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n-Homogeneous dynamical systems and n-ary algebras

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Abstract. A bijective correspondence between the classes of center-affinely equivalent n-homogeneous equations ($n \geq 2$) and the classes of isomorphic commutative n-ary algebras is established. It generates a correspondence between the properties of these equations and the structural properties of the associated n-ary algebras.

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1 Introduction

Let $E(\|\cdot\|)$ be a Banach space over the field K (here K is $\mathbb{R}$ or $\mathbb{C}$). A *n-homogeneous dynamical system* comes from a n-homogeneous differential equation (shortly, nHDE) of the form (S) $\dfrac{dX}{dt} = F(X)$, where $F : E \to E$ is a continuous n-monomial vector form. The polar form for F is a symmetric n-linear vector form which allows us to define a n-ary algebra on E; this algebra is commutative and nonassociative (more exactly, it is not necessarily an associative algebra).

The nHDE (S) is said to be *center-affinely equivalent* (CA-equivalent) with another nHDE (S_1) defined on a Banach space E' if and only if there exists an invertible continuous linear mapping $h : E' \to E$ such that $X = h(Y)$ is a solution for (S) as long as Y is a solution for (S_1). Then, the following result holds: (S) *is CA-equivalent with* (S_1) *if and only if their associated* n*-ary algebras are continuously isomorphic.* According to this result, one gets: *there exists a bijection between the set of all classes of CA-equivalent nHDEs and the set of all classes of isomorphic commutative* n*-ary* K*-algebras.* It means that a classification up to an isomorphism of commutative n-ary K-algebras gives the classification up to a CA-equivalence of all nHDEs.

Actually, the structure of the associated algebra allows us to determine some features of the analyzed nHDE as well as of its space of solutions. As examples, we quote the following results:

1. semisimple algebras give a decoupling of the initial equation into equations occurring in simple algebras,
2. solvable algebras give solutions via a subset of linear differential equations,
3. the n-degree nilpotents N (i.e., with $N^n = [N, N, ..., N] = 0$) are *steady-state points* or *equilibria*, i.e., they are the *constant* solutions,

4. the n-degree idempotents e (i.e., with $e^n = [e, e, ..., e] = e$) give the *ray* solutions,

5. the origin is never asymptotically stable and the existence of an idempotent implies that the origin is actually an instable steady state.

Recall that the automorphisms of the associated algebra keep invariant all equilibria, periodic orbits, and domains of attraction.

Some quantitative results can be obtained, too. Firstly, it must be remarked that, F being an analytic function then the solution of every CAUCHY problem for (S) is an analytical one. Besides, if the n-ary algebra associated with the nHDE (S) is power-associative, then there exists a formula giving the solution of every CAUCHY problem for (S).

Several new results can be obtained in the particular case of the nHDEs defined on finite dimensional spaces. This time, any nHDE becomes really an n-homogeneous differential system of equations (shortly, nHSDE).

NOTE. We have preferred to work in a BANACH space not only to generalize some known results but, mainly, because in this frame a good understanding of the facts is necessary (facts which - in the finite dimensional case - are hidden behind of some bushy computations).

2 Preliminaries

Polynomial mappings. Throughout this paper the following notations will be used:

E – a BANACH space,

$C_n(E)$ – the space of all continuous n-homogeneous functions from E to E,

$L(E^n, E)$ – the BANACH space of all continuous n-multilinear forms from E^n to E (endowed with the norm induced by the one of E),

$L_s(E^n, E) \subset L(E^n, E)$ – the BANACH space of all continuous n-multilinear symmetric forms from E^n to E,

$\Delta : E \to E^n$ – the *n-diagonal mapping* on E (i.e., $\Delta(x) = (\underbrace{x, x, ..., x}_{n\,times})$, $\forall x \in E$).

