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Regular maps and quasilinear total differential equations

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Abstract. The bounded and S-concordant solutions of the quasilinear total differential equations with a real parameter by the nonlinearity and with a regular homogeneous part are investigated.

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1 Introduction

In the paper the quasilinear total differential equations [3] of the form

$$y'h = a(t)hy + (b(t) + \lambda g(t, y))h \quad (h \in E) \tag{1}$$

with a real parameter λ by the nonlinearity and with a regular homogeneous part are investigated. Here: $a \in C(P, L(E, L(T, T)))$, $b \in C(P, L(E, T))$, $g \in C(P \times T, L(E, T))$, $y \in C(P, T)$ is an unknown map; E is a normed real space; T is a Banach space (real or complex); $P \subset E$ is an open set; the prime $'$ means the operation of taking a bounded derivative (derivative Frechet). By $C(X, Y)$ we designate the space of all continuous maps of the space X in the space Y endowed with the uniform structure of compact convergence (compact open topology); by $L(X, Y)$ we designate the space of all linear continuous maps of the normed space X in the normed space Y with the natural operator norm.

For such equations some sufficient conditions of the existence of bounded, compact, Lagrange stable, concordant and uniformly concordant solutions are established. Earlier similar problems for ordinary differential equations ($E = P = R$) and multidimensional differential equations, for $E = P = R^n$ and $T = R^m$, were considered in [1,4,8]. The Lagrange stability, concordance and uniform concordance are considertd relative to some semigroup $S \subset P$, in contrast to [1, 4, 8], where $S = E$. The peculiarity of our researches is that dynamical systems (transformation groups or semigroups) are not used.

We propose also some general approach to the research of quasilinear equations, based on the concept of a regular, generally speaking, many-valued map. We consider regular maps in the first section of this paper. In the second section we indicate some applications of results obtained in the first section to the quasilinear equations (1).

2 Regular maps

The concept of a regular map naturally arises when abstracting from the concrete type of the regular equation. Under a regular ordinary differential equation we understand such an ordinary differential equation $y' = a(t)y$ that for an arbitrary bounded function $f : R \to R^n$ there exists a unique bounded solution $\varphi : R \to R^n$ of the equation $y' = a(t)y + f(t)$. It is known [7] (Theorem 51. A) that for the regular equation $y' = a(x)y$ there is a constant $r > 0$ such that $\sup_{t\in R} \|\varphi(t)\| \le r \sup_{t\in R} \|f(t)\|$, where φ is a bounded solution of the equation $y' = a(t)y + f(t)$ with the bounded function f. Close connection between a regular and an exponential dichotomy is known, too [2, 7].

By the research of quasilinear equations $y' = a(t)y + b(t) + f(t, y)$ sometimes one of crucial is the following property of linear equations: if φ_i is a solution of the equation $y' = a(t)y + b_i(t)$ $(i = 1, 2)$, then $\varphi_1 - \varphi_2$ is a solution of the equation $y' = a(t)y + (b_1 - b_2)(t)$. Besides with homogeneous equations of the form $y' = a(t)y$, generally speaking, a many-valued map naturally associates that to each function b puts in correspondence the set of solutions of the equation $y' = a(t)y + b(t)$. The last two facts in combination with definition of a regular homogeneous equation will be taken as a basis in the definition of a regular map.

Definition 1. *Let X, Y be normed real spaces, 2^Y be a family of all subsets of Y, $r > 0$. A map $q : X \to 2^Y$ is called weakly r-regular if:*

1) $\forall x \in X \ q(x) \ne \emptyset$;

2) $\forall x, y \in X \ q(x) - q(y) \subset q(x - y)$;

3) $\forall x \in X \ \forall y \in q(x) \quad \|y\| \le r \cdot \|x\|$.

A weakly r-regular map is called r-regular if it is a one-valued map.

Let's give some examples of regular maps.

Example 1. Let $X = Y$ be the space of bounded maps from $C(R,\ R^m)$ with the norm *sup* and the map $a \in C(R,\ L(R^m,\ R^m))$ be such that the differential equation $y' = a(t)y$ is regular. Then there is a positive number r such that the map $q : X \to 2^Y$ defined by the rule

$$\varphi \in q(f) \Longleftrightarrow \varphi'(t) = a(t)\varphi(t) + f(t) \quad (t \in R)$$

is r-regular.

Example 2. Let E and T be Banach spaces, $a \in L(E, L(T, T))$ be a permutable operator (i.e. $ahak = akah$ for $\forall h, k \in E$) such that (Sp a)e does not intersect with the imaginary axis of the complex plane for some vector $e \in E$ of the unit norm; X be the space of all continuously differentiable bounded maps $f : E \to L(E, T)$ with the norm *sup* which satisfy the condition $\wedge\{ahf(t)k - f'(t)kh\} = 0$ for $\forall h, k, t \in E$; Y be the space of all continuous bounded maps $E \to T$ with the norm *sup*. And let

$r = 2c/\alpha$, where $c > 0$ and $\alpha > 0$ are constants for which $\|G_e(t)\| \leq c \cdot \exp(-\alpha|t|)$ $(t \in R)$; G_e be the main Green function of the operator ae [3]. Then, as it follows from the theorem 12.2 of [3], the map $q : X \to 2^Y$ defined by the rule

$$\varphi \in q(f) \Longleftrightarrow \varphi'(t)h = a(t)h\varphi(t) + f(t)h \quad (t,\ h \in E)$$

is r-regular (by the symbol $\wedge$ everywhere in the paper we designate the operation of taking the skew-symmetric part of bilinear operator: $\wedge\{Chk\} = 1/2(Chk - Ckh)$ $(C \in L_2(E, F))$).

Example 3. Let $1 \leq p,\ q < \infty,\ 1/p + 1/q = 1$; $D \subset R^n$ be a bounded closed set, $M_0 = (mesD)^{\frac{1}{p}}$; $X = Y = L_p(D)$; $K : D \times D \to R$ be a measurable function such that for some number $M > 0$ and for almost all $t \in D$,

$$\Big(\int_D |K(t,s)|^q\ ds\Big)^{\frac{1}{q}} \leq M;$$

the number λ is such that $|\lambda| < 1/(MM_0)$; $r = 1/(1 - |\lambda|MM_0)$. Then the map $q : X \to 2^Y$ defined by the rule

$$\varphi \in q(f) \Longleftrightarrow \varphi(t) = f(t) + \lambda \int_D K(t,s)\varphi(s)ds \quad (t \in D)$$

is r-regular.

Theorem 1. *Let Y be a complete space; $q : X \to 2^Y$ be a weakly r-regular map; $b \in X$ and the map $B : Y \to X$ satisfies the Lipschitz condition with the constant L. Then for $\forall\lambda, |\lambda| < 1/(rL)$, in Y there is an element x_λ for which $x_\lambda \in q(b+\lambda B(x_\lambda))$; in addition $\|x_\lambda\| \leq r \cdot \|b + \lambda B(x_\lambda)\|$. The element x_λ is determined uniquely if q is an r-regular map. In the last case x_λ may be found as the limit of the sequence*

$$y_1, y_2, \cdots, y_n, \cdots \tag{2}$$

for $\forall y_1 \in Y$ and for any $n > 1$ $y_n = q(b + \lambda B(y_{n-1}))$.

