open problems that have appeared in the study of the algebraic limit cycles of the real planar polynomial vector fields.

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1 Introduction

We divide this brief presentation of several open problems on the algebraic limit cycles of the real planar polynomial vector fields into the following sections:

2. Invariant algebraic curves.
3. Algebraic limit cycles.
4. A unique irreducible invariant algebraic curve.
5. Quadratic polynomial vector fields.
6. Cubic polynomial vector fields.
7. Configurations of algebraic limit cycles.

2 Invariant algebraic curves

Since Darboux [12] has found in 1878 connections between algebraic curves and the existence of first integrals of planar polynomial vector fields, invariant algebraic curves are a central object in the theory of integrability of these vector fields. Today after more than one century of investigations the theory of invariant algebraic curves is still full of open questions.

A *real planar polynomial differential system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

where P and Q are real polynomials in the variables x and y. The dependent variables x and y, the independent variable t, and the coefficients of the polynomials P and Q are all real because in this paper we are interested in the real algebraic limit cycles of system (1). The *degree* n of the polynomial system (1) is the maximum of the degrees of the polynomials P and Q.

Associated to the (real) polynomial differential system (1) there is the *(real) polynomial vector field*

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y},$$

or simply $\mathcal{X} = (P,Q)$.

Let $f = f(x,y)$ be a (real) polynomial in the variables x and y. The algebraic curve $f(x,y) = 0$ of $\mathbb{R}^2$ is an *invariant algebraic curve* of the vector field $\mathcal{X}$ if for some polynomial $K \in \mathbb{R}[x,y]$ we have

$$\mathcal{X} f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{2}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. We note that since the polynomial system has degree n, then any cofactor has at most degree $n-1$.

Since on the points of the algebraic curve $f = 0$ the gradient $(\partial f/\partial x,\, \partial f/\partial y)$ of the curve is orthogonal to the vector field $\mathcal{X} = (P,Q)$ (see (2)), the vector field $\mathcal{X}$ is tangent to the curve $f = 0$. Hence the curve $f = 0$ is formed by orbits of the vector field $\mathcal{X}$. This justifies the name of invariant algebraic curve given to the algebraic curve $f = 0$ satisfying (2) for some polynomial K, because it is *invariant* under the flow defined by $\mathcal{X}$.

The next result tell us that we can restrict our attention to the irreducible invariant algebraic curves.

Proposition 1. *We suppose that* $f \in \mathbb{R}[x,y]$ *and let* $f = f_1^{n_1} \cdots f_r^{n_r}$ *be its factorization in irreducible factors over* $\mathbb{R}[x,y]$. *Then for a polynomial vector field* $\mathcal{X}$, $f = 0$ *is an invariant algebraic curve with cofactor* K_f *if and only if* $f_i = 0$ *is an invariant algebraic curve for each* $i = 1, \ldots, r$ *with cofactor* K_{f_i}. *Moreover* $K_f = n_1 K_{f_1} + \ldots + n_r K_{f_r}$.

For a proof of this proposition see for instance [25].

3 Algebraic limit cycles

We recall that a *limit cycle* of a polynomial vector field $\mathcal{X}$ is an isolated periodic orbit in the set of all periodic orbits of $\mathcal{X}$. An *algebraic limit cycle* of degree m of $\mathcal{X}$ is an oval of an irreducible invariant algebraic curve $f = 0$ of degree m which is a limit cycle of $\mathcal{X}$.

We must remark that when we are interested in the invariant algebraic curves for questions different from the algebraic limit cycles (like for instance integrability, multiplicity and others) it is important to consider complex invariant algebraic curves (i.e. $f = f(x, y)$ is a complex polynomial in the variables x and y), because the natural background for all the Darboux theory of integrability is the ring of the complex polynomials. But the limit cycles and in particular the algebraic limit cycles is a real phenomena, so in all this paper we restrict our attention to the real Darboux theory of integrability. In any case many of the results used here also hold inside the theory of complex polynomial differential systems.

A first question related with this subject is whether a polynomial vector field has or does not have invariant algebraic curves. The answer is not easy, see the large section in Jouanolou's book [21], or the long paper [31] devoted to show that one particular polynomial system has no invariant algebraic solutions. Even one of the more studied limit cycles, the limit cycle of the van der Pol system, until 1995 it was unknown that it is not algebraic [32].

One of the nice results in the theory of invariant algebraic curves is the following result.