The mapping $\mathcal{P} : L_s(E^n, E) \to C_n(E)$ defined by

$$\mathcal{P}(G) = G \circ \Delta, \quad \forall G \in L_s(E^n, E)$$

is the so-called *polynomial projection*, while $\mathcal{P}(G)$ is called the *n-homogeneous polynomial* associated with G (or a *monomial* of degreee n, or a *n-monomial*). Any (finite) linear combination of monomials on E is called a *polynomial*; the *degree of a polynomial* is the maximum among the degrees of its monomial components.

Actually, the polynomial projection $\mathcal{P}$ is a bijection between $L_s(E^n, E)$ and the space $\mathcal{P}_n(E)$ of all n-monomials on E. Indeed, for any $F \in \mathcal{P}_n(E)$, the mapping

$G \in L_s(E^n, E)$ defined by

$$G(x_1, x_2, ..., x_n) = \frac{1}{n!}\left[F\left(\sum_{i=1}^{n} x_i\right) - \sum_{j=1}^{n} F\left(\sum_{\substack{i=1\\ i\neq j}}^{n} x_i\right) + \right.$$
$$\left. + \sum_{\substack{j,k=1\\ j<k}}^{n} F\left(\sum_{\substack{i=1\\ i\notin\{j,k\}}}^{n} x_i\right) + ... + (-1)^n \sum_{i=1}^{n} F(x_i)\right], \quad \forall x_1, x_2, ..., x_n \in E$$

satisfies $F = G \circ \Delta$; G is called the *polar form* of F.

***n*-ary Algebras.** Let E be a K-vector space (finite dimensional or not).

Definition 1. *A n-ary algebra is any K-vector space E endowed with a n-multilinear vector mapping $[., ..., .] : E^n \to E$*

$$(x_1, x_2, ..., x_n) \to [x_1, x_2, ..., x_n], \quad \forall (x_1, x_2, ..., x_n) \in E^n.$$

We denote it by $E([., ..., .])$.

In this case, the mapping $G : E^n \to E$ defined by

$$G(x_1, x_2, ..., x_n) = [x_1, x_2, ..., x_n], \quad \forall x_1, x_2, ..., x_n \in E$$

is a $(1, n)$-tensor, i.e., $G \in E^{*\otimes n} \otimes_K E$. Actually, the set of all n-ary K-algebras on E is identifiable with the tensor product $E^{*\otimes n} \otimes_K E$.

Recall that, for any algebra $E([., ..., .])$ and any $x_1, x_2, ..., x_{n-1} \in E$ we can define the i-multiplication $M^i_{x_1,x_2,...,x_{n-1}} : E \to E$ $(i \in \{1, 2, ..., n\})$ by

$$x \to M^i_{x_1,x_2,...,x_{n-1}}(x) = [x_1, x_2, ..., x_{i-1}, \underset{i}{x}, x_i, ..., x_{n-1}], \quad \forall x \in E.$$

$L_{x_1,x_2,...,x_{n-1}} = M^n_{x_1,x_2,...,x_{n-1}}$ is called the *left multiplication* by $(x_1, x_2, ..., x_{n-1})$, and $R_{x_1,x_2,...,x_{n-1}} = M^1_{x_1,x_2,...,x_{n-1}}$ is called the *right multiplication* by $(x_1, x_2, ..., x_{n-1})$.

The algebra $E([., ..., .])$ is said to be *associative* if

$$[[x_1, x_2, ..., x_n], y_2, ..., y_n] = [x_1, [x_2, ..., x_n, y_2], y_3, ..., y_n] = ... =$$
$$= [x_1, x_2, ..., x_{n-1}, [x_n, y_2, ..., y_n]], \quad \forall x_1, x_2, ..., x_n, y_2, ..., y_n \in E.$$

It results that $E([., ..., .])$ is associative if and only if the following equations hold

$$L_{x_1,x_2,...,x_{n-1}} \circ L_{y_1,...,y_{n-1}} = L_{[x_1,x_2,...,x_{n-1},y_1],y_2,y_3,...,y_{n-1}} =$$
$$= L_{x_1,[x_2,...,x_{n-1},y_1,y_2],y_3,...,y_{n-1}} = ... = L_{x_1,x_2,...,x_{n-2},[x_{n-1},y_1,y_2,...,y_{n-1}]},$$
$$\forall x_1, x_2, ..., x_n, y_1, ..., y_{n-1} \in E.$$

The associativity of a n-ary algebra can be similarly characterized by means of the right multiplications.