Proof. Let λ be such that $|\lambda| < 1/(rL)$. Let's designate by $H_\lambda : Y \to Y$ the choice function for the composition $f_\lambda \circ q$ where $f_\lambda : Y \to X$ is defined by the rule: $f_\lambda(x) = b + \lambda B(x)$. We shall prove that H_λ is a contraction map. Let $x_1, x_2 \in Y$. Then

$$H_\lambda(x_1) - H_\lambda(x_2) \in q(b + \lambda B(x_1)) - q(b + \lambda B(x_2)) \subset q(\lambda(B(x_1) - B(x_2))),$$

i.e. $H_\lambda(x_1) - H_\lambda(x_2) \in q(\lambda B(x_1) - B(x_2))$. Then

$$\|H_\lambda(x_1) - H_\lambda(x_2)\| \leq r\|\lambda(B(x_1) - B(x_2))\| \leq r|\lambda|L\|x_1 - x_2\|,$$

i.e. $\|H_\lambda(x_1) - H_\lambda(x_2)\| \leq r|\lambda|L\|x_1 - x_2\|$. Since $r|\lambda|L < 1$, then the map H_λ is contracting. By virtue of the completeness of the space Y, according to the Banach contracting principle, there exists a unique x_λ such that $H_\lambda(x_\lambda) = x_\lambda$. Therefore $x_\lambda \in q(b + \lambda B(x_\lambda))$. It is clear that x_λ does not depend on the choice function H_λ, hence, it is determined uniquely as the limit of the sequence (2) if q is a r-regular map.

Theorem 2. *Let Y be a complete space; $q : X \to 2^Y$ be a weakly r-regular map; $b \in X$; $x \in q(b)$; $\delta > 0$; V_x^δ be a closed δ-neighbourhood of x and the map $B : Y \to X$ satisfies the Lipschitz condition on V_x^δ with the constant L. Then for $\forall\lambda, |\lambda| < \lambda_0 = \delta/(rL\delta + rl_1)$, where $l_1 = \|B(x)\|$, in V_x^δ there is an element x_λ for which $x_\lambda \in q(b + \lambda B(x_\lambda))$; in addition $\|x_\lambda\| \le r \cdot \|b + \lambda B(x_\lambda)\|$. The element x_λ is determined uniquely if q is an r-regular map. In the last case x_λ may be found as the limit of the sequence* (2) *which begins at an arbitrary point $y_1 \in V_x^\delta$.*

Proof. Let $\lambda \in]-\lambda_0,\ \lambda_0[$, $\ z \in V_x^\delta$ and $y \in q(b + \lambda B(z))$. Since $x \in q(b)$, then $y - x \in q(\lambda B(z))$. Hence

$$\|y - x\| \le r|\lambda|\|B(z)\| \le r|\lambda|(\|B(z) - B(x)\| + \|B(x)\|) \le$$
$$\le r|\lambda|(L\|z - x\| + l_1) \le r|\lambda|(L\delta + l_1) < \delta,$$

i.e. $\|y - x\| < \delta$, therefore $y \in V_x^\delta$. We have proved that for $\forall z \in V_x^\delta$, $q(b + \lambda B(z)) \subset V_x^\delta$. Therefore the composition $f_\lambda \circ q$, where $f_\lambda : V_x^\delta \to X$ is defined by the rule: $f_\lambda(x) = b + \lambda B(x)$, is a map $V_x^\delta \to 2^{V_x^\delta}$. Let's designate by $H_\lambda : V_x^\delta \to V_x^\delta$ the choice function for the composition $f_\lambda \circ q$. We shall also prove that H_λ is a contraction map. Let $x_1, x_2 \in V_x^\delta$. Then

$$H_\lambda(x_1) - H_\lambda(x_2) \in q(b + \lambda B(x_1)) - q(b + \lambda B(x_2)) \subset q(\lambda(B(x_1) - B(x_2))),$$

i.e. $H_\lambda(x_1) - H_\lambda(x_2) \in q(\lambda(B(x_1) - B(x_2)))$. Then

$$\|H_\lambda(x_1) - H_\lambda(x_2)\| \le r\|\lambda(B(x_1) - B(x_2))\| \le r|\lambda|L\|x_1 - x_2\|,$$

i.e. $\|H_\lambda(x_1) - H_\lambda(x_2)\| \le r|\lambda|L\|x_1 - x_2\|$. Since $r|\lambda|L < 1$, then the map H_λ is contracting. According to the Banach contracting principle there is a unique $x_\lambda \in V_x^\delta$ such that $H_\lambda(x_\lambda) = x_\lambda$. It is clear that x_λ is the required element.

Definition 2. *Let $X \subset Z^P$ and $F : Y \times P \to W$ be a map. The map $f : P \times W \to Z$ is called F-admissible if for $\forall x \in Y$, the map f^x, where $f^x(t) = f(t, F(x,t))$ $(t \in P)$, belongs to X.*

From Theorems 1 and 2 for $X \subset Z^P$ and $B(y) = f^y$ $(y \in Y)$ as a corollary we obtain the following two theorems.

Theorem 3. *Let $X \subset Z^P$, $q : X \to 2^Y$ be a weakly r-regular map, $b \in X$, $f : P \times W \to Z$ be a F-admissible map, $L \in R$ and the following conditions be satisfied:*

1) *Y is a complete space;*

2) *$\forall y, z \in Y$ $\|f^y - f^z\| \le L \cdot \|y - z\|$.*

Then for $\forall\lambda, |\lambda| < 1/(rL)$, in Y there is an element x_λ such that $x_\lambda \in q(b + \lambda f^{x_\lambda})$; in addition $\|x_\lambda\| \le r \cdot \|b + \lambda f^{x_\lambda}\|$. The element x_λ is determined uniquely if q is a r-regular map. In the last case x_λ may be found as the limit of the sequence $y_1, y_2, \cdots, y_n, \cdots$, where y_1 is an arbitrary element from Y and for any $n > 1$, $y_n = q(b + \lambda f^{y_{n-1}}))$.

Theorem 4. *Let $X \subset Z^P$, $q : X \to 2^Y$ be a weakly r-regular map, $b \in X$, $f : P \times W \to Z$ be an F-admissible map, $L \in R$, $\delta > 0$ and the following conditions are satisfied:*

1) *Y is a complete space;*
2) *V_x^δ is a closed δ-neighbourhood of $x \in q(b)$;*
3) *$\forall y, z \in V_x^\delta$ $\|f^y - f^z\| \leq L \cdot \|y - z\|$.*

Then for $\forall \lambda, |\lambda| < \delta/(r(L\delta + l_1))$, where $l_1 = \|f^x\|$, in V_x^δ there is an element x_λ for which $x_\lambda \in q(b + \lambda f^{x_\lambda})$; in addition $\|x_\lambda\| \leq r \cdot \|b + \lambda f^{x_\lambda}\|$. The element x_λ is determined uniquely if q is an r-regular map. In the last case x_λ may be found as the limit of the sequence $y_1, y_2, \cdots, y_n, \cdots$, where y_1 is an arbitrary element from V_x^δ and for any $n > 1$, $y_n = q(b + \lambda f^{y_{n-1}}))$.

Theorem 5. *Let Z, W be normed spaces, $X \subset Z^P$, $q : X \to 2^Y$ be a weakly r-regular map, $f : P \times W \to Z$ be an F-admissible map, $b \in X$, $x \in q(b)$, $V^\delta_{F(x,P)}$ be a closed δ-neighbourhood of $F(x,P)$; $A_x^\delta = \{y \mid y \in Y$ and $F(y,P) \subset V^\delta_{F(x,P)}\}$. If the following conditions are valid:*

1) *A_x^δ is a complete subset of the space Y;*
2) *$\forall g \in X$ $\|g\| \leq \sup_{t \in P} \|g(t)\|$;*
3) *f satisfies the Lipschitz condition in the second argument with the constant L_1 on the set $V^\delta_{F(x,P)}$ and F satisfies the Lipschitz condition in the first argument with the constant L_2;*
4) *$\sup_{(t,s) \in P \times P} \|f(t, F(x,s))\| \leq l_1 \in R$,*

then for $\forall \lambda, |\lambda| < \lambda_0 = \delta/(rL_2(L_1\delta + l_1))$, in the set A_x^δ there is an element x_λ for which $x_\lambda \in q(b + \lambda f^{x_\lambda})$; in addition $\|x_\lambda\| \leq r \cdot \|b + \lambda f^{x_\lambda}\|$. The element x_λ is determined uniquely if q is an r-regular map. In the last case x_λ may be found as the limit of the sequence $y_1, y_2, \cdots, y_n, \cdots$, where y_1 is an arbitrary element from A_x^δ and for any $n > 1$, $y_n = q(b + \lambda f^{y_{n-1}}))$.

