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On theory of surfaces defined by the first order systems of equations

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Abstract. The properties of surfaces defined by spatial systems of differential equations are studied. The Monge equations connected with the first order nonlinear p.d.e. are investigated. The properties of Riemannian metrics defined by the systems of differential equations having applications in theory of nonlinear dynamical systems with regular and chaotic behaviour are considered.

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1 Introduction

An investigation of the properties of spatial systems of differential equations having three degrees of freedom represented by the dynamical variables x, y and z

$$\frac{dx}{ds} = P(x,y,z), \quad \frac{dy}{ds} = Q(x,y,z), \quad \frac{dz}{ds} = R(x,y,z) \tag{1}$$

is an important task of modern mathematics.

The Lorenz

$$\frac{dx}{ds} = \sigma(y-x), \quad \frac{dy}{ds} = rx - y - xz, \quad \frac{dz}{ds} = xy - bz, \tag{2}$$

and the Rössler

$$\frac{dx}{ds} = -(y+z), \quad \frac{dy}{ds} = x + ay, \quad \frac{dz}{ds} = b + xz - cz \tag{3}$$

are the most famous examples of the systems of equations having regular and chaotic behavior of trajectories at some values of parameters.

To study the properties of the systems (1) we propose geometrical approach founded on consideration of the surfaces of the form $z = z(x,y)$, $x = x(y,z)$ or $y = y(x,z)$ in R^3-space which are connected naturally with such type of systems.

In result we get from the system (1) a set of nonlinear of the first order partial differential equation for every pair of variables.

As example, in the case $z = z(x,y)$ we can write the equation

$$\frac{\partial z(x,y)}{\partial x} P(x,y,z(x,y)) + \frac{\partial z(x,y)}{\partial x} Q(x,y,z(x,y)) - R(x,y,z(x,y)) = 0. \tag{4}$$

Its solutions the surfaces in the R^3-space are locally presented.

A studies of surfaces defined by spatial systems of equations like (1) can be useful for understanding of their properties.

2 The examples of surfaces corresponding to the Lorenz system

For the system (2) we consider the variable z as a function of variables x and y i.e. $z = z(x, y)$.

As result we get the partial first order differential equation

$$\sigma(y-x)\frac{\partial z(x,y)}{\partial x} + (rx - y - xz(x,y))\frac{\partial z(x,y)}{\partial x} - xy + bz(x,y) = 0 \tag{5}$$

determines the surface $z = z(x, y, \sigma, b, r)$ depending on parameters.

To the integration of this equation we present it in the equivalent form

$$\begin{gathered}\sigma\, y\left(\frac{\partial}{\partial x}z(x,y)\right) + ((r-1)\,x - (\sigma+1)\,y - xz(x,y))\left(\frac{\partial}{\partial y}z(x,y)\right) - \\ -yx - x^2 + bz(x,y) = 0\end{gathered} \tag{6}$$

which is connected with the previous form by the change of variable

$$y = Y + x.$$

In fact after such a substitution the Lorenz system looks as

$$\frac{dx}{ds} = \sigma Y, \quad \frac{dY}{ds} = rx - Y - x - \sigma Y - xz, \quad \frac{dz}{ds} = x^2 + xY - bz \tag{7}$$

and the corresponding equation for the function $z = z(x, Y)$ takes the form (6), where we conserve old name of variable $Y = y$.

2.1 Simplest solutions

1. The substitution

$$z(x, y) = A(x)$$

into the equation (6) leads to the conditions

$$A(x) = \frac{x^2}{2\sigma}, \quad b = 2\sigma.$$

From the system (7) in the case $z = z(x)$ we get the equation

$$y(x)\frac{d}{dx}y(x) + \frac{(\sigma+1)}{\sigma}y(x) - \frac{(r-1)}{\sigma}x + \frac{1}{2\sigma^2}x^3 = 0.$$

2. The substitution of more general form

$$z(x, y) = A(x)(1 + B(x)y + C(x)y^2)$$

gives rise to the expression

$$z(x,y) = A(x)\left(1 + \left(2\,\frac{\sigma}{x} - 2\,x^{-1}\right) y\,(A(x))^{-1} - \frac{\sigma\,y^2}{x^2 A(x)}\right), \quad r = 2\sigma - 1,$$

$$b = 6\sigma - 2,$$

where

$$A(x) = -1/4\,\frac{4 - x^2 + 4\,\sigma^2 - 8\,\sigma}{\sigma}.$$

In an explicit form we have

$$z(x,y) = -\frac{\sigma\,y^2}{x^2} + 2\,\frac{(\sigma - 1)\,y}{x} + 1/4\,\frac{x^2}{\sigma} - \frac{(\sigma - 1)^2}{\sigma}. \tag{8}$$

Returning to the system (6) we get from the system (7) the Abel equation

$$\frac{d}{dx}y(x) - 1/4\,\frac{x^4 + \left(-12\,\sigma^2 - 4 + 12\,\sigma\right)x^2 + \left(-4\,\sigma + 8\,\sigma^2\right)y(x)x - 4\,\sigma^2\,(y(x))^2}{x\,(-y(x) + x)\,\sigma^2} = 0 \tag{9}$$

for the function $y = y(x)$.

