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Attractors in affine differential systems with impulsive control *

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Abstract. In this paper we prove that an asymptotic equilibrium of an affine system of differential equations can become a strange attractor under affine impulsive control. The linear oscillator is studied as example.

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1 Introduction

We are concerned with the system of affine differential equations

$$\dot{x} = Ax + b \quad (x \in \mathbb{R}^m), \tag{1}$$

where A is a nonsingular matrix.

Suppose that at the moments $t = n \in \mathbb{N}^*$ instantaneous control actions occur, which change the state of system as follows:

$$\Delta x\big|_{t=n} := x(n+0) - x(n-0) = C_{i_n} x(n-0) + d_{i_n} \quad (n \in \mathbb{N}^*), \tag{2}$$

where the matrices C_{i_n} and vectors d_{i_n} belong to given sets (finite or infinite).

Between any two consecutive kicks the motion of the system obeys (1). At the moment $t = n$ the elements C_{i_n} and d_{i_n}, which determine the jump by (2), are chosen, say randomly. For convenience, we will consider that all solutions of system (1)–(2) are right continuous at the moments $t \in \mathbb{N}$.

Let $\mathcal{F}$ be the set of all affine maps $\{F_{i_n} : x \mapsto C_{i_n} x + d_{i_n}\}_{n \in \mathbb{N}^*}$. For simplicity, we assume that the set $\mathcal{F}$ is finite and contains only r distinct elements, say $\mathcal{F} = \{F_1, F_2, \dots, F_r\}$. Denote $\tilde{F_n} = E + F_n$ $(1 \le n \le r)$, where E is the identity operator.

Assume that the spectra of the operators $(E + C_n)e^A$ $(1 \le n \le r)$ are located strictly inside of the unit circle. Sometimes, if necessary, it is required that all operators $E + C_n$ $(1 \le n \le r)$ are invertible.

It is known (see, e.g., [1]) that if the sequence $\{F_{i_n}\}_{n \in \mathbb{N}^*}$ is periodic, then the behavior of the system is quite simple: there exists a globally attracting periodic cycle, corresponding to a periodic motion. This situation may, however, be changed essentially in the general case.

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In what follows we show that there exists a global attractor in the extended phase space and for "typical" sequence $\{F_{i_n}\}_{n\in\mathbb{N}^*}$ this invariant set has a non-integer Hausdorff dimension (see, e.g., [2]) and represents a fractal. Moreover, the motions on the attractor are chaotic by Li-Yorke (in the meaning that every trajectory on the attractor is dense) and, as a consequence, every solution of the impulsive system (1)–(2) tends to a chaotic one.

This paper represents an application of general results from [3, 4].

2 Invariant sets

If one denotes by $x(n)$ the state of the system immediately after the n-th kick, then by a straightforward calculation we end up with a sequence of affine maps $\Phi_{i_n} : x(n) \mapsto x(n+1)$, $n \in \mathbb{N}^*$. More precisely,

$$\Phi_{i_n} : x \mapsto (E + C_{i_{n+1}})e^A x + (E + C_{i_{n+1}})(e^A - E)A^{-1}b + d_{i_{n+1}}.$$

We call the sequence $\{\Phi_{i_n}\}_{n\in\mathbb{N}}$ the *Poincaré system* associated with the impulsive system (1)–(2). Under the above assumptions, this Poincaré system is generated by r affine maps $\{\Phi_1, \Phi_2, \ldots, \Phi_r\}$.

Let $\mathcal{P}_{cp}(\mathbb{R}^m)$ be the set of all nonempty compact subsets of $\mathbb{R}^m$, endowed with the Hausdorff-Pompeiu metrics (see, e.g.,[2]). Denote by Φ the *Nadler-Hutchinson* operator [6] on the space $\mathcal{P}_{cp}(\mathbb{R}^m)$, defined by $\Phi(M) := \bigcup_{n=1}^{r} \Phi_n[M]$, $M \in \mathcal{P}_{cp}(\mathbb{R}^m)$.