Proof. Let $z \in A_x^\delta$, $|\lambda| < \lambda_0$ and $y \in q(b + \lambda f^z)$. Then $y - x \in q(\lambda f^z)$. Therefore $\|y - x\| \leq r|\lambda|\|f^z\|$. Since $z \in A_x^\delta$, for arbitrary $t \in P$ there exists $p_t \in P$ such that $\|F(z,t) - F(x,p_t)\| \leq \delta$. Then for $\forall s \in P$

$$\begin{aligned}
&\|F(y,s) - F(x,s)\| \leq L_2\|y - x\| \leq r|\lambda|L_2\|f^z\| \leq rL_2|\lambda| \sup_{t \in P} \|f(t, F(z,t))\| \leq \\
&\leq rL_2|\lambda| \sup_{t \in P}(\|f(t, F(z,t)) - f(t, F(x,p_t))\| + \|f(t, F(x,p_t))\|) \leq \\
&\leq rL_2|\lambda|(L_1 \sup_{t \in P} \|F(z,t) - F(x,p_t)\| + l_1) \leq rL_2|\lambda|(L_1\delta + l_1) < \delta,
\end{aligned}$$

i.e. $\|F(y,s) - F(x,s)\| \leq \delta$. Hence $y \in A_x^\delta$. We have proved that $q(b + \lambda f^z) \subset A_x^\delta$ for $\forall z \in A_x^\delta$. Therefore the composition $f_\lambda \circ q$, where $f_\lambda : A_x^\delta \to X$ is defined by the rule: $f_\lambda(z) = b + \lambda f^z$, is a map $A_x^\delta \to 2^{A_x^\delta}$. Let's designate by $H_\lambda : A_x^\delta \to A_x^\delta$ choice function of this map. We shall also prove that H_λ is a map of contraction. Let $x_1, x_2 \in A_x^\delta$. Then

$$H_\lambda(x_1) - H_\lambda(x_2) \in q(b + \lambda f^{x_1}) - q(b + \lambda f^{x_2}) \subset q(\lambda(f^{x_1} - f^{x_2})),$$

i.e. $H_\lambda(x_1) - H_\lambda(x_2) \in q(\lambda(f^{x_1} - f^{x_2}))$. Therefore

$$\begin{aligned}
&\|H_\lambda(x_1) - H_\lambda(x_2)\| \leq r\|\lambda(f^{x_1} - f^{x_2})\| \leq \\
&\leq r|\lambda| \sup_{t\in P} \|f(t, F(x_1, t)) - f(t, F(x_2, t))\| \leq \\
&\leq r|\lambda| L_1 \sup_{t\in P} \|F(x_1, t) - F(x_2, t\| \leq r|\lambda| L_1 L_2 \|x_1 - x_2\|,
\end{aligned}$$

i.e. $\|H_\lambda(x_1) - H_\lambda(x_2)\| \leq r L_1 L_2 |\lambda| \|x_1 - x_2\|$. Since $r L_1 L_2 |\lambda| < 1$ then the map H_λ is contracting. By virtue of the completeness of the set A_x^δ there exists unique $x_\lambda \in A_x^\delta$ for which $H_\lambda(x_\lambda) = x_\lambda$. It is clear that x_λ is the required element.

3 Quasilinear equations

Lemma 1. *Let the space E be quasicomplete; P be a connected convex set if the space E is finite-dimensional and $P = E$ if the space E is infinite-dimensional; K be a compact set from T; $\lambda \in R$; I be a directional set and for $\forall i \in I$, φ_i is a solution of the total differential equation*

$$y'h = a_i(X)hy + (b_i(x) + \lambda g_i(x, y))h \ (h \in E), \tag{3}$$

and $\overline{\varphi_i(P)} \subset K$ and $\lim_i(a_i, b_i, g_i) = (a, b, g)$ in $C(P, L(E, L(T, T))) \times C(P, L(E, T))$ $\times C(P \times T, L(E, T))$. Then:

1) *the set $\overline{\{\varphi_i \,|\, i \in I\}}$ is compact;*

2) *the limit φ of an arbitrary subnet of the net $\{\varphi_i\}$ is a solution of the total differential equation*

$$y'h = a(X)hy + (b(x) + \lambda g(x, y))h \ (h \in E) \tag{4}$$

and $\overline{\varphi(P)} \subset K$.

Proof. We shall define for $\forall i \in I$ the maps $f_i : P \times T \to L(E, T)$ and $f : P \times T \to L(E, T)$ by the rules: for $\forall (x, y) \in P \times T$, $\forall h \in E$, $f_i(x, y)h = a_i(x)hy + (b_i(x) + \lambda g_i(x, y))h$ and $f(x, y)h = a(x)hy + (b(x) + \lambda g(x, y))h$. It is clear that for $\forall i \in I$, the equation (3) is equivalent to the equation

$$y' = f_i(x, y) \tag{5}$$

and the equation (4) is equivalent to the equation

$$y' = f(x, y). \tag{6}$$

Let's prove that $\lim_i f_i = f$ in $C(P \times T, L(E,T))$. Let ε be an arbitrary positive number, $Q \times M$ be an arbitrary compact set from $P \times T$, $m^* = \sup_{m \in M} \|m\|$, $\varepsilon_1 = \varepsilon/(m^* + |\lambda| + 1)$. Since $\lim_i(a_i, b_i, g_i) = (a, b, g)$ then there exists $i_0 \in I$ such that for arbitrary $i > i_0$, $(t,m) \in Q \times M$ and $h \in E$, $\|h\| \leq 1$, the following relations are fulfilled

$$\|(a_i(t) - a(t))h\| < \varepsilon_1, \ \|(b_i(t) - b(t))h\| < \varepsilon_1 \text{ and } \|(g_i(t,m) - g(t,m))h\| < \varepsilon_1.$$

From these relations for $i > i_0$, $(t,m) \in Q \times M$ and $h \in E$, $\|h\| \leq 1$, we shall receive

$$\begin{aligned}
&\|(f_i(t,m) - f(t,m))h\| = \\
&= \|a_i(t)hm + (b_i(t) + \lambda g_i(t,m))h - a(t)hm - (b(t) + \lambda g(t,m))h\| \leq \\
&\leq \|(a_i(t) - a(t))hm\| + \|(b_i(t) - b(t))h\| + |\lambda| \|(g_i(t,m) - g(t,m))h\| < \\
&< \varepsilon_1 m^* + \varepsilon_1 + |\lambda|\varepsilon_1 = \varepsilon,
\end{aligned}$$

i.e. $\|(f_i(t,m) - f(t,m))h\| < \varepsilon$. The proof also means that $\lim_i f_i = f$. At this point, since the equations (3) is equivalent to the equation (5) and the equation (4) is equivalent to the equation (6), then our lemma follows from Lemma 2 [5].

Further by S we designate a subsemigroup of the group E, $0 \in S \subset P$ and $S + P \subset P$. To each map f from the spaces of maps under consideration and to every $s \in S$ with the help of the shift σ in the argument from P we put in correspondence some map f_s which is defined as follows. If $f : P \to Y$ then $f_s(p) = f(s+p)$ $(p \in P)$. If $f : P \times T \to Y$ then $f_s(p,t) = f(s+p,t)$ $(p \in P, \ t \in T)$. By fS we designate the set $\{f_s \mid s \in S\}$.