The n-ary algebra $E([.,...,.])$ is said to be *commutative* or *symmetric* if

$$[x_1, x_2, ..., x_n] = [x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}], \quad \forall x_1, x_2, ..., x_n \in E, \ \forall \sigma \in S_n$$

(here S_n denotes the symmetric group of n elements).

$e \in E$ is said to be an *identity element* for $E([.,...,.])$ if

$$[e, e, ..., e, x] = [e, ..., e, x, e] = [e, x, e, ..., e] = [x, e, ..., e] = x, \ \forall x \in E.$$

An identity element for $E([.,...,.])$, if it exists, is not necessarily unique (in contrast with the case of binary algebras).

Any n-ary algebra $E([.,...,.])$ having no identity element can be naturally embedded into a n-ary algebra with identity element, namely $\overline{E} = E \oplus K$ endowed with n-ary composition

$$[x_1 \oplus \lambda_1, x_2 \oplus \lambda_2, ..., x_n \oplus \lambda_n] = ([x_1, x_2, ..., x_n] + \lambda_2 ... \lambda_n x_1 + ... + \\ + \lambda_1 ... \lambda_{n-1} x_n, \lambda_1 \lambda_2 ... \lambda_n), \quad \forall x_1 \oplus \lambda_1, x_2 \oplus \lambda_2, ..., x_n \oplus \lambda_n \in \overline{E};$$

obviously, $0 \oplus 1$ is an identity element for $\overline{E}$.

$e \in E \setminus 0$ is said to be an *idempotent element* for $E([.,...,.])$ if $[e, e, ..., e] = e$.

$e \in E \setminus 0$ is said to be a *nilpotent element* for $E([.,...,.])$ if $[e, e, ..., e] = 0$.

For any fixed $x \in E$ one considers the left/right powers defined recurrently by:
left powers: $x^n = [x, x, ..., x], \ \ x^{n+k(n-1)} = [x^{n+(k-1)(n-1)}, x, ..., x], n \geq 2,$
right powers: $x^{[n]} = [x, x, ..., x], \ \ x^{[n+k(n-1)]} = [x, ..., x, x^{[n+(k-1)(n-1)]}], n \geq 2.$

In a commutative algebra, left and right powers of any element are coincident.

$E([.,...,.])$ is named *power-associative* or *mono-associative* if

$$[x^{n+m_1(n-1)}, x^{n+m_2(n-1)}, ..., x^{n+m_n(n-1)}] = x^{n+(m_1+m_2+...+m_n+n)(n-1)}, \\ \forall m_1, m_2, ..., m_n \in \mathbb{N}^*, \ \forall x \in E.$$

$E([.,...,.])$ is power-associative if and only if

$$L^k_{\underbrace{x, x, ..., x}_{n-1\, times}}(x) = L_{x^{n+(k-2)(n-1)}, x, ..., x}(x) = L_{x, x^{n+(k-2)(n-1)}, ..., x}(x) = ... = \\ = L_{x, ..., x, x^{n+(k-2)(n-1)}}(x), \quad \forall x \in E, \ k \geq 2.$$

In any power-associative n-ary algebra both left and right powers of any element are coincident.

3 Polynomial equations

Definition 2. *a) A n-polynomial differential equation ($nPoDE$) on E is every differential equation of the form*

$$\frac{dX}{dt} = P_n(X) \tag{1}$$

where P_n is a polynomial of degree n and $X : U \to E$, with U an interval of $\mathbb{R}$, is the unknown vector function.

b) A n-homogeneous differential equation (nHDE) on E is every differential equation of the form

$$\frac{dX}{dt} = F(X) \tag{2}$$

where F is a n-degree monomial on E.