Let $\psi(\cdot, \tau, x_0)$ stand for the solution of the system (1)–(2) with the initial condition $x(\tau) = x_0$. Since the system (1)–(2) is affine, there is a unique such solution, defined on $\mathbb{R}$ (see, e.g., [1]).

Lemma 1. *For every $x \in \mathbb{R}^m$ and $t_1, t_2, t_3 \in \mathbb{R}$ one has*

$$\psi(t_1, t_2, \psi(t_2, t_3, x)) = \psi(t_1, t_3, x). \tag{3}$$

Proof. This follows immediately from the uniqueness of the solution of respective Cauchy problem. □

Lemma 2. *Every solution of the system (1)–(2) can be written as follows:*

$$\psi(t,\tau,x) = e^{(t-\tau)A}x, \quad \text{if } [t] = [\tau], \quad \text{or}$$

$$\psi(t,\tau,x) = e^{(t-[t])A}\tilde{F}_{i_{[t]}}e^{([t]-\tau)A}x, \quad \text{if } [t] = [\tau]+1, \quad \text{or}$$

$$\psi(t,\tau,x) = e^{(t-[t])A}\left(\prod_{j=[\tau]+2}^{[t]} \Phi_{i_j}\right)\tilde{F}_{i_{[\tau]+1}}e^{([\tau]+1-\tau)A}x, \quad \text{if } [t] > [\tau]+1, \quad \text{or}$$

$$\psi(t,\tau,x) = e^{(t-[\tau])A}\tilde{F}^{-1}_{i_{[\tau]}}e^{([\tau]-\tau)A}x, \quad \text{if } [t] = [\tau]-1, \quad \text{or}$$

$$\psi(t,\tau,x) = e^{(t-[t]-1)A}\tilde{F}^{-1}_{i_{[t]+1}}\left(\prod_{j=[\tau]}^{[t]+2} \Phi^{-1}_{i_j}\right)e^{([\tau]-\tau)A}x, \quad \text{if } [t] < [\tau]-1,$$

where $[\cdot]$ denotes the integral part.

Proof. The proof is straightforward. □

Remark 1. In fact, the impulsive system (1)–(2) is nonautonomous and we can consider only its integral curves. Even in the case of periodic sequence $\{F_{i_n}\}_{n\in\mathbb{N}^*}$ we cannot factorize the system to obtain a (autonomous) system on the direct product $S^1 \times \mathbb{R}^m$. However, since the impulse actions occur at the moments $t \in \mathbb{N}$, we can obtain a foliation on the cylinder $S^1 \times \mathbb{R}^m$ by factorization on time. This foliation consists of pieces of integral curves of the system (1)–(2).

Project the system (1)-(2) to the cylinder $S^1 \times \mathbb{R}^m$, using the projection $\pi : (t, y) \mapsto (t \pmod 1, y)$.

We will say that a set $V \subset S^1 \times \mathbb{R}^m$ is *positive invariant* (*invariant*) with respect to the system (1)-(2), if for every point $(\tau, x) \in V$ and every natural k one has

$$(t \pmod 1, \psi(t, \tau + k, x)) \in V \quad \text{for} \quad t \geq \tau + k \quad (\text{for} \quad t \in \mathbb{R}). \tag{4}$$

In other words, V consists of pieces of integral curves.

By definition, such a set $V \subset S^1 \times \mathbb{R}^m$ covers the whole base S^1 by projection. Denote by $(t, V_t) := \{(t, x) \in S^1 \times \mathbb{R}^m \mid (t, x) \in V\}$ the *fiber* over the point $t \in [0, 1)$. For convenience, we will identify this fiber with V_t. Moreover, in the sequel the notation V_t for $t \in \mathbb{R}$ will mean $V_{t \pmod 1}$.