Definition 3. *The solution φ of the equation* (1) *is called to compact if the set $\overline{\varphi(P)}$ is compact.*

If X is the set of maps on which the operation of a semigroup S is defined with the help of the shift σ, then the map $f \in X$ is called Lagrange S-stable if the set $\overline{fS}$ is compact.

The following proposition contains some sufficient conditions of Lagrange S-stable solutions of the equation (1).

Proposition 1. *Let the map f be defined by the rule: $f(x,y)h = a(x)hy + (b(x) + \lambda g(x,y))h$ $((x,y) \in P \times T, \ h \in E)$. A compact solution φ of the equation* (1) *is Lagrange S-stable if one of the conditions is valid:*

1) *the map φ is uniformly continuous.*

2) *the set $f(P, \overline{\varphi(P)})$ is bounded.*

3) *f is Lagrange S-stable, the space E is quasicomplete, P is a connected convex set if the space E is finite-dimensional and $P = E$ if the space E is infinite-dimensional.*

Lemma 2. *Let $K \subset T$ be a compact set, $g : P \times T \to L(E,T)$. If for $\forall k \in K$ the set $\overline{g(P,k)}$ is compact and the map g satisfies the Lipschitz condition in the second argument on K, then the set $\overline{g(P,K)}$ is compact. The set $\overline{G(P,K)}$ for $\forall G \in \overline{gS}$ is compact, too.*

Proof. Let L be the Lipschitz constant of g. For the proof of the compactness of the set $\overline{g(P,K)}$ it is sufficient to prove that from every sequence $\{g(t_i,k_i)\}$, where $(t_i,k_i) \in P \times K$, it is possible to single out a subsequence of Cauchy. Let $\varepsilon > 0$ be an arbitrary number, $\{(t_i,k_i)\} \subset P \times K$. By virtue of the compactness of K there is a subsequence $\{k_{i_l}\} \subset K$ such that $\lim_l k_{i_l} = k \in K$. Then for number $\varepsilon/4L$ there is a number $l_1 \in N$ such that for $\forall l > l_1$,

$$\|k_{i_l} - k\| < \varepsilon/4L. \tag{7}$$

By virtue of the compactness of the set $\overline{g(P,k)}$ we consider that $\exists \lim_l g(t_{i_l},k) = g_0$. Then for the number $\varepsilon/4$ there is $l_2 \in N$ such that for $\forall l > l_2$,

$$\|g(t_{i_l},k) - g_0\| < \varepsilon/4. \tag{8}$$

Let $l_0 = \max(l_1,l_2)$ and $l > l_0\, p > l_0$. With the account of relations (7) and (8) we obtain

$$\begin{aligned}
&\|g(t_{i_l},k_{i_l}) - g(t_{i_p},k_{i_p})\| \le \|g(t_{i_l},k_{i_l}) - g(t_{i_l},k)\| + \\
&+\|g(t_{i_l},k) - g_0\| + \|g(t_{i_p},k) - g_0\| + \|g(t_{i_p},k) - g(t_{i_p},k_{i_p})\| < \\
&< L \cdot \|k_{i_l} - k\| + \varepsilon/4 + \varepsilon/4 + L \cdot \|k_{i_p} - k\| < \varepsilon,
\end{aligned}$$

i.e. $\|g(t_{i_l},k_{i_l}) - g(t_{i_p},k_{i_p})\| < \varepsilon$ for $\forall l,p > l_0$. The proof means that $\{g(t_{i_l},k_{i_l})\}$ is a Cauchy sequence. So, the set $\overline{g(P,K)}$ is compact. If $G \in \overline{gS}$, then $G = \lim g_{t_i}$ for some net $\{t_i\} \subset S$. Therefore for $\forall (t,k) \in P \times K$, $G(t,k) = \lim g_{t_i}(t,k) \in \overline{g(P,K)}$, hence, the set $\overline{G(P,K)}$ is compact.

Lemma 3. *Let $W \subset T$ and $g|_W$ be the contraction of the map $g : P \times T \to L(E,T)$ on the set $P \times W$. If:*

1) *for $\forall y \in W$, the map g is uniformly continuous on $P \times \{y\}$ and the set $\overline{g(P,y)}$ is compact;*

2) *g satisfies the Lipschitz condition in the second argument on W,*

then the map $g|_W$ is Lagrange S-stable.

Proof. By virtue of Ascoli theorem it is sufficient to prove equicontinuity of the family of maps $\{g_t|_W \mid t \in S\}$ on each compact set from $P \times W$. Beforehand we shall prove that for an arbitrary compact set $K \subset W$ the map g is uniformly continuous on the set $P \times K$. Let $K \subset W$ be a compact set and the map g be non-uniformly continuous on the set $P \times K$. Then there is an $\varepsilon_0 > 0$ such that for an arbitrary natural i there are elements (t_1^i, k_1^i) and (t_2^i, k_2^i) in $P \times K$ for which the following relations are fulfilled

$$\|(t_1^i, k_1^i) - (t_2^i, k_2^i)\| < 1/i \tag{9}$$

and

$$\|g(t_1^i, k_1^i) - g(t_2^i, k_2^i)\| \geq \varepsilon_0. \tag{10}$$

Let L be a Lipschitz constant of g. From the relation (9) the relation $\|k_1^i - k_2^i\| < 1/i$ follows. Therefore by virtue of the compactness of K we may consider that $\lim_i k_1^i = \lim_i k_2^i = k \in K$. In that case for number $\varepsilon_0/(3L)$ there will be a natural number i_1 such that for arbitrary $i > i_1$ the following relations are fulfilled

$$\|k_1^i - k\| < \varepsilon_0/(3L), \quad \|k_2^i - k\| < \varepsilon_0/(3L). \tag{11}$$

From the relation (9) also follows the relation

$$\|t_1^i - t_2^i\| < 1/i. \tag{12}$$

Since the map g is uniformly continuous on $P \times \{k\}$, then for $\varepsilon_0/3$ there is a number $\delta > 0$ such that for $\forall (t_1, k), (t_2, k) \in P \times \{k\}$ from the relation $\|(t_1, k) - (t_2, k)\| < \delta$ the following relation follows

$$\|g(t_1, k) - g(t_2, k)\| < \varepsilon_0/3. \tag{13}$$

Let a natural number i_2 be such that $1/i_2 < \delta$. Then for an arbitrary $i > i_2$, by virtue of the relations (12) and (13), the following relation is fulfilled

$$\|g(t_1^i, k) - g(t_2^i, k)\| < \varepsilon_0/3. \tag{14}$$

Let $i_0 = \max(i_1, i_2)$. For $i > i_0$, the relations (11) and (14) are simultaneously fulfilled. Therefore for $i > i_0$

$$\begin{aligned}
&\|g(t_1^i, k_1^i) - g(t_2^i, k_2^i)\| \leq \|g(t_1^i, k_1^i) - g(t_1^i, k)\| + \|g(t_1^i, k) - g(t_2^i, k)\| + \\
&+\|g(t_2^i, k) - g(t_2^i, k_2^i)\| < L\|k_1^i - k\| + \varepsilon_0/3 + L\|k_2^i - k\| < \\
&< L\varepsilon_0/(3L) + \varepsilon_0/3 + L\varepsilon_0/(3L) = \varepsilon_0,
\end{aligned}$$

i.e. $\|g(t_1^i, k_1^i) - g(t_2^i, k_2^i)\| < \varepsilon_0$. The obtained relation contradicts the relation (10). So, the map g is uniformly continuous on the set $P \times K$ for an arbitrary compact set $K \subset W$. Let D be an arbitrary compact set from $P \times W$ and the compact set $M \times K \subset P \times W$ is such that $D \subset M \times K$. And let $\varepsilon > 0$, $t \in S$, $(m, k) \in D$ and δ be a number corresponding to the number ε by virtue of an uniform continuity of g on the set $P \times K$. We shall assume that $(m_1, k_1) \in D$ and $\|(m_1, k_1) - (m, k)\| < \delta$. Then $\|(t + m_1, k_1) - (t + m, k)\| < \delta$. Therefore $\|g_t(m_1, k_1) - g_t(m, k)\| < \varepsilon$. The proof means equicontinuity of the family of maps $\{g_t|_W \mid t \in S\}$ at the point $(m, k) \in D$. In that case the family of maps $\overline{\{g_t|_W \mid t \in S\}}$ is equicontinuous on D.