3. The next example is the solution of equation (6) in the form

$$z(x,y) = 1/4\,\frac{x^4 + 4\,x^2 - 4\,rx^2 - 4\,y^2}{x^2 + 4 - 4\,r},$$

where

$$\sigma = 1, \quad b = 4, \quad r \quad is \quad arbitrary.$$

Remark 1. To integrate the partial nonlinear first order differential equation

$$F(x, y, z(x,y), z_x, z_y) = 0 \tag{10}$$

a following method can be applied.

We use the change of variables

$$z(x,y) \to u(x,t), \quad y \to v(x,t), \quad z_x \to u_x - \frac{v_x}{v_t}u_t, \quad v_y \to \frac{u_t}{v_t}. \tag{11}$$

In result instead of the equation (10) one gets the relation between the new variables $u(x,t)$ and $v(x,t)$ and their partial derivatives

$$\Phi(u, v, u_x, u_t, v_x, v_t) = 0. \tag{12}$$

In some cases to solve the last equation is more simple problem than to solve the equation (10).

To illustrate this method let us consider an example.

The equation

$$\frac{\partial}{\partial x}z(x,y) - \left(\frac{\partial}{\partial y}z(x,y)\right)^2 = 0 \tag{13}$$

is transformed into the following form

$$\frac{\partial}{\partial x}u(x,t) - \frac{\left(\frac{\partial}{\partial t}u(x,t)\right)\frac{\partial}{\partial x}v(x,t)}{\frac{\partial}{\partial t}v(x,t)} - \frac{\left(\frac{\partial}{\partial t}u(x,t)\right)^2}{\left(\frac{\partial}{\partial t}v(x,t)\right)^2} = 0.$$

Using the substitution

$$u(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t), \quad v(x,t) = \frac{\partial}{\partial t}\omega(x,t)$$

we find the equation for $\omega(x,t)$

$$\frac{\partial}{\partial x}\omega(x,t) + t^2 = 0.$$

Its integration leads to

$$\omega(x,t) = -t^2 x + F_1(t)$$

where $F_1(t)$ is an arbitrary function.

Now with the help of $\omega(x,t)$ we find the functions $u(x,t)$ and $v(x,t)$

$$u(x,t) = -t^2 x + t\frac{d}{dt}F_1(t) - F_1(t), \quad v(x,t) = -2\,tx + \frac{d}{dt}F_1(t)$$

or

$$u(x,t) = ty + t^2 x - F_1(t), \quad y = -2\,tx + \frac{d}{dt}F_1(t).$$

After the choice of arbitrary function $F_1(t)$ and the elimination of the parameter t from these relations we get the function $z(x,y)$, satisfying the equation (13).

We apply this method for the study of the surfaces connected with the Lorenz model (7) in the case $y = y(x,z)$.

The corresponding partial differential equation is

$$\begin{gathered}\left(\frac{\partial}{\partial x}y(x,z)\right)\sigma\, y(x,z) + \left(\frac{\partial}{\partial z}y(x,z)\right)\left(xy(x,z) + x^2 - bz\right) + \\ +\sigma\, y(x,z) - rx + y(x,z) + x + xz = 0.\end{gathered} \tag{14}$$

In new variables it looks as

$$\begin{gathered}\left(\frac{\partial}{\partial x}u(x,t) - \frac{\left(\frac{\partial}{\partial t}u(x,t)\right)\frac{\partial}{\partial x}v(x,t)}{\frac{\partial}{\partial t}v(x,t)}\right)\sigma\, u(x,t) + \frac{\left(\frac{\partial}{\partial t}u(x,t)\right)\left(xu(x,t) + x^2 - bv(x,t)\right)}{\frac{\partial}{\partial t}v(x,t)} + \\ +\sigma\, u(x,t) - rx + u(x,t) + x + xv(x,t) = 0.\end{gathered} \tag{15}$$