Theorem 3. *The set $V \subset S^1 \times \mathbb{R}^m$ is positive invariant (invariant) with respect to the system (1)-(2) if and only if it satisfies the following conditions:*

1. $e^{(t-\tau)A}V_\tau \subset V_t$ *for* $0 \leq \tau \leq t < 1$ $\quad (e^{(t-\tau)A}V_\tau = V_t$ *for* $\tau, t \in [0, 1))$*;*
2. $\bigcup_{n=1}^{r} \tilde{F}_n e^{(1-\tau)A} V_\tau \subset V_0 \quad (\bigcup_{n=1}^{r} \tilde{F}_n e^{(1-\tau)A} V_\tau = V_0)$ *for* $0 \leq \tau < 1$.

Proof. Necessity. Assume that the set $V \subset S^1 \times \mathbb{R}^m$ is positive invariant (invariant). If $x \in V_\tau$, then $(\tau, x) \in V$, and by Lemma 2 one has for $0 \leq \tau \leq t < 1$

$$e^{(t-\tau)A}x = \psi(t, \tau, x) \in V_t. \tag{5}$$

At the same time, for every natural n there is a natural k_n such that

$$\tilde{F}_n e^{(1-\tau)A}x = \psi([\tau] + k_n + 1, \tau + k_n, x) \in V_{[t]+k_n+1} = V_0.$$

In the case of invariance the relation (5) holds for $\tau, t \in [0, 1)$. Moreover, for every $z \in V_t$ there is $y = \psi(\tau, t, z) \in V_\tau$ which verifies $e^{(t-\tau)A}y = z$.

Analogously, for every $0 \leq \tau < 1$, $z \in V_0$ and $1 \leq n \leq r$ there is a natural k_n such that $\tilde{F}_{k_n} = \tilde{F}_n$ and there is $y = \psi(\tau + k_n - 1, k_n, z) \in V_{\tau+k_n-1} = V_\tau$, verifying $\tilde{F}_n e^{(1-\tau)A}y = \psi(k_n, \tau + k_n - 1, y) = z \in V_0$.

Sufficiency. Assume that conditions 1)–2) hold. Let $(\tau, x) \in V$. To proof (5) take firstly $t \geq \tau + k$, $k \in \mathbb{N}^*$.

If $[t] = [\tau] + k$ or $[t] = [\tau] + k + 1$, then (4) is a consequence of the conditions 1)–2) and Lemma 2.

If $[t] = q, [\tau] + k = p$, $q > p + 1$, then by Lemmas 1 and 2 the conditions 1)–2) imply for every $x \in V_\tau$:

$$\psi(t, \tau + k, x) = e^{(t-q)A} \prod_{j=p+2}^{q} \Phi_{i_j} \tilde{F}_{i_{p+1}} e^{(p+1-\tau-k)A} x \in V_t,$$

$$\psi(t, \tau + k, x) = \psi(t, q, \psi(q, p + 1, \psi(p + 1, \tau + k, x))) \in V_t.$$

In the case of invariance we consider in addition $t < [\tau] + k$. In this case there is $y \in V_0$ such that $x \in V_\tau = V_{\tau+k}$ may be represented as $x = \psi(\tau + k, [\tau] + k, y)$. In turn, there is $z \in V_{[t]} = V_0$ such that $y = \psi([\tau] + k, [t], z)$. By Lemma 1, $\psi(t, \tau + k, x) = \psi(t, [\tau] + k, y) = \psi(t, [t], z) \in V_t$. This completes the proof. □

Corollary 4. *If $V \subset S^1 \times \mathbb{R}^m$ is positive invariant (invariant), then V_0 is positive invariant (invariant) with respect to the Nadler-Hutchinson operator Φ, i.e. $\Phi(V_0) = \bigcup_{n=1}^{r} \Phi_n[V_0] \subset V_0 \quad (\Phi(V_0) = V_0)$.*

3 IFS and attractors

Since the eigenvalues of all matrices $(E + C_n)e^A$ $(1 \le n \le r)$ are located strictly inside the unit circle, all operators Φ_n $(1 \le n \le r)$ are contracting.

We associate to the system (1)-(2) a hyperbolic *Iterated Function System* (IFS) $\{\mathbb{R}^m; \Phi_1, \Phi_2, \ldots, \Phi_r\}$ (see, e.g., [2]), consisting of affine contractions. This IFS determines in $\mathbb{R}^m$ a global compact attractor K, which is the unique fixed point of the corresponding contractive Nadler-Hutchinson operator Φ.