Every polynomial P_n on E can be considered as the restriction on $E(\equiv E \times 1)$ of a n-homogeneous polynomial P_n^h defined on $E \times K$ by

$$P_n^h(X, Y) = Y^n P_n(X/Y)$$

(here Y has nonzero values, only). Thus, (1) can be always transformed in a n-homogeneous equation on $E \times K$ namely

$$\begin{cases} \dfrac{dX}{dt} = P_n^h(X, Y), \\ \dfrac{dY}{dt} = 0, \end{cases} \tag{3}$$

for which the only solutions of interest will be the ones satisfying the condition $Y(t_0) = 1$. Consequently, the study of any nPoDEs can be reduced to that of nHDEs, i.e. it is enough to study the nHDEs.

Let us consider the n-homogeneous equation

$$\frac{dY}{dt} = F_1(Y) \tag{4}$$

on the Banach space E'.

Definition 3. *It is said that (2) is (center-) affinely equivalent (shortly, CA-equivalent) with (4) if there exists a continuous invertible linear mapping $h : E' \to E$ such that $X = h(Y)$ is a solution for (2) as long as Y is a solution for (4); h is called a CA-equivalence. A CA-equivalence of (2) with itself is called an automorphism for (2).*

Theorem 1. *The nHDE (2) is CA-equivalent with (4) if and only if there exists a continuously invertible linear mapping $h : E' \to E$ such that*

$$h \circ F_1 = F \circ h. \tag{5}$$

Proof. Let $y_o \in E'$ be an arbitrarily chosen (but fixed) element and $Y(t)$ be the solution of the Cauchy problem

$$\frac{dY}{dt} = F_1(Y), \qquad Y(t_0) = y_0. \tag{6}$$

Since (2) is equivalent to (4) there exists an invertible continuous linear transformation $h : E' \to E$ such that $X(t) = h(Y(t))$ is the solution of (2) with the initial condition $x_o = h(y_o)$. Consequently, the following equations hold:

$$\begin{aligned} &\frac{dX(t)}{dt} = F(X(t)) = \\ &= F(h(Y(t))) = h\left(\frac{dY(t)}{dt}\right) = h(F_1(Y(t)), \ \ \forall t \in I_1, \end{aligned} \tag{7}$$

i.e., $(F \circ h)(y_o) = (h \circ F_1)(y_o)$ (here $I_1 \subset \mathbb{R}$ is the domain of $Y(t)$). As y_o was arbitrarily chosen in E it follows that (5) holds. Conversely, if $Y(t)$ is a solution of problem (4) and (5) holds, then the equations

$$\frac{dX(t)}{dt} = h\left(\frac{dY(t)}{dt}\right) = (h \circ F_1)(Y(t)) = (F \circ h)(Y(t))) = F(X(t)), \tag{8}$$

also hold, i.e., $X(t)$ is a solution of equation (2) with the same domain as $Y(t)$.

As (5) is equivalent to $F_1 \circ h^{-1} = h^{-1} \circ F$ (and h^{-1} is continuous), it follows

Corollary 1. *If (2) is equivalent to (4), then (4) is also equivalent to (2).*

Obviously, E' can be identified, via h, with E so that it is enough to analyze the set of all nHDEs given on a fixed Banach space E, only. Thus, any CA-equivalence is really an equivalence on the set of all nHDEs on a fixed Banach space (i.e. it is a reflexive, symmetric and transitive binary relation).

4 The algebra associated with a nHDE

The polar form $G : E^n \to E$ for F in (2) is a continuous and symmetric n-linear vector form on E. The n-ary algebra $E([., ..., .])$ defined by the n-ary operation

$$[x_1, x_2, ..., x_n] = G(x_1, x_2, ..., x_n), \quad \forall x_1, x_2, ..., x_n \in E$$

is a *commutative* (or, *symmetric*) n-ary algebra; it is a non-associative algebra, i.e. it is not necessarily an associative one.