After the substitution

$$u(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t), \quad v(x,t) = \frac{\partial}{\partial t}\omega(x,t)$$

we get the equation for the function $\omega(x,t)$

$$-\left(\frac{\partial}{\partial x}\omega(x,t)\right)\sigma\, t\frac{\partial}{\partial t}\omega(x,t)+\left(\frac{\partial}{\partial x}\omega(x,t)\right)\sigma\,\omega(x,t)+xt^2\frac{\partial}{\partial t}\omega(x,t)-tx\omega(x,t)-tb\frac{\partial}{\partial t}\omega(x,t)+$$

$$+tx^2+\sigma\, t\frac{\partial}{\partial t}\omega(x,t)-\sigma\,\omega(x,t)-rx+t\frac{\partial}{\partial t}\omega(x,t)-\omega(x,t)+x+x\frac{\partial}{\partial t}\omega(x,t)=0.$$

The simplest solution of this equation has the form

$$\omega(x,t)=A(t)x,$$

where

$$A(t)=1+\sqrt{t^2+1}C_1,\quad b=1,\sigma=1,r=C_1^2.$$

With the help of this solution we find the functions $u(x,t)$ and $v(x,t)$

$$u(x,t)=-\frac{x\sqrt{r}}{\sqrt{t^2+1}}-x,\quad v(x,t)=\frac{\sqrt{r}tx}{\sqrt{t^2+1}}.$$

Elimination the parameter t from these relations we find the corresponding solution of the equation (14)

$$y(x,z)=-\sqrt{-z^2+x^2r}-x,$$

with

$$b=1,\quad \sigma=1,\quad and\quad r\quad is\quad arbitrary.$$

For the surfaces in the form

$$x=x(y,z)$$

we get the equation

$$\begin{aligned}&\left(\frac{\partial}{\partial y}x(y,z)\right)(-\sigma\, y+rx(y,z)-y-x(y,z)-x(y,z)z)+\\&+\left(\frac{\partial}{\partial z}x(y,z)\right)\left((x(y,z))^2-bz+x(y,z)y\right)-\sigma\, y=0.\end{aligned}\tag{16}$$

The simplest solution of this equation is

$$x(y,z)=\sqrt{2\,rz-z^2}-y,\quad b=1,\quad r=0,\quad \sigma\quad is\quad arbitrary.$$

Another type of solution is

$$x(y,z)=-1/2\,\frac{z^2}{y}-1/2\,y,\quad b=1,\quad \sigma=1,\quad r=1.$$

More general solutions in the case $x=x(y,z)$ can be obtained with the help of transformation like (11).

On the first step with the help of relations

$$\frac{\partial}{\partial z}x(y,z)=\frac{\frac{\partial}{\partial t}u(y,t)}{\frac{\partial}{\partial t}v(y,t)},\quad \frac{\partial}{\partial y}x(y,z)=\frac{\partial}{\partial y}u(y,t)-\frac{\left(\frac{\partial}{\partial t}u(y,t)\right)\frac{\partial}{\partial y}v(y,t)}{\frac{\partial}{\partial t}v(y,t)},$$

$$x(y,z)=u(y,t),\quad z=v(y,t)$$

we get the equation

$$\left(\frac{\partial}{\partial y}u(y,t)-\frac{\left(\frac{\partial}{\partial t}u(y,t)\right)\frac{\partial}{\partial y}v(y,t)}{\frac{\partial}{\partial t}v(y,t)}\right)(-\sigma\,y+ru(y,t)-y-u(y,t)-u(y,t)v(y,t))+$$
$$+\frac{\left(\frac{\partial}{\partial t}u(y,t)\right)\left((u(y,t))^2-bv(y,t)+u(y,t)y\right)}{\frac{\partial}{\partial t}v(y,t)}-\sigma\,y=0$$

and then with the help of substitutions

$$u(y,t)=t\frac{\partial}{\partial t}\omega(y,t)-\omega(y,t),\quad v(y,t)=\frac{\partial}{\partial t}\omega(y,t)$$

we find the equation for the function $\omega(y,t)$

$$\left(\frac{\partial}{\partial y}\omega(y,t)\right)\sigma\,y-\left(\frac{\partial}{\partial y}\omega(y,t)\right)rt\frac{\partial}{\partial t}\omega(y,t)+\left(\frac{\partial}{\partial y}\omega(y,t)\right)r\omega(y,t)+\left(\frac{\partial}{\partial y}\omega(y,t)\right)y+$$
$$+\left(\frac{\partial}{\partial y}\omega(y,t)\right)t\frac{\partial}{\partial t}\omega(y,t)-\left(\frac{\partial}{\partial y}\omega(y,t)\right)\omega(y,t)+\left(\frac{\partial}{\partial y}\omega(y,t)\right)t\left(\frac{\partial}{\partial t}\omega(y,t)\right)^2-$$
$$-\left(\frac{\partial}{\partial y}\omega(y,t)\right)\left(\frac{\partial}{\partial t}\omega(y,t)\right)\omega(y,t)+t^3\left(\frac{\partial}{\partial t}\omega(y,t)\right)^2-2\,t^2\left(\frac{\partial}{\partial t}\omega(y,t)\right)\omega(y,t)+$$
$$+t\,(\omega(y,t))^2-tb\frac{\partial}{\partial t}\omega(y,t)+yt^2\frac{\partial}{\partial t}\omega(y,t)-ty\omega(y,t)-\sigma\,y=0.$$