Given the natural k we say that the sequence $\{F_{i_n}\}_{n\in\mathbb{N}^*}$ is *k-universal* if it contains every word of the length k from the alphabet of $\mathcal{F} = \{F_1, F_2, \ldots, F_r\}$. We say that the sequence $\{F_{i_n}\}_{n\in\mathbb{N}^*}$ is *universal* if it is *k-universal* for every natural k.

By Lemma Borel-Cantelli [5], if the sequence $\{F_{i_n}\}_{n\in\mathbb{N}^*}$ is chosen randomly with a uniform distribution, then with probability 1 it is universal.

If the sequence $\{F_{i_n}\}_{n\in\mathbb{N}}$ is universal, then the orbit of each point in K is dense on K (is chaotic by Li-Yorke).

Recall some notions (see, e.g., [2]). A set is called *totally disconnected* if for every its point the connected component, containing this point, is the point itself. A set S is called *perfect* if it is closed and every point $p \in S$ is the limit of points $q_n \in S \setminus \{p\}$. A set is called a *Cantor set* if it is totally disconnected, perfect and compact.

Theorem 5. *If the spectra of the operators $(E + C_n)e^A$ $(1 \le n \le r)$ are located strictly inside the disk of radius $\frac{1}{r}$, then the attractor K is totally disconnected.*

Proof. It is known [7] that if an IFS consists of r contractions, each of them with the contraction coefficient s, and $rs < 1$, then the attractor K of this IFS is totally disconnected. It is sufficient to say that in our case $s = \max_{1\le n\le r} \|(E + C_n)e^A\| < \frac{1}{r}$. □

Remark 2. The hypotheses of Theorem 5 are far from being also necessary conditions for the attractor K to be totally disconnected.

Denote the distance from the point $x \in \mathbb{R}^m$ to the compact $M \subset \mathbb{R}^m$ by $\varrho(x, M) := \min\{d(x, y) \,|\, y \in M\}$.

A bounded subset $V \subset S^1 \times \mathbb{R}^m$ is called an *attractor* of the system (1)-(2) if it is positive invariant and for every solution $\psi(\cdot, \tau, x)$ one has $\varrho(\psi(t, \tau, x), V_t) \to 0$ as $t \to +\infty$.

Theorem 6. *There exists an attractor of the system (1)-(2)*

$$K^* = \{(t, e^{tA}x + (e^A - E)A^{-1}b) \,|\, t \in [0, 1), x \in K\} \subset S^1 \times \mathbb{R}^m$$

with the Hausdorff dimension $DH(K^*)$, verifying the inequalities:

$$1 < DH(K^*) \le 1 - \frac{\ln r}{\ln s}, \tag{6}$$

where s is the smallest radius of a disc centered at the origin of coordinates, which contains the spectra of the operators $(E + C_n)e^A$ $(1 \le n \le r)$.

Proof. By Theorem 3, the set K^* is positive invariant. Since K is compact, every fiber $K_t = e^{tA}K + (e^A - E)A^{-1}b$ $(0 \le t < 1)$ is compact as well. The compact K attracts every compact M in the fiber $0 \times \mathbb{R}^m$ under the actions of the Nadler-Hutchinson operator Φ. As a result every fiber $K_t = e^{\nu A}K + (e^A - E)A^{-1}b$, where $\nu = t \pmod 1$, attracts the image $e^{(t-[t])A}\Phi^n e^{([t]+1-t)A}M$ as $n \to +\infty$ uniformly on $t \in \mathbb{R}$.