Recall that there exists $\|G\|_1$ such that $\|G(x_1, x_2, ..., x_n)\|_1 \le \|G\|_1 \|x_1\|_1 \|x_2\|_1 ... \|x_n\|_1$. Then, the norm $\|\cdot\|$ on E defined by $\|x\| = \sqrt[n-1]{\|G\|_1} \cdot \|x\|_1$ has the property

$$\|[x_1, x_2, ..., x_n]\| \le \|x_1\| \cdot \|x_2\| ... \|x_n\|.$$

The n-ary algebra $E([., ..., .])$ endowed with norm $\|\cdot\|$ is called the B-*algebra* associated with (2).

Theorem 2. *The nHDE (2) is CA-equivalent with (4) if and only if the n-ary algebras associated with them as before are continuously isomorphic.*

Proof. If (2) and (4) are equivalent equations then there exists, according to Theorem 1, an invertible continuous linear mapping $h : E' \to E$ which satisfies (5). By passing to the polar forms for F and F_1, it follows that h is an algebra isomorphism between $E'(\{\cdot,\cdot,...,\cdot\})$ and $E([\cdot,\cdot,...,\cdot])$. Conversely, if $h : E'(\{\cdot,...,\cdot\}) \to E([\cdot,...,\cdot])$ is a continuous algebra isomorphism, then (5) holds, i.e. h is an equivalence of the nHDEs (2) and (4).

Remark 1. Theorem 2 ensures that there exists a bijection between the classes of affinely equivalent nHDEs on E and the classes of isomorphic commutative n-ary algebras on E. Consequently, there exists a correspondence between certain qualitative properties of a nHDE (2) and the invariant properties under an isomorphism of its associated n-ary commutative algebra.

5 Solving nHDEs

Let us suppose that the n-ary algebra $E([\cdot,...,\cdot])$ associated with (2) has no identity element. Then, $E([\cdot,...,\cdot])$ can be embedded into the n-ary algebra $\overline{E}([\cdot,...,\cdot])$ whose operation has the n-monomial form

$$\overline{F}(x \oplus \lambda) = (F(x) + n\lambda^{n-1}x) \oplus \lambda^n, \quad \forall x \oplus \lambda \in \overline{E},$$

which suggests us to consider the following nHDE on $\overline{E}$

$$\begin{cases} \dfrac{dX}{dt} = n\lambda^{n-1}X + F(X), \\ \dfrac{d\lambda}{dt} = \lambda^n. \end{cases} \tag{9}$$

Then, with every Cauchy problem

$$\frac{dX}{dt} = F(X), \qquad X(t_0) = x_0 \tag{10}$$

we associate the following Cauchy problem

$$\begin{cases} \dfrac{dX}{dt} = n\lambda^{n-1}X + F(X), \\ \dfrac{d\lambda}{dt} = \lambda^n, \end{cases} \qquad \begin{cases} X(t_0) = x_0, \\ \lambda(t_0) = 0, \end{cases} \tag{11}$$

for (9). Consequently, there exists the 1-1 correspondence

$$X(t) \Leftrightarrow X(t) \oplus \{0\}$$

between the sets of solutions for (2) and the set of solutions of (9) with $\lambda \equiv 0$, respectively. That is why, in what follows, we shall study only nHDEqs for which the associated (B-)algebra has at least an identity element.

In order to imply the n-ary algebra in the study of nHDE (2), it is suitable to use the equality $F(x) = x^n$, and then (2) becomes

$$\frac{dX}{dt} = X^n. \tag{12}$$

CAUCHY-KOWALEWSKAIA Theorem assures us that there exists a unique analytic solution for CAUCHY problem (10).