Theorem 2 (Jouanolou's Theorem [21])**.** *A polynomial vector field of degree n has less than $[n(n+1)/2]+2$ irreducible invariant algebraic curves, or it has a rational first integral.*

For a shorter proof of this result see [9] or [10].

Jouanolou's Theorem shows that for a given polynomial vector field $\mathcal{X}$ of degree n the maximum degree of its irreducible invariant algebraic curves is bounded, because either $\mathcal{X}$ has a finite number of invariant algebraic curves less than $[n(n+1)/2]+2$, or $\mathcal{X}$ has rational first integral $f(x, y)/g(x, y)$. In this last case all the orbits of $\mathcal{X}$ are contained in the invariant algebraic curves $af(x, y) + bg(x, y) = 0$ for some $a, b \in \mathbb{R}$.

Thus for each polynomial vector field there is a natural number N which bounds the degree of all its irreducible invariant algebraic curves. A natural question, going back to Poincaré [33] and which for some people in this area is now known as the *Poincaré problem*, is to give an effective procedure to find N. There are only partial answers to this question, see for instance [2–4, 36], ... We must mention here that the actual Poincaré problem is to determine when a polynomial differential system over the complex plane has a rational first integral, and that the previous called Poincaré problem is a main step according with Poincaré for solving the actual Poincaré problem.

Of course if we know for a polynomial vector field the maximum degree of its invariant algebraic curves, then it is possible (at least in theory) to compute its invariant algebraic curves.

We are interested in algebraic limit cycles of polynomial vector fields, and if a polynomial vector field has a rational first integral it cannot have limit cycles. Unfortunately for the class of polynomial vector fields with fixed degree n having finitely many invariant algebraic curves (i.e. having no rational first integrals), there does not exist a uniform upper bound $N(n)$ for N as it was shown in [11, 30]. This implies that there are polynomial vector fields with a fixed degree having irreducible invariant algebraic curve of arbitrary degree. Therefore a priori it is possible the existence of polynomial vector fields with a fixed degree having algebraic limit cycle of arbitrary degree. But it may be worse than that.

We shall need the next well known result.

Theorem 3 (Harnack's Theorem)**.** *The maximum number of ovals of a real algebraic curve of degree m is $[(m-1)(m-2)/2]+1$.*

Summarizing, a polynomial vector field of degree n with finitely many irreducible invariant algebraic curves has at most $[n(n+1)/2]+1$ of such curves, but we do not have a bound for the degree of these invariant algebraic curves. Consequently due to the Harnack's Theorem we do not have a uniform bound for the number of algebraic limit cycles that any polynomial vector field of degree n can have. So the second part of the 16–th Hilbert problem [20] (see also [19, 22]) which asks for finding a uniform bound for the number of limit cycles that any polynomial vector field of degree n can have, remains also open if we restrict our attention to the limit cycles which are algebraic.

Open problem 1. *Is there a uniform bound for the number of algebraic limit cycles that a polynomial vector field of degree n could have*?

From the previous paragraphs it is clear that a uniform positive answer to the Poincaré problem inside the class of all polynomial vector fields of degree n, i.e. to provide a uniform bound $N(n)$ for the degrees of the invariant algebraic curves of all polynomials vector fields of degree n, will provide also a uniform bound for the number of algebraic limit cycles of all polynomials vector fields of degree n.

4 A unique irreducible invariant algebraic curve

In this section we shall need the following result.

Theorem 4 (Bautin–Christopher–Dolov–Kuzmin Theorem)**.** *Let $f=0$ be a non–singular algebraic curve of degree m, and D a first degree polynomial, chosen so that the line $D=0$ lies outside all bounded components of $f=0$. Choose the constants a and b so that $aD_x+bD_y\neq 0$, then the polynomial differential system*

$$\dot{x}=af-Df_y, \qquad \dot{y}=bf+Df_x,$$

of degree m has all the bounded components of $f=0$ as hyperbolic limit cycles. Furthermore the vector field has no other limit cycles.

It seems that the main result in the paper of Bautin [1] is similar to the previous theorem. However the paper contains a mistake which was corrected in [13] and generalized in [14]. A proof of the statement of theorem like it is presented here appeared in [8].

The next proposition provides the maximum number of algebraic limit cycles that a polynomial vector field having a unique irreducible invariant algebraic curve can have in function of the degree of that curve. This proposition is well known in the area, we write it here for completeness.