It is known (see, e.g.,[2]) that the Hausdorff dimension of K verifies the inequality $DH(K) \le -\frac{\ln r}{\ln s}$. This implies that $DH(K^*) = 1 + DH(K) \le 1 - \frac{\ln r}{\ln s}$. □

Theorem 7. *Let the spectra of the operators $(E + C_n)e^A$ $(1 \le n \le r)$ be located strictly inside the disk of radius $\frac{1}{r}$ and let the sequence $\{F_{i_n}\}_{n\in\mathbb{N}}$ be universal. Then the attractor K^* is homeomorphic to the direct product of a half-open interval and a Cantor set, its Hausdorff dimension verifying the inequalities: $1 < DH(K^*) < 2$.*

Proof. It is easily seen from Theorem 6 that the attractor K^* is homeomorphic to the direct product $[0, 1) \times K$. Under the given hypothesis, the compact K is perfect and by Theorem 5 is totally disconnected, and, as a consequence, it is a Cantor set. In this case the inequalities (6) become: $1 < DH(K^*) < 2$. □

Lemma 8. *There exist $L > 0, \gamma > 0$ such that for any $y \in \mathbb{R}^m$ there exists $x \in K$, satisfying*

$$\|\psi(t, 0, y) - \psi(t, 0, x)\| \le Le^{-\gamma t}\|y - x\| \quad (t \ge 0).$$

Proof. This follows immediately from Lemma 2 and Theorem 6. □

Theorem 9. *If the sequence $\{F_{i_n}\}_{n\in\mathbb{N}}$ is universal, then every integral curve of the system (1)–(2), starting in K^*, is chaotic by Li-Yorke, i.e. is dense in K^*.*

Proof. If the sequence $\{F_{i_n}\}_{n\in\mathbb{N}}$ is universal, then for every solution $\psi(\cdot,\tau,x)$ with $(\tau,x)\in K^*$, the sequence $\{\psi(j,\tau,x)\}_{j\geq[\tau]+1}$ is the orbit of the point $\psi([\tau]+1,\tau,x)\in K$ under the IFS $\{\mathbb{R}^m;\Phi_1,\dots,\Phi_r\}$. Since this orbit is dense on K, the integral curve of the solution $\psi(\cdot,\tau,x)$ is dense on K^*. □

Corollary 10. *If the sequence $\{F_{i_n}\}_{n\in\mathbb{N}}$ is universal, then every integral curve of the system (1)-(2) is chaotic by Li-Yorke or tends to a chaotic one.*

Proof. This follows from Lemma 8 and Theorem 9. □

Remark 3. If the impulses occur only in some integer moments, i.e.

$$\Delta x\big|_{t=\tau_n} := x(\tau_n+0)-x(\tau_n-0)=C_{i_n}x(\tau_n-0)+d_{i_n} \quad (\tau_n\in\mathbb{N}^*), \tag{7}$$

then the system (1),(7) may be considered as a particular case of the system (1)-(2) by supplementing the set $\mathcal{F}$ with the null operator $F=0$ for other integer moments.

Remark 4. Analogously, if the spectra of all operators $(E+C_n)e^A$ $(1\leq n\leq r)$ are located strictly outside the unit circle, we can say about the repeller of the system (1)–(2).

Remark 5. Many classical fractals may be represented as attractors of affine IFS on $\mathbb{R}^2$. Fig. 1 shows some of them as attractors K of impulsive differential equations on $\mathbb{C}$, for example:

- the *Sierpinski triangle* in $\dot z=-z\cdot\ln 2, \Delta z|_{t=n}=i\exp\frac{2\pi ni}{3}$ $(n\in\mathbb{N})$;
- the *pentagasket* in $\dot z=-z\cdot\ln(\frac{3+\sqrt5}{2}), \Delta z|_{t=n}=i\exp\frac{2\pi ni}{5}$ $(n\in\mathbb{N})$;
- the *hexagasket* in $\dot z=-z\cdot\ln 3, \Delta z|_{t=n}=\exp\frac{\pi ni}{3}$ $(n\in\mathbb{N})$.