Let $X(t)$ be the saturated solution for (10). Then we get

$$X(t_0) = x_0, \qquad \frac{dX(t_0)}{dt} = x_0^n$$

Further, we get recurrently:

$$\left\| \frac{d^i X(t_0)}{dt^i} \right\| \leq ((i-1)(n-1)+1)\underset{n-1\,times}{!...!} \cdot \|x_0\|^{i(n-1)+1}, \quad \forall i \geq 1,$$

where $((i-1)(n-1)+1)\underset{n-1\,times}{!...!} = 1\cdot(1(n-1)+1)\cdot(2(n-1)+1)\cdot...\cdot((i-1)(n-1)+1)$. Consequently, the series

$$\|x_0\| + \frac{|t-t_0|}{1!}\left\|\frac{dX(t_0)}{dt}\right\| + \frac{|t-t_0|^2}{2!}\left\|\frac{d^2X(t_0)}{dt^2}\right\| + ... + \frac{|t-t_0|^k}{k!}\left\|\frac{d^kX(t_0)}{dt^k}\right\| + ...$$

is upper bounded by the numerical series for

$$\frac{\|x_0\|}{\sqrt[n-1]{1-(n-1)\|x_0\|\cdot|t-t_0|}}.$$

Thus, the TAYLOR series for $X(t)$, around t_0, is absolutely and uniformly convergent for $(n-1)\|x_0\|\cdot|t-t_0| < 1$.

In the next section we shall use these computations to find a formula for solving (10) in case when the associated algebra satisfies a "weak" associativity axiom (e.g., the monoassociativity). Obviously, these computations allow also to prove again the analyticity of the solution of (10).

6 The case of nHDEs with power-associative algebras

Let us consider the CAUCHY problem (10). Suppose the corresponding n-ary B-algebra is a power-associative algebra. Then, its solution $X(t)$ satisfies the conditions

$$\begin{aligned} &\frac{dX(t_0)}{dt} = x_0^n = L_{x_0,...,x_0}(x_0), \\ &\frac{d^kX(t_0)}{dt^k}\| = ((k-1)(n-1)+1)\underset{n-1\,times}{!...!} \cdot L^k_{x_0,...,x_0}x_0, \quad k > 1 \end{aligned}$$

(here $L_{x_0,...,x_0}$ is instead of $L\underbrace{x_0,...,x_0}_{n-1\ times})$. Consequently, one gets

$$\begin{aligned} X(t) = &\left(I + \frac{t-t_0}{1!}L_{x_0,...,x_0} + \frac{n!...!(t-t_0)^2}{2!}L^2_{x_0,...,x_0} + ... + \right. \\ &+ \left. \frac{((k-1)(n-1)+1)!...!(t-t_0)^k}{k!}L^k_{x_0,...,x_0} + ... \right)(x_0) = \\ &= (I-(n-1)(t-t_0)L_{x_0,...,x_0})^{-\frac{1}{n-1}}(x_0) \end{aligned}$$

where I denotes the identity mapping on E and !...! is used instead of $\underbrace{!...!}_{n-1\,times}$. By a direct checking one gets that the analytic mapping

$$Y(t) = (I - (n-1)(t-t_0)L_{x_0,\ldots,x_0})^{-\frac{1}{n-1}}(x_0),$$

defined for all t such that $(n-1)^{-1}(t-t_0)^{-1}$ does not belong to the spectrum of $L_{x_0,\ldots,x_0}$, satisfies the equation

$$\frac{dY(t)}{dt} = Y^n(t)(= G(Y(t), Y(t), ..., Y(t))).$$

Theorem 3. *If the B-algebra* $E([.,\ldots,.])$ *associated with* (*2*) *is power-associative, then the saturated solution for* (*10*) *is*

$$X(t) = (I - (n-1)(t-t_0)L_{x_0,\ldots,x_0})^{-\frac{1}{n-1}}(x_0). \tag{13}$$

Remark 2. The computations performed for proving that $Y(t)$ is the solution of (10) can be similarly performed in the case when x_0 has associative powers, only. Consequently, (13) is the solution for (10) as long as x_0 has associative powers.