Proposition 5. *Suppose that $f = 0$ of degree m is the unique irreducible invariant algebraic curve of a polynomial vector field X. Then X can have at most $[(m-1)(m-2)/2]+1$ algebraic limit cycles. Moreover choosing that $f = 0$ has the maximal number of ovals for the irreducible algebraic curves of degree m, there exist vector fields X of degree m having exactly $[(m-1)(m-2)/2]+1$ algebraic limit cycles.*

Proof. The first part of the proposition follows directly from the Harnack's Theorem, and the second part again from the Harnack's theorem and using Christopher's Theorem. □

5 Quadratic polynomial vector fields

In 1958 Qin Yuan–Xun [35] proved that quadratic (polynomial) vector fields can have algebraic limit cycles of degree 2, and when such a limit cycle exists then it is the unique limit cycle of the system.

Evdokimenco in [15–17] proved that quadratic vector fields do not have algebraic limit cycles of degree 3, for two different shorter proofs see [6, 25].

In 1966 Yablonskii [34] found the first class of algebraic limit cycles of degree 4 inside the quadratic vector fields. The second class was found in 1973 by Filiptsov [18]. More recently two new classes has been found and in [7] the authors proved that there are no other algebraic limit cycles of degree 4 for quadratic vector fields. The uniqueness of these limit cycles has been proved in [5]. Some other results on the algebraic limit cycles of quadratic vector fields can be found in [27, 28].

Doing convenient birational transformation of the plane to quadratic vector fields having algebraic limit cycles of degree 4 in [7] the authors obtained algebraic limit cycle of degrees 5 and 6 for quadratic vector fields. Of course in general a birational transformation does not preserve the degree of the polynomial vector field.

Open problems 2. *The following questions related with the algebraic limit cycles of quadratic polynomial vector fields remain open, see for instance[25].*

(i) *What is the maximum degree of an algebraic limit cycle of a quadratic polynomial vector field?*

(ii) *Does there exist a chain of rational transformations of the plane (as in [7]) which gives examples of quadratic systems with algebraic limits cycles of arbitrary degree, or at least of degree larger than 6?*

(iii) *Is 1 the maximum number of algebraic limit cycles that a quadratic system can have?*

6 Cubic polynomial vector fields

It is known that there are cubic polynomial vector fields having algebraic limit cycles of degrees 2 and 3, see for instance [23, 24]. In [29] we provide cubic systems having algebraic limit cycles of degrees 4, 5 and 6 respectively, and an example of a cubic system having two algebraic limit cycles.

Open problems 3. *The following questions related with the algebraic limit cycles of cubic polynomial vector fields remain open, see for instance [29].*

(i) *What is the maximum degree of an algebraic limit cycle of a cubic polynomial vector field?*
(ii) *Does there exist a chain of rational transformations of the plane (as in [7]) which gives examples of cubic polynomial vector fields with algebraic limits cycles of arbitrary degree, or at least of degree larger than* 6*?*
(iii) *Is* 2 *the maximum number of algebraic limit cycles that a cubic polynomial vector fields can have?*

7 Configurations of algebraic limit cycles

In 1900 Hilbert not only proposed in the second part of his 16–th problem (see [20]) to estimate a uniform upper bound for the number of limit cycles of all polynomial vector fields of a given degree, but he also asked about the possible distributions or configurations of the limit cycles in the plane. This last question has been solved using algebraic limit cycles.

A *configuration of limit cycles* is a finite set $C = \{C_1, \ldots, C_n\}$ of disjoint simple closed curves of the plane such that $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Two configurations of limit cycles $C = \{C_1, \ldots, C_n\}$ and $C' = \{C'_1, \ldots, C'_m\}$ are *(topologically) equivalent* if there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h\left(\cup_{i=1}^n C_i\right) = \left(\cup_{i=1}^m C'_i\right)$. Of course for equivalent configurations of limit cycles C and C' we have that $n = m$.

We say that a polynomial vector field $\mathcal{X}$ *realizes* the configuration of limit cycles C if the set of all limit cycles of X is equivalent to C.

Theorem 6. *Let $C = \{C_1, \ldots, C_n\}$ be an arbitrary configuration of limit cycles. Then the configuration C is realizable with algebraic limit cycles by a polynomial vector field.*

This theorem is proved in [26]. Looking at the way in which it is proved you can provide an alternative proof using the Bautin–Christopher–Dolov–Kuzmin Theorem.

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