The separation of variables in this equation leads to the solution

$$\omega(y,t)=1/4\,\frac{y\left(2+4\,C_1+2\,\sqrt{1+2\,C_1+2\,t^2C_1}\right)}{C_1},$$

where

$$r=(1+2\,C_1)^{-1},\quad b=1,\quad \sigma=1.$$

Using this solution we get the functions $u(y,t)$ and $v(y,t)$ and then after the elimination of the parameter t from the relations

$$x=u(y,t),\quad z=v(y,t)$$

obtain the family of solutions $x(y,z)$ of equation (16)

$$x(y,z)=1/4\,\frac{-4\,yC-2\,y+2\,\sqrt{2\,y^2C+y^2-2\,C\,z^2-4\,z^2C^2}}{C},$$

$$r = (1 + 2\,C)^{-1}\,, \quad b = 1, \quad \sigma = 1.$$

The next example.

With the help of substitution

$$v(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t), \quad u(x,t) = \frac{\partial}{\partial t}\omega(x,t)$$

into the relation

$$\sigma\ (y - x)\,\frac{\partial}{\partial x}z(x,y) + (rx - y - xz(x,y))\,\frac{\partial}{\partial y}z(x,y) - yx + bz(x,y) = 0$$

we get the equation

$$\left(-\sigma\,\frac{\partial}{\partial x}\omega(x,t) + tx + 1\right)\omega(x,t) + \left(-x\sigma + \sigma\,t\frac{\partial}{\partial t}\omega(x,t)\right)\frac{\partial}{\partial x}\omega(x,t) +$$

$$\left(bt - x - xt^2 - t\right)\frac{\partial}{\partial t}\omega(x,t) + rx = 0.$$

It has the solution

$$\omega(x,t) = 1/2\,\frac{x^2\left(1 + t^2\right)}{t}$$

by the conditions

$$\sigma = 1/2, \quad b = 1, \quad r = 0.$$

From here we get the equation of the corresponding surface

$$z(x,y) = 1/2\,\frac{x^4 - y^2}{x^2}.$$

3 Spatial homogeneous quadratic first order systems of equations

The system of equations

$$\begin{aligned}\frac{dx}{ds} &= a_0 + a_1x + a_2y + a_{11}x^2 + a_{12}xy + a_{22}y^2,\\ \frac{dy}{ds} &= b_0 + b_1x + b_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2\end{aligned} \tag{17}$$

where a_i, a_{ij} and b_i, b_{ij} are parameters after the extension on the projective plane takes the form of Pfaff equation

$$\left(x\tilde{Q} - \tilde{P}y\right)dz - dx\,z\tilde{Q} + \tilde{P}\,dy\,z = 0 \tag{18}$$

where the functions $\tilde{P}, \tilde{Q}$ are homogeneous polinomials.

In the explicit form we get the expression

$$\left(xb\_0\,z^2 + b\_1\,x^2z + xb\_2\,yz + b\_\{11\}\,x^3 + b\_\{12\}\,x^2y + xb\_\{22\}\,y^2 - ya\_0\,z^2\right)dz -$$

$$-\left(ya\_1\,xz - a\_2\,y^2z - ya\_\{11\}\,x^2 - a\_\{12\}\,xy^2 - a\_\{22\}\,y^3\right)dz+$$
$$+\left(z^2a\_2\,y + za\_\{11\}\,x^2 + z^3a\_0 + z^2a\_1\,x + za\_\{12\}\,xy + za\_\{22\}\,y^2\right)dy+$$
$$+\left(-z^3b\_0 - z^2b\_1\,x - zb\_\{12\}\,xy - zb\_\{22\}\,y^2 - z^2b\_2\,y - zb\_\{11\}\,x^2\right)dx = 0.$$