Lemma 4. *If the map $g : P \times T \to L(E,T)$ satisfies the Lipschitz condition on the second argument in a set W from T with the Lipschitz constant L, then any map from $\overline{g|_W S}$ satisfies the Lipschitz condition in the second argument on W with the Lipschitz constant L.*

Proof. The proof is obvious.

Let's introduce the concept of concordance of maps and we shall describe shortly its purpose.

Let X and Y be some spaces of maps on which the operation of the semigroup S is defined with the help of shift σ, $\mathcal{U}[X]$ and $\mathcal{U}[Y]$ be uniform structures of spaces X and Y respectively; $\varphi \in X$, $f \in Y$.

Definition 4. *We say that φ is S-concordant with f if for every index $\alpha \in \mathcal{U}[X]$ there is an index $\gamma \in \mathcal{U}[Y]$ such that $s \in S$ and $(f, f_s) \in \gamma$ implies $(\varphi, \varphi_s) \in \alpha$. We say that φ is uniformly S-concordant with f if for every index $\alpha \in \mathcal{U}[X]$ there is an index $\gamma \in \mathcal{U}[Y]$ such that $s, t \in S$ and $(f_t, f_s) \in \gamma$ imply $(\varphi_t, \varphi_s) \in \alpha$.*

The essence of the concept of S-concordance is that if φ is S-concordant with f and f has certain property of the recursiveness, then φ has this property of the recursiveness, too. Let's explain this in more details.

Let $[S]$ be some class of subsets from S, $f \in X$ (or $f \in Y$).

The map f is called $[S]$-recursive if for an arbitrary index $\alpha \in \mathcal{U}[X]$ there is a set $A \in [S]$ for which $(f, f_a) \in \alpha$, for all $a \in A$. The set fS is called $[S]$-recursive if for an arbitrary index $\alpha \in \mathcal{U}[X]$ there is a set $A \in [S]$ for which $(f_s, f_{s+a}) \in \alpha$, for all $s \in S$ and $a \in A$.

And let $\varphi \in X$, $f \in Y$. Then: 1) If φ is S-concordant with f and the map f is $[S]$-recursive, then the map φ is $[S]$-recursive, too. 2) If φ is uniformly S-concordant with f and the set fS is $[S]$-recursive, then the set φS is $[S]$-recursive, too.

The last definitions and proposition are well concordant with the facts known for dynamic systems [1].

As concrete definitions of $[S]$-recursivenesses various types of Poisson stability of maps, in particular, Poisson SQ-stability, Poisson $S\mathcal{P}$-stability, SQ-recurrentness in sense of Birkhoff, SQ-almost periodicity in sense of Bohr (here Q is a subset from S, $\mathcal{P}$ is some family of subset of S). We shall give corresponding definitions, for $\varphi \in C(P,T)$ (for more details see [1, 6]).

A map φ is Poisson SQ-stable if for arbitrary $\varepsilon > 0$, a compact set A from P and arbitrary $q \in Q$ there is $p \in Q$ for which $\|\varphi(a) - \varphi(a+q+p)\| < \varepsilon$, for all $a \in A$.

If a map is Poisson SQ-stable for arbitrary $Q \in \mathcal{P}$, then it is called as Poisson $S\mathcal{P}$-stable.

A map φ is SQ-recurrent in sense of Birkhoff if for arbitrary $\varepsilon > 0$ and a compact set A from P there is a compact set $K \subset Q$ such that for $\forall q \in Q$ $\exists k \in K$ for which $\|\varphi(a) - \varphi(a+q+k)\| < \varepsilon$, for all $a \in A$.

A map φ is SQ-almost periodic in sense of Bohr if for arbitrary $\varepsilon > 0$ and a compact set A from P there is a compact set $K \subset Q$ such that for $\forall q \in Q$ $\exists k \in K$ for which $\|\varphi(s+a) - \varphi(s+a+q+k)\| < \varepsilon$, for all $s \in S$ and $a \in A$.

Thus, if it is established that some solution φ of the equations is S-concordant (uniformly S-concordant) with the right-hand side f of this equation and the map f is Poisson SQ-stable, or Poisson $S\mathcal{P}$-stable, or SQ-recurrent in sense of Birkhoff (SQ-almost periodic in sense of Bohr), then the solution φ is respectively Poisson SQ-stable, or Poisson $S\mathcal{P}$-stable, or SQ-recurrent in sense of Birkhoff (SQ-almost periodic in sense of Bohr), too.

Definition 5. *Let $a \in C(P, L(E, L(T,T)))$. The total differential equation*

$$y'h = a(t)hy \quad (h \in E) \tag{15}$$

is called weakly regular (regular) of type 1 with the constant $r > 0$ if for an arbitrary bounded map $b \in C(P, L(E,T))$ the equation

$$y'h = a(t)hy + b(t)h \quad (h \in E) \tag{16}$$

has a compact (unique compact) solution $x \in C(P,T)$. In addition, for an arbitrary compact solution x of the equation (16) is valid the estimation

$$\sup_{t\in P} \|x(t)\| \le r \cdot \sup_{t\in P} \|b(t)\|. \tag{17}$$

Theorem 6. *Let for the equation (1) the following conditions be fulfilled:*

1) *the equation (15) is weakly regularly of type 1 with the constant r;*

2) *the map b is bounded;*

3) *there is $t_0 \in T$ for which the set $g(P, t_0)$ is bounded;*

4) *the map g satisfies the Lipschitz condition in the second argument with the Lipschitz constant L.*

Then for an arbitrary λ, $|\lambda| < 1/(rL)$, the equation (1) has a compact solution $x_\lambda \in C(P,T)$ and for it the estimation is valid

$$\sup_{t\in P} \|x_\lambda(t)\| \le r \cdot \sup_{t\in P} \|b(t) + \lambda g(t, x_\lambda(t))\|. \tag{18}$$

If in addition to the conditions 1) – 4) of our theorem the following condition is fulfilled:

5) *the map a is bounded and the set $\overline{g(P,y)}$ is compact for $\forall y \in T$,*

then the solution x_λ is Lagrange S-stable.

If in addition to conditions 1) – 4) of our theorem the following conditions are fulfilled:

6) *the equation* (15) *is regular of type of* 1 *with the constant* r*;*

7) *the space* E *is quasicomplete,* P *is a connected convex set if the space* E *is finite-dimensional and* $P = E$ *if the space* E *is infinite-dimensional,*

then the solution x_λ *is* S*-concordant with* (a, b, g)*.*

Proof. Let the conditions 1) – 4) of our theorem be fulfilled. We shall use Theorem 3 for: X is the space of bounded maps from $C(P, L(E, T))$ with the norm *sup*, Y is the space of compact maps from $C(P, T)$ with the norm *sup*, q is the map that to every $p \in X$ puts in correspondence the set of all solutions of the equation

$$y'h = a(t)hy + p(t)h \quad (h \in E)$$

contained in Y. As F we shall take the map $Y \times P \to T$ according to the rule $F(\varphi, t) = \varphi(t)$, and as f we shall take the map g. Since the equation (15) is weakly regular of type 1 with the constant r, then the map q is weakly r-regular. It is directly checked that Y is a complete space.