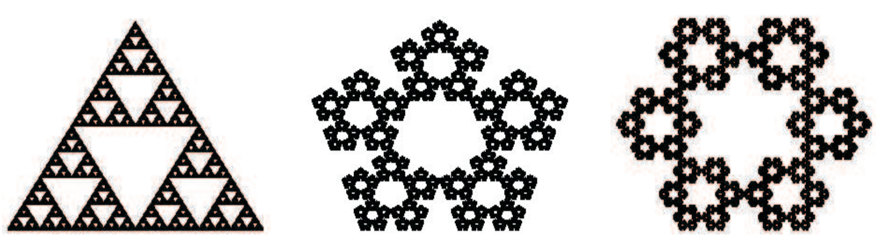

Figure 1. Fractals: the Sierpinski triangle, pentagasket and hexagasket as attractors K of impulsive affine systems

4 Linear oscillator

Let us consider, as an example, the linear oscillator with impulsive actions

$$\ddot{x} + c\dot{x} + kx = 0 \quad (c > 0, k > 0), \tag{8}$$

$$\Delta\dot{x}\Big|_{t=n} := \dot{x}(n+0) - \dot{x}(n-0) = \xi_{i_n} \quad (n \geq 1). \tag{9}$$

Assume that the range of the sequence $\{\xi_{i_n}\}_{n\geq 1}$ contains only r distinct elements.

We can reduce the equations (8)–(9) to an affine system of impulsive differential equations (1)–(2) in the phase space $x_1 = x, x_2 = \dot{x}$. In this case we obtain some analogues of the previous theorems.

Theorem 11. *[8] There exists an attractor $K^* \subset S^1 \times \mathbb{R}^2_{(x,\dot{x})}$ of the system (8)-(9) with the Hausdorff dimension*

$$DH(K^*) = 1 + \frac{2\sqrt{2}\ln r}{\sqrt{2}c - \sqrt{c^2 - 4k + |c^2 - 4k|}}.$$

Theorem 12. *[8] If*

$$2\ln r < c < \frac{k}{\ln r} + \ln r$$

and the sequence $\{\xi_{i_n}\}_{n\in\mathbb{N}}$ is universal, then the attractor K^* is homeomorphic to the direct product of a half-open interval and a Cantor set, its Hausdorff dimension verifying the inequalities: $1 < DH(K^*) < 2$.

Fig. 2 represents the respective attractors $K \subset \mathbb{R}^2_{(x,\dot{x})}$ for distinct values of parameters for two impulsive differential equations (8)–(9): on the left for $c = 5/2$, $k = 2$ and $r = 3$ (K is totally disconnected), on the right for $c = 1$, $k = 5/4$ and $r = 3$ (K is connected).

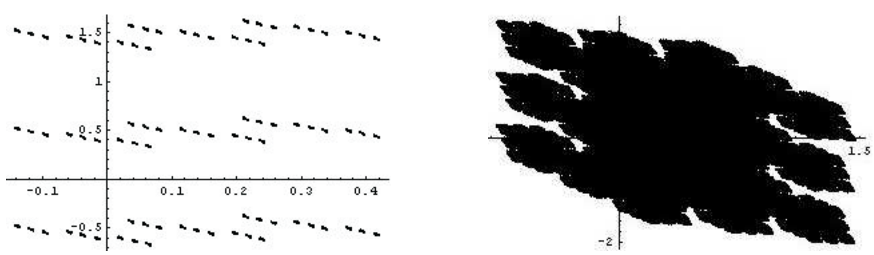

Figure 2. Attractor K for: $c = 5/2$, $k = 2$, $r = 3$ (left) and $c = 1$, $k = 5/4$, $r = 3$ (right)

Remark 6. If the range of sequence $\{\xi_{i_n}\}_{n\geq 1}$ is infinite but bounded, then the system (8)–(9) admits an attractor as well.

Fig. 3 represents the attractor K (of an infinite IFS) for an impulsive differential equation (8)–(9) with $c = 2$, $k = 2$, where the values $\{\xi_{i_n}\}_{n\geq 1}$ are randomly chosen from $[0, 1]$.

All calculations and graphic objects have been done using the Computer Algebra System *Mathematica*.

Figure 3. Attractor K for $c = 2$, $k = 2$, $\xi \in [0, 1]$

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