The spatial first order system of equations

$$\frac{dx}{ds} = P(x,y,z), \quad \frac{dy}{ds} = Q(x,y,z), \quad \frac{dz}{ds} = R(x,y,z), \tag{19}$$

connected with a given Pfaff equation has the following form

$$\frac{dx}{ds} = Q_z - R_y, \quad \frac{dy}{ds} = R_x - P_z, \quad \frac{dz}{ds} = P_y - Q_x,$$

and in our case looks as

$$\frac{d}{ds}x(s) = 4\,a\_0\,z^2 + (4\,a\_2\,y + (3\,a\_1\, - b\_2\,)\,x)\,z + 4\,a\_\{22\}\,y^2+$$
$$+(3\,a\_\{12\} - 2\,b\_\{22\})\,xy + (2\,a\_\{11\} - b\_\{12\})\,x^2,$$
$$\begin{aligned}\frac{d}{ds}y(s) = 4\,b\_0\,z^2 + ((3\,b\_2\, - a\_1\,)\,y + 4\,b\_1\,x)\,z + (2\,b\_\{22\} - a\_\{12\})\,y^2+\\ +(-2\,a\_\{11\} + 3\,b\_\{12\})\,xy + 4\,b\_\{11\}\,x^2,\end{aligned} \tag{20}$$
$$\frac{d}{ds}z(s) = (-b\_2\, - a\_1\,)\,z^2 + ((-2\,b\_\{22\} - a\_\{12\})\,y - b\_\{12\}\,x - 2\,a\_\{11\}\,x)\,z.$$

For such a system of equations the condition

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0$$

is fulfilled.

The system of equations (20) is equivalent to the first order equation connected with the system (17) in coordinates $\chi(s)$ and $\eta(s)$

$$\xi(s) = \frac{x(s)}{z(s)}, \quad \eta(s) = \frac{y(s)}{z(s)}$$

or

$$\frac{d\xi}{d\eta} = \frac{a_0 + a_1\xi + a_2\eta + a_{11}\xi^2 + a_{12}\xi\eta + a_{22}\eta^2}{b_0 + b_1\xi + b_2\eta + b_{11}\xi^2 + b_{12}\xi\eta + b_{22}\eta^2}.$$

The equation of the surfaces defined by the system (20) has the form

$$\left(\frac{\partial}{\partial x}z(x,y)\right)P(x,y,z) + \left(\frac{\partial}{\partial y}z(x,y)\right)Q(x,y,z) - R(x,y,z) = 0.$$

Let us consider some examples.

For the system of equations

$$\frac{d}{dt}x(t) + 4\,z(t)y(t) - 3\,z(t)lx(t) - 4\,\left(y(t)\right)^2 - 3\,mx(t)y(t) + 20\,\left(x(t)\right)^2 + \left(x(t)\right)^2 n = 0,$$

$$\frac{d}{dt}y(t) - 4\,z(t)x(t) + z(t)ly(t) - 4\,\left(x(t)\right)^2 - 3\,x(t)y(t)n - 20\,x(t)y(t) + m\left(y(t)\right)^2 = 0,$$

$$\frac{d}{dt}z(t) + l\left(z(t)\right)^2 + z(t)my(t) + z(t)x(t)n - 20\,z(t)x(t) = 0,$$

which is connected with the projective extension of the planar system that cab be found in the theory of limit cycles

$$\frac{d}{dt}x(t) - lx + y + 10\,x^2 - mxy - y^2 = 0,$$

$$\frac{d}{dt}y(t) - x - x^2 - nxy = 0,$$

the equation of surface $z = z(x, y)$ takes the form

$$\left(4\,z(x,y)x - z(x,y)ly + 4\,x^2 + 3\,xyn + 20\,xy - my^2\right)\frac{\partial}{\partial y}z(x,y) +$$
$$+\left(-4\,yz(x,y) + 3\,z(x,y)lx + 4\,y^2 + 3\,mxy - 20\,x^2 - x^2 n\right)\frac{\partial}{\partial x}z(x,y) +$$
$$+z(x,y)my + z(x,y)xn - 20\,z(x,y)x + l\left(z(x,y)\right)^2 = 0.$$

A simplest solution of this equation can be obtained with the help of u, v-transformation with the conditions

$$u(x,t) = t\frac{\partial}{\partial t}\omega(x,t) - \omega(x,t), \quad v(x,t) = \frac{\partial}{\partial t}\omega(x,t). \tag{21}$$

As result we get the equation

$$\left(\left(-3\,xlt - 4\,\omega(x,t) - 3\,xm\right)\frac{\partial}{\partial t}\omega(x,t) + (4\,t-4)\left(\frac{\partial}{\partial t}\omega(x,t)\right)^2\right)\frac{\partial}{\partial x}\omega(x,t) +$$
$$+\left(3\,xl\omega(x,t) + (n+20)\,x^2\right)\frac{\partial}{\partial x}\omega(x,t) +$$
$$+\left(-tl\omega(x,t) + 4\,xt^2 - m\omega(x,t) + 4\,txn\right)\frac{\partial}{\partial t}\omega(x,t) -$$
$$-xn\omega(x,t) - 4\,tx\omega(x,t) + l\left(\omega(x,t)\right)^2 + 20\,x\omega(x,t) + 4\,tx^2 = 0.$$

A simplest solution of this equation can be presented in the form

$$\omega(x,t) = A(t) + kxt$$

with parameter k.