Since for $\forall x \in Y$ and $\forall t \in P$

$$\begin{aligned} &\|g(t, x(t))\| \le \|g(t, x(t)) - g(t, t_0)\| + \|g(t, t_0)\| \le \\ &\le L(\sup_{s \in P} \|x(s)\| + \|t_0\|) + \sup_{s \in P} \|g(s, t_0)\| \equiv l \in R, \end{aligned}$$

i.e. $\|g(t, x(t))\| \le l$, then the map g^x is bounded. Therefore the map g is F-admissible.

From the condition 4) of our theorem the condition 2) of Theorem 3 follows.

According to Theorem 3 for an arbitrary λ, $|\lambda| < 1/(rL)$, there is an $x_\lambda \in q(b + \lambda g^{x_\lambda})$. By the definition of the maps q and g^{x_λ} the map x_λ is compact and satisfies the equality $x'_\lambda(t)h = a(t)hx_\lambda(t) + (b(t) + \lambda g(t, x_\lambda(t))h$ for an arbitrary $t \in P$, $h \in E$. It also means that x_λ is a compact solution of the equation (1). The estimation (18) follows from the estimation for x_λ from Theorem 3.

Let's assume that the condition 5) of our theorem is also fulfilled, and we shall prove the Lagrange S-stability of the solution x_λ. Let's designate by g^* the map $P \times T \to L(E, T)$ according to the rule: $g^*(t, y)h = a(t)hy + (b(t) + \lambda g(t, y))h$ $((t, y) \in P \times T,\ h \in E)$. And let $K \subset T$ be a compact set. By Lemma 2 the set $\overline{g(P, K)}$ is compact. Since for $\forall (t, y) \in P \times K$

$$\|g^*(t, y)\| = \sup_{\|h\|=1} \|g^*(t, y)h\| \le \|a(t)\|\|y\| + \|b(t)\| + |\lambda|\|g(t, y)\|,$$

then the map g^* is bounded on the set $P \times K$. In that case the solution x_λ is Lagrange S-stable by Proposition 1.

Let's assume that the conditions 1) – 4) and 6) – 7) of our theorem are fulfilled. Then x_λ is a unique compact solution of the equation (1). Suppose that x_λ is not S-concordant with (a, b, g). Then there is an index α of the uniform structure of the space $C(P, T)$ such that for an arbitrary index γ of the uniform structure of the space $C(P, L(E, L(T, T))) \times C(P, L(E, T)) \times C(P \times T, L(E, T))$ there is an element $s_\gamma \in S$ such that

$$((a, b, g), (a_{s_\gamma}, b_{s_\gamma}, g_{s_\gamma})) \in \gamma \tag{19}$$

and

$$(x_\lambda, (x_\lambda)_{s_\gamma}) \notin \alpha. \tag{20}$$

From the relation (19) it follows that $\lim_\gamma (a_{s_\gamma}, b_{s_\gamma}, g_{s_\gamma}) = (a, b, g)$. By virtue of Lemma 1 from the net $\{(x_\lambda)_{s_\gamma}\}$ it is possible to single out a subnet $\{(x_\lambda)_{s_\beta}\}$ converging to some compact solution ψ of the equation (1). According to the proved above $\psi = x_\lambda$, therefore $\lim_\beta (x_\lambda)_{s_\beta} = x_\lambda$. The obtained relation contradicts (20). The contradiction says that the solution x_λ is S-concordant with (a, b, g).

Theorem 7. *Let E be a Banach space, $P = E$ and for the equation* (1) *the following conditions are fulfilled:*

1) *for $\forall t \in E$, $a(t) = a$ is a permutable operator such that $(Sp\ a)e$ does not intersect the imaginary axis of the complex plane for some vector $e \in E$ of the unit norm;*

2) *the map b is bounded, continuously differentiable and $\wedge\{ahb(t)k - b'(t)kh\} = 0$ for $\forall h, k, t \in E$;*

3) *for an arbitrary bounded map $y \in C(E, T)$ the map g^y is continuously differentiable and $\wedge\{ahg^y(t)k - (g^y)'(t)kh\} = 0$ for $\forall h, k, t \in E$ (here and further, g^y is the map according to the rule $g^y(t) = g(t, y(t))$ for $\forall t \in E$);*

4) *the map g satisfies the Lipschitz condition in the second argument with the Lipschitz constant L;*

5) *there is $t_0 \in T$ for which the set $g(E, t_0)$ is bounded.*

Then for an arbitrary λ, $|\lambda| < 1/(rL)$, where r is the constant from the Example 2 of regular maps, the equation (1) *has a unique bounded solution $x_\lambda \in C(E, T)$ and the estimation also is valid*

$$\sup_{t \in E} \|x_\lambda(t)\| \le r \cdot \sup_{t \in E} \|b(t) + \lambda g(t, x_\lambda(t))\|. \tag{21}$$

If in addition to the conditions 1) – 5) *of our theorem the set $\overline{g(E, y)}$ is compact for $\forall y \in T$, then the solution x_λ is Lagrange E-stable.*

If in addition to the conditions 1) – 5) *of our theorem $T = R^m$, then the solution x_λ is E-concordant with (a, b, g).*

Proof. We relate the equation (1) to the r-regular map from the Example 2 of regular maps, and the proof of the theorem is done by the proof scheme of Theorem 6 taking into account that for $T = R^m$, the solution x_λ is compact.

Theorem 8. *Let for the equation* (1) *the following conditions be fulfilled:*

1) *the equation* (15) *is weakly regular of type* 1 *with the constant* r*;*
2) *the map* b *is bounded;*
3) x *is a compact solution of the equation* (16) *and* $V^{\delta}_{x(P)}$ *is a closed* δ*-neighbourhood of the set* $x(P)$ $(\delta > 0)$*;*
4) *the map* g *satisfies the Lipschitz condition in the second argument on* $V^{\delta}_{x(P)}$ *with the Lipschitz constant* L*;*
5) *for some* $t_0 \in P$, $\sup_{t\in P} \|g(t, x(t_0))\| = l \in R$.

Then for an arbitrary λ, $|\lambda| < \lambda_0 = \delta/(r(L(d+\delta)+l))$, *where* d *is the diameter of the set* $x(P)$, *the equation* (1) *has a compact solution* $x_\lambda : P \to V^{\delta}_{x(P)}$ *and for it the estimation* (18) *is valid. In addition, if* $\lim_{i\to\infty} \lambda_i = 0$, *then* $\lim_{i\to\infty} x_{\lambda_i} = x$ *uniformly on* P.

If in addition to the conditions 1) – 5) *of our theorem the map* a *is bounded, then the solution* x_λ *is Lagrange* S*-stable.*

If in addition to the conditions 1) – 5) *of our theorem the following conditions are fulfilled:*

6) *the equation* (15) *is regular of type* 1 *with the constant* r*;*
7) *the space* E *is quasicomplete,* P *is a connected convex set if the space* E *is finite-dimensional and* $P = E$ *if the space* E *is infinite-dimensional.*

Then the solution x_λ *is* S*-concordant with* (a, b, g).

Proof. Let the conditions 1) – 5) of our theorem be fulfilled. We shall use Theorem 5 for: X is the space of bounded maps from $C(P, L(E,T))$ with the norm *sup*, Y is the space of compact maps from $C(P,T)$ with the norm *sup*, q is the map that to every $p \in X$ puts in correspondence the set of all solutions of the equation

$$y'h = a(t)hy + p(t)h \quad (h \in E)$$

contained in Y. As F we shall take the map $Y \times P \to T$ according to the rule $F(\varphi, t) = \varphi(t)$ and as f the map g. Since the equation (15) is weakly regular of type 1 with the constant r, then the map q is weakly r-regular. Since for $\forall y \in Y$

$$\begin{aligned}&\sup_{t\in P} \|g(t, y(t))\| \le \sup_{t\in P} \|g(t, y(t)) - g(t, x(t_0))\| + \sup_{t\in P} \|g(t, x(t_0))\| \le \\ &\le L(\sup_{t\in P} \|y(t)\| + \|x(t_0)\|) + \sup_{t\in P} \|g(t, x(t_0))\| \in R,\end{aligned}$$

i.s. $\sup_{t\in P} \|g(t, y(t))\| \in R$, then the map g^y is bounded. Therefore the map g is F-admissible.

Let $A_x^\delta = \{y \mid y \in Y \text{ and } y(P) \subset V_{x(P)}^\delta\}$. It is clear that the set A_x^δ is closed, hence, it is complete, as closed subset of a complete space Y. Since

$$\sup_{(t,s)\in P\times P} \|g(t,x(s))\| \leq \sup_{(t,s)\in P\times P} \|g(t,x(s)) - g(t,x(t_0))\| + \\ + \sup_{t\in P} \|g(t,x(t_0))\| \leq L \sup_{t\in P} \|x(t) - x(t_0)\| + l \leq Ld + l,$$

i.e. $\sup_{t\in P} \|g(t,x(t))\| \leq Ld + l$, then taking as l_1 in the condition 4) of Theorem 5 the number $Ld + l$, we shall receive that our number λ_0 coincides with λ_0 from Theorem 5. According to Theorem 5 for an arbitrary λ, $|\lambda| < \lambda_0$, there exists $x_\lambda \in q(b + \lambda g^{x_\lambda}) \cap A_x^\delta$. By the definition of maps q, g^{x_λ} and of set A_x^δ the map $x_\lambda : P \to V_{x(P)}^\delta$ is compact and satisfies the equality $x'_\lambda(t)h = a(t)hx_\lambda(t) + (b(t) + \lambda g(t, x_\lambda(t))h$ for an arbitrary $t \in P,\ h \in E$. It also means that $x_\lambda : P \to V_{x(P)}^\delta$ is a compact solution of the equation (1). The estimation (18) follows from the estimation of x_λ from Theorem 5.

Let's prove that if $\lim_{i\to\infty} \lambda_i = 0$, then $\lim_{i\to\infty} x_{\lambda_i} = x$ is uniform on P. Let $\varepsilon > 0$ be an arbitrary number. Since $\lim_{i\to\infty} \lambda_i = 0$, then there is a number i_0 such that for all $i > i_0$ $|\lambda_i| < \varepsilon/(r(Ld + l))$. Let $i > i_0$. Because $x_{\lambda_i} - x$ is a compact solution of the equation

$$y'h = a(t)hy + \lambda_i g(t, x_{\lambda_i}(t))h \quad (h \in E)$$

with a bounded map $\lambda_i g(t, x_{\lambda_i}(t))$ $(t \in P)$, then using the conditions 1) and 5) of our theorem, we have

$$\sup_{t\in P} \|x_{\lambda_i}(t) - x(t)\| \leq r|\lambda_i| \sup_{t\in P} \|g(t, x_{\lambda_i}(t))\| \leq r|\lambda_i|(Ld + l) < \varepsilon,$$

i.e. $\sup_{t\in P} \|x_{\lambda_i}(t) - x(t)\| < \varepsilon$. The proof also means that $\lim_{i\to\infty} x_{\lambda_i} = x$ is uniform on P.

Let's assume that the map a is bounded and we shall prove that x_λ is Lagrange S-stable. Let's designate by g^* the map $P \times T \to L(E,T)$ by the rule: $g^*(t,y)h = a(t)hy + (b(t) + \lambda g(t,y))h$ $((t,y) \in P \times T,\ h \in E)$. Since $\overline{x_\lambda(P)} \subset V_{x(P)}^\delta$ and for $\forall y \in \overline{x_\lambda(P)}$

$$\sup_{t\in P} \|g^*(t,y)\| = \sup_{t\in P} \sup_{\|h\|=1} \|g^*(t,y)h\| \leq \sup_{t\in P}(\|a(t)\|\|y\| + \|b(t)\| + \\ + |\lambda|\|g(t,y)\|) \leq \sup_{t\in P}(\|a(t)\|\|y\| + \|b(t)\|) + |\lambda|(Ld + l) = m \in R,$$

i.e. $\sup_{t\in P} \|g^*(t,y)\| \leq m \in R$, then the map g^* is bounded on the set $P \times \overline{x_\lambda(P)}$. In that case the solution x_λ is Lagrange S-stable according to Proposition 1.

If the conditions 1) – 7) of our theorem are fulfilled, then the S-concordance of x_λ with (a, b, g) is proved as in Theorem 6.

Theorem 9. *Let E be a Banach space, $P = E$ and for the equation* (1) *the following conditions are fulfilled:*

1) *for* $\forall t \in E$ $a(t) = a$ *is an operator such that* $(Sp\ a)e$ *does not intersect the imaginary axis of the complex plane for some vector* $e \in E$ *of the unit norm;*

2) *the map* b *is bounded, continuously differentiable and* $\wedge\{ahb(t)k - b'(t)kh\} = 0$ *for arbitrary* $h, k, t \in E$*;*

3) *for an arbitrary bounded map* $y \in C(E,T)$*, the map* g^y *is continuously differentiable and* $\wedge\{ahg^y(t)k - (g^y)'(t)kh\} = 0$ *for arbitrary* $h, k, t \in E$*;*

4) x *is a bounded solution of the equation* (16) *and* $V^{\delta}_{x(E)}$ *is a closed* δ*-neighbourhood of the set* $x(E)$ $(\delta > 0)$*;*

5) *the map* g *satisfies the Lipschitz condition in the second argument on* $V^{\delta}_{x(E)}$ *with the Lipschitz constant* L*;*

6) *for some* $t_0 \in E$ $\sup_{t\in E} \|g(t, x(t_0))\| = l \in R$*;*

7) $\lambda_0 = \delta/(r(L(d+\delta)+l))$*, where* r *is the constant from the Example 2 of regular maps and* d *is the diameter of the set* $x(E)$*.*

Then for an arbitrary λ, $|\lambda| < \lambda_0$*, the equation* (1) *has a unique bounded solution* $x_\lambda : E \to V^{\delta}_{x(E)}$*. This solution is Lagrange* E*-stable and for it the estimation* (21) *is valid. Besides, if* $\lim_{i\to\infty} \lambda_i = 0$*, then* $\lim_{i\to\infty} x_{\lambda_i} = x$ *is uniform on* E*.*

If in addition to the conditions 1) – 7) *of our theorem* $T = R^m$*, then the solution* x_λ *is* E*-concordant with* (a, b, g)*.*

Proof. We connect with the equation (1) the r-regular map from the Example 2 of regular maps, and the proof of the theorem is done by the proof scheme of Theorem 8 taking into account that for $T = R^m$ the solution x_λ is compact.

Alongside with the equation (1) we also consider the limiting equations

$$y'h = A(t)hy + (B(t) + \lambda G(t, y))h \quad (h \in E), \tag{22}$$

where $A \in \overline{aS}$, $B \in \overline{bS}$, $G \in \overline{gS}$.