After the substitution of this expression we find possible values of parameters

$$k = -1/9, \quad m = \frac{85}{36}, \quad n = -5/4, \quad l = 0$$

and the expression for the function $A(t)$

$$A(t) = t^{\frac{85}{16}} \_C1\, (-1+t)^{-\frac{69}{16}}.$$

After the elimination of variable t from the relations (21) we get the expression for the function $z(x,y)$

$$4477456\, (x + 9\,y - 9\,z(x,y))^5 \left(-23\,\frac{x + 9\,y - 9\,z(x,y)}{23\,x + 207\,y + 48\,z(x,y)}\right)^{\frac{5}{16}} -$$

$$-3234611728125\,\_C1\, 255^{\frac{5}{16}}\, (z(x,y))^4 \left(\frac{z(x,y)}{23\,x + 207\,y + 48\,z(x,y)}\right)^{\frac{5}{16}} = 0.$$

In general case the equation of the surfaces looks as

$$z_x \left(4\,a_0 z^2 + (4\,a_2\,y + (3\,a_1 - b_2)\,x)\,z + 4\,a_{22}\,y^2 + (3\,a_{12} - 2\,b_{22})\,xy + (2\,a_{11} - b_{12})\,x^2\right) +$$

$$+ z_y \left(4\,b_0\,z^2 + ((3\,b_2 - a_1)\,y + 4\,b_1\,x)\,z + (2\,b_{22} - a_{12})\,y^2 + (-2\,a_{11} + 3\,b_{12})\,xy + 4\,b_{11}\,x^2\right) +$$

$$+ (b_2 + a_1)\,(z)^2 + ((2\,b_{22} + a_{12})\,y + (2\,a_{11} + b_{12})\,x)\,z = 0.$$

After the u,v-transformation (21) with the function $\omega(x,t) = A(t)x$ we find from here the equation on the function $A(t)$

$$\left(a_0\,t^2 A(t) + a_2\,tA(t) - b_2\,t^2 - b_{22}\,t + a_{22}\,A(t) - t^3 b_0\right)\left(\frac{d}{dt}A(t)\right)^2 +$$

$$+ \left(-(2\,a_0\,t + a_2)\,(A(t))^2 + (a_{12} - t^2 b_1 + a_1\,t - b_{12}\,t + b_2\,tA + 2\,t^2 b_0)\,A(t)\right)\frac{d}{dt}A(t) -$$

$$-a_1\,(A(t))^2 + a_{11}\,A(t) + a_0\,(A(t))^3 + tb_1\,A(t) - tb_{11} - tb_0\,(A(t))^2.$$

A genus of this equation depends on the values of the parameters and can be $g = 1$ or $g = 0$.

In the first case to integration of the equation on $A(t)$ may be used parametrization by elliptic functions an by rational functions in the second case.

Another way of investigation of the properties of this equation follows from its geometrical interpretation as equation of asymptotical lines on the some surface.

4 Monge equations in the theory of the first order nonlinear p.d.e.'s

The Monge equations

$$\Phi(x, y, z, dx, dy, dz) = 0 \qquad (22)$$

are equations homogeneous with respect to the differentials dx, dy, dz.

They are naturally connected with equations of the form

$$F(x, y, z, z_x, z_y) = F(x, y, z, p, q) = 0 \qquad (23)$$

and can be used for the study of their properties.

To construct the equation (22) for the equation (23) it is necessary to eliminate the variables p and q from the system of equations

$$F(x, y, z, p, q) = 0, \quad \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{pP + qQ},$$

where

$$P = \frac{\partial F(x, y, z, p, q)}{\partial q}, \quad Q = \frac{\partial F(x, y, z, p, q)}{\partial q}.$$

Let us consider some example.

For the equation

$$F(x, y, z, p, q) = p^2 + qy^2 - 1 \qquad (24)$$

the system

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{pP + qQ},$$

leads to

$$dx\, y^2 - 2\, dy\, p = 0, \quad 2\, dy\, p^2 + dy\, qy^2 - dz\, y^2 = 0. \qquad (25)$$

After the elimination of the variables p and q from the equations (24-25) we find the corresponding Monge equation

$$-\left(\frac{d}{ds}x(s)\right)^2 (y(s))^4 + 4\left(\frac{d}{ds}z(s)\right)(y(s))^2 \frac{d}{ds}y(s) - 4\left(\frac{d}{ds}y(s)\right)^2 = 0. \qquad (26)$$

Its general solution has the form

$$z(s) = -s\frac{d}{ds}B(s) + 1/2\, s^2 \frac{d^2}{ds^2}B(s) + 1/2\, \frac{d^2}{ds^2}B(s) + B(s),$$

$$y(s) = -2\left(\frac{d^2}{ds^2}B(s)\right)^{-1}, \; x(s) = -\frac{d}{ds}B(s) + s\frac{d^2}{ds^2}B(s)$$

where $B(s)$ is an arbitrary function and it can be obtained with the help of complete integral of equation (24)

$$z(x, y) = Ax + \frac{(A^2 - 1)}{y} + B,$$

where A and B are parameters.

5 From Monge equations to the Riemann geometry

Riemann space with the metric

$$ds^2 = g_{ij} dx^i dx^j \tag{27}$$

has geodesic satisfying the system of equations

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0,$$

where Γ^i_{jk} are the Christoffel symbols of the metrics (27).

These equations have the first integral of the form

$$g_{ij} \frac{dx^j}{ds} \frac{dx^k}{ds} = \mu$$

or

$$g_{ij} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$

by the condition $\mu = 0$.

In the three-dimensional case such integral has the form of the Monge equation

$$g_{11}(x,y,z)\left(\frac{dx}{ds}\right)^2 + 2g_{12}(x,y,z)\frac{dx}{ds}\frac{dy}{ds} + g_{22}(x,y,z)\left(\frac{dy}{ds}\right)^2 + 2g_{13}(x,y,z)\frac{dx}{ds}\frac{dz}{ds} + \\ + 2g_{23}(x,y,z)\frac{dy}{ds}\frac{dz}{ds} + g_{33}(x,y,z)\left(\frac{dz}{ds}\right)^2 = 0. \tag{28}$$

and can be considered as quadratic first integral of null-geodesic of some three-dimensional space endowed with the metric

$$ds^2 = g_{11}(x,y,z)dx^2 + 2g_{12}(x,y,z)dxdy + g_{22}(x,y,z)dy^2 + 2g_{13}(x,y,z)dxdz + \\ + 2g_{23}(x,y,z)dydz + g_{33}(x,y,z)dz^2.$$

From this point of view the methods of Riemann geometry can be used for the investigation of the properties of Monge equations and the corresponding first order nonlinear p.d.e.

In particular, scalar invariants and the theory of surfaces of such type of spaces can be used with this aim.

We consider two-dimensional surfaces of the translation $y^\nu = [x(u,v),\ y(u,v),\ z(u,v)]$ in a three-dimensional Riemann space which corresponds to the Monge equation of the form (26) as it was described above.

The equations for determination of translation surfaces have the form

$$\frac{\partial y^\alpha}{\partial u \partial v} + \Gamma^\alpha_{\beta\gamma} \frac{\partial y^\beta}{\partial u} \frac{\partial y^\gamma}{\partial v} = 0. \tag{29}$$

Let us consider some examples.

For the equation (24) the Monge equation is (26).

The metric of corresponding Riemann space is

$$ds^2 = -y^4\,dx^2 - 4\,dy^2 + 4\,y^2 dy\,dz. \tag{30}$$

The geodesic equations of this space are of the form

$$\left(\frac{d^2}{ds^2}x(s)\right)y(s) + 4\,\left(\frac{d}{ds}y(s)\right)\frac{d}{ds}x(s) = 0,$$

$$\left(\frac{d^2}{ds^2}y(s)\right)y(s) + 2\,\left(\frac{d}{ds}y(s)\right)^2 = 0,$$

$$\left(\frac{d^2}{ds^2}z(s)\right)(y(s))^3 + (y(s))^4\left(\frac{d}{ds}x(s)\right)^2 + 4\,\left(\frac{d}{ds}y(s)\right)^2 = 0.$$

Taking in consideration the equation (26) we get the solutions of geodesic equations

$$x(s) = \_C3 + \frac{\_C4}{\sqrt[3]{3\,\_C1\,s + 3\,\_C2}}, \quad y(s) = \sqrt[3]{3\,\_C1\,s + 3\,\_C2},$$

and

$$z(s) = -1/4\,\frac{4 + \_C4^{\,2}}{\sqrt[3]{3\,\_C1\,s + 3\,\_C2}} + \_C6.$$

The equations (29) for the surfaces of translation of the space with metric (30) looks as

$$\left(\frac{\partial^2}{\partial u\partial v}x(u,v)\right)y(u,v) + 2\,\left(\frac{\partial}{\partial u}x(u,v)\right)\frac{\partial}{\partial v}y(u,v) + 2\,\left(\frac{\partial}{\partial u}y(u,v)\right)\frac{\partial}{\partial v}x(u,v) = 0,$$