Definition 6. *The equation* (15) *is called weakly regular* (*regular*) *of type* 2 *with the constant* $r > 0$ *if for an arbitrary* $A \in \overline{aS}$ *and an arbitrary bounded map* $b \in C(P, L(E,T))$ *the equation*

$$y'h = A(t)hy + b(t)h \quad (h \in E). \tag{23}$$

has a compact (*unique compact*) *solution* $x \in C(P,T)$*. In addition, for an arbitrary compact solution* x *of the equation* (23) *the estimation* (17) *takes place.*

Theorem 10. *Let for the equation* (1) *the following conditions be fulfilled:*

1) *the equation* (15) *is weakly regular of type* 2 *with the constant* r*;*

2) *the map* b *is bounded;*

3) *there exists* $t_0 \in T$ *for which the set* $g(P, t_0)$ *is bounded;*

4) *the map* g *satisfies the Lipschitz condition in the second argument with the Lipschitz constant* L.

Then for an arbitrary λ, $|\lambda| < 1/(rL)$, *and an arbitrary triple* $(A, B, G) \in \overline{(a, b, g)S}$, *the equation* (22) *has a compact solution* x_λ *and for it the estimation is valid*

$$\sup_{t \in P} \|x_\lambda(t)\| \leq r \cdot \sup_{t \in P} \|B(t) + \lambda G(t, x_\lambda(t))\|. \tag{24}$$

If in addition to the conditions 1) – 4) *of our theorem the following condition is fulfilled:*

5) *the map* a *is bounded and the set* $\overline{g(P, y)}$ *is compact for* $\forall y \in T$,

then the solution x_λ *is Lagrange* S*-stable.*

If in addition to the conditions 1) – 4) *of our theorem the following conditions are fulfilled:*

6) *the equation* (15) *is regular of type* 2 *with the constant* r*;*

7) *the space* E *is quasicomplete,* P *is a connected convex set if the space* E *is finite-dimensional and* $P = E$ *if the space* E *is infinite-dimensional,*

then the solution x_λ *is* S*-concordant with* (A, B, G).

If in addition to the conditions 1) – 4) *and* 6) – 7) *of our theorem the following conditions are fulfilled:*

8) *the map* (a, b) *is Lagrange* S*-stable;*

9) *for* $\forall y \in T$, *the map* g *is uniformly continuous on the set* $P \times \{y\}$ *and the set* $\overline{g(P, y)}$ *is compact,*

then the solution x_λ *is uniformly* S*-concordant with* (A, B, G).

Proof. Let the conditions 1) – 4) of theorem be fulfilled and $(A, B, G) \in \overline{(a, b, g)S}$. Since in the conditions of our theorem the map B is bounded, then the set $G(P, t_0)$ is bounded and the map G satisfies, according to Lemma 4, the Lipschitz condition in the second argument with the constant L, then the conclusion of our theorem follows from Theorem 6. If the conditions 1) – 5) of the theorem are valid, then the conclusion of our theorem follows from Theorem 6, so in our case A is a bounded map and the set $\overline{G(P, y)}$ is compact for an arbitrary $y \in T$.

If the conditions 1) – 4) and 6) – 7) of our theorem are fulfilled, then the conclusion of our theorem follows from Theorem 6.

Let the conditions 1) – 4) and 6) – 9) of our theorem be fulfilled. According to Lemma 3 the map g is Lagrange S-stable. Therefore the maps (a, b, g) and $(A, B, G) \in \overline{(a, b, g)S}$ are Lagrange S-stable, too. If the equation (15) is regular of type 2 with the constant r, then each equation (22) for an arbitrary λ, $|\lambda| <$

$1/(rL)$, has a unique compact solution x_λ. Suppose that x_λ is not uniformly S-concordant with (A,B,G). Then there exists an index α of the uniform structure of the space $C(P,T)$ such that for an arbitrary index β the uniform structure of the space $C(P,L(E,L(T,T)))\times C(P,L(E,T))\times C(P\times T,L(E,T))$ there are elements $s_\beta,t_\beta\in S$ such that

$$((A_{t_\beta},B_{t_\beta},G_{t_\beta}),(A_{s_\beta},B_{s_\beta},G_{s_\beta}))\in\beta \tag{25}$$

and

$$((x_\lambda)_{t_\beta},(x_\lambda)_{s_\beta})\notin\alpha. \tag{26}$$

By virtue of the compactness of the set $\overline{(A,B,G)S}$ we may consider that

$$\lim_\beta(A_{t_\beta},B_{t_\beta},G_{t_\beta})=(A_1,B_1,G_1)$$

and

$$\lim_\beta(A_{s_\beta},B_{s_\beta},G_{s_\beta})=(A_2,B_2,G_2).$$

In this case from the relation (25) we obtain that $(A_1,B_1,G_1)=(A_2,B_2,G_2)=(A_0,B_0,G_0)$. By virtue of Lemma 1 we suppose that $\lim_\beta(x_\lambda)_{t_\beta}=\psi_1$ and $\lim_\beta(x_\lambda)_{s_\beta}=\psi_2$, in addition, ψ_1 and ψ_2 are solutions of the equation

$$y'h=A_0(t)hy+(B_0(t)+\lambda G_0(t,y))h\quad(h\in E).$$

Since this equation has a unique compact solution then $\psi_1=\psi_2$, that contradicts (26). The contradiction indicates that the solution x_λ is uniformly S-concordant with (A,B,G).

Theorem 11. *Let for the equation* (1) *the following conditions be fulfilled:*

1) *the equation* (15) *is weakly regular of type* 2 *with the constant* r*;*

2) *the map* b *is bounded;*

3) x *is a compact solution of the equation* (16) *and* $V^\delta_{x(P)}$ *is a closed* δ*-neighbourhood of* $x(P)$ $(\delta>0)$*;*

4) *the map* g *satisfies the Lipschitz condition in the second argument from* $V^\delta_{x(P)}$ *with the Lipschitz constant* L*;*

5) *for some* $t_0\in P$, $\sup_{t\in P}\|g(t,x(t_0))\|=l\in R$.

Then for an arbitrary λ, $|\lambda|<\lambda_0=\delta/r(L(d+\delta)+l)$*, where* d *is the diameter of the set* $x(P)$*, and for* $\forall(A,B,G)\in\overline{(a,b,g|_{V^\delta_{x(P)}})S}$*, where* $g|_{V^\delta_{x(P)}}$ *is the contraction of the map* g *on the set* $P\times V^\delta_{x(P)}$*, the equation* (22) *has a compact solution* $x_\lambda:P\to V^\delta_{x(P)}$ *and for it the estimation* (24) *is valid. Besides if* $\lim_{i\to\infty}\lambda_i=0$*, then* $\lim_{i\to\infty}x_{\lambda_i}=z$ *is uniform on* P *for some compact solution* z *of the equation*

$$y'h=A(t)hy+B(t)h\quad(h\in E)$$

(z *exists by virtue of the condition* 1) *of our theorem*).

If in addition to the conditions 1) – 5) *of our theorem the map* a *is bounded, then the solution* x_λ *is Lagrange* S*-stable.*

If in addition to the conditions 1) – 5) *of our theorem the following conditions are fulfilled:*

6) *the equation* (15) *is regular of type* 2 *with the constant* r*;*

7) *the space* E *is quasicomplete,* P *is a connected convex set if the space* E *is finite-dimensional and* $P = E$ *if the space* E *is infinite-dimensional,*

then the solution x_λ *is* S*-concordant with* (A, B, G).

If in addition to the conditions 1) – 7) *of our theorem the map* (a, b) *is Lagrange* S*-stable, for an arbitrary* $y \in V^{\delta}_{x(P)}$ *the map* g *is uniformly continuous on the set* $P \times \{y\}$ *and the set* $\overline{g(P, y)}$ *is compact, then the solution* x_λ *is uniformly* S*-concordant with* (A, B, G).

Proof. The proof is similar to the proof of Theorem 10 using Theorem 8 instead of Theorem 6.

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