$$\left(\frac{\partial^2}{\partial u\partial v}y(u,v)\right)y(u,v) + 2\,\left(\frac{\partial}{\partial u}y(u,v)\right)\frac{\partial}{\partial v}y(u,v) = 0,$$

$$\left(\frac{\partial^2}{\partial u\partial v}z(u,v)\right)(y(u,v))^3 + (y(u,v))^4\left(\frac{\partial}{\partial u}x(u,v)\right)\frac{\partial}{\partial v}x(u,v) +$$

$$+4\,\left(\frac{\partial}{\partial u}y(u,v)\right)\frac{\partial}{\partial v}y(u,v) = 0.$$

They can be integrated without problems.

The simplest solution is

$$x(u,v) = u - v, \quad y(u,v) = \sqrt[3]{u+v},$$

$$z(u,v) = \_F2(u) + \_F1(v) + \frac{9}{28}\,(u+v)^{7/3} - \frac{1}{\sqrt[3]{u+v}}.$$

In particular case $\_F2(u) = u$, $\_F1(v) = 0$ with the help of relations

$$u = 1/2\,x + 1/2\,y^3, \quad v = -1/2\,x + 1/2\,y^3$$

we get the surface with equation

$$z(x,y) = \frac{9}{28}\,y^7 + 1/2\,y^3 + 1/2\,x - y^{-1}.$$

6 On the surfaces connected with the Rössler system

Let us consider examples of surfaces corresponding to the Rössler system of equations (3)

One of them has the form

$$\left(\frac{\partial}{\partial y}x(y,z)\right)(x(y,z)+ay)+\left(\frac{\partial}{\partial z}x(y,z)\right)(b+(x(y,z)-c)\,z)+y+z. \qquad (31)$$

In the case $a=0,\ b=0,\ c=0$ we find the solution

$$x(y,z)=\sqrt{-y^2-2\,z+\_F1(\frac{z}{e^y})}.$$

where $\_F1(\frac{z}{e^y})$ is an arbitrary function.

To construct the Monge equation corresponding the equation (31) we present its in a new notations as

$$\left(\frac{\partial}{\partial u}y(u,v)\right)(y(u,v)+au)+\left(\frac{\partial}{\partial v}y(u,v)\right)(b+(y(u,v)-c)\,v)+u+v=0.$$

After the change of variables

$$\frac{\partial}{\partial u}y(u,v)=\frac{\frac{\partial}{\partial x}\omega(x,v)}{\frac{\partial}{\partial x}\lambda(x,v)},\quad \frac{\partial}{\partial v}y(u,v)=\frac{\partial}{\partial v}\omega(x,v)-\frac{\left(\frac{\partial}{\partial x}\omega(x,v)\right)\frac{\partial}{\partial v}\lambda(x,v)}{\frac{\partial}{\partial x}\lambda(x,v)},$$

$$y(u,v)=\omega(x,v),\quad u=\lambda(x,v)$$

with conditions

$$\lambda(x,v)=x\frac{\partial}{\partial x}\rho(x,v)-\rho(x,v)\omega(x,v)=\frac{\partial}{\partial x}\rho(x,v)$$

one gets the equation equivalent to (31)

$$\left(1+ax+x^2+v\frac{\partial}{\partial v}\rho(x,v)\right)\frac{\partial}{\partial x}\rho(x,v)+(-cv+b)\frac{\partial}{\partial v}\rho(x,v)-x\rho(x,v)+$$
$$+vx-a\rho(x,v)=0.$$

For this p.d.e. the Monge equation is defined by the condition

$$dz^2y^2+\left(\left(2\,yx^2+2\,yax+2\,y\right)dy+\left(2\,by-2\,cy^2\right)dx\right)dz+$$
$$+\left(2\,x^2+2\,ax+x^4+1+2\,ax^3+a^2x^2\right)dy^2+(cy-b)^2\,dx^2$$
$$+\left(2\,axcy-2\,b-2\,axb+4\,xy^2+2\,x^2cy-2\,x^2b-4\,xzy-4\,azy+2\,cy\right)dx\,dy=0$$

and it may be considered as the first integral of geodesic of the corresponding three-dimensional Riemann space.

Another aproach to the study of the Rössler system gives us the investigation of the integral manifolds of corresponding Pfaff equation

$$-(y+z)dx+(x+ay)dy+(b+(x-c)zdz=0.$$

It is reduced to the form

$$-(y+z)dx+(x+ay)dy+(b+(x-c)zdz=dU+VdW$$

and so determines one-dimensional integral manifolds.

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