7 Properties of n-ary commutative algebras reflected in those of nHDE

The correspondence between the classes of CA-equivalent nHDEs and the classes of continuously isomorphic n-ary algebras induces the existence of a correspondence between the qualitative properties of nHDEs and those of n-ary algebras.

We shall present below some results on this line.

1) $x_0 \in E$ is a critical point (or, a steady state point) for (2) if and only if $x_0^n = 0$, i.e. if and only if it is a nilpotent for $E([.,.,.])$. If x_0 is a critical point for (2), then λx_0 is also a critical point for (2), for every $\lambda \in K$. $0 \in E$ is always a critical point for (2); it is an isolated critical point if and only if $E([.,.,.])$ has no nilpotent element.

2) If $e \in E$ is an idempotent element for $E([.,.,.])$, then it has associative powers and

$$X(t) = (1 - (n-1)(t-t_0))^{-\frac{1}{n-1}} \cdot e \tag{14}$$

is the unique solution of the Cauchy problem (10) with $x_0 = e$. The idempotent elements identify unbounded solutions of (2).

Let us consider the Cauchy problem (10) with $x_0 = P$, where $P^n = aP$, $a \in \mathbb{R}$. Then P has associative powers and

$$X(t) = a(1 - (n-1)(t-t_0))^{-\frac{1}{n-1}} \cdot P \tag{15}$$

is the solution of (2)+($X(t_0) = P$).

Following step by step the proof of Proposition 3.4 [2] and using (15) one gets the result:

Proposition 1. *If* $E([.,...,.])$ *has an idempotent, then the origin* $0 \in E$ *is an unstable critical point for* (2). *Consequently, if the origin is stable for* (2) *then* $E([.,...,.])$ *has no idempotent and, in particular, no identity element.*

As an immediate consequence of the fact that $N_0 \neq 0$ is a nilpotent element ($N_0^n = 0$) implies $N = \lambda N_0$ is also a nilpotent (for every $\lambda \in K$), we can readily prove the following proposition.

Proposition 2. *If* $E([.,...,.])$ *has a* (*nonzero*) *nilpotent, then the origin* $0 \in E$ *is not asymptotically stable steady state point for* (2).

Following the idea of proving Theorem 1 [1] it can be proved

Proposition 3. *If* $E([.,...,.])$ *is a real commutative finite dimensional* n*-ary algebra, with* n *an even number, it has a* (*nonzero*) *idempotent or a* (*nonzero*) *nilpotent.*

Proof. Suppose $E = \mathbb{R}^m$. Let us consider the mapping $F : \mathbb{R}^m \to \mathbb{R}^m$ defined by $F(x) = x^n$. If $E([.,...,.])$ has an idempotent e, then it is a fixed point for F which still keeps invariant the axe $\{\lambda e \mid \lambda \in \mathbb{R}\}$. Moreover, if $F(x_0) = cx_0$ and c is positive, then $e = \dfrac{1}{\sqrt[n-1]{c}} \cdot x_0$ is an idempotent (in case when c is negative and n is even number, then $e = -\dfrac{1}{\sqrt[n-1]{-c}} \cdot x_0$ is also an idempotent), i.e. F has necessarily a fixed point. Supposing that F has no fixed points and n is an even number, it results

$$(1-\lambda)F(x) \neq \lambda x, \ \ \forall x \neq 0, \ \forall \lambda \in \mathbb{R}.$$

If in addition $E([.,...,.])$ has no nilpotent, then $F(x) \neq 0$ for all $x \neq 0$, and it induces a function $g : S^{m-1} \to S^{m-1}$ defined by $g(x) = \dfrac{F(x)}{\|F(x)\|}$ for all $x \in S^{m-1}$. Let us define also the uniparametric family of functions $G(\cdot,\lambda) : S^{m-1} \to S^{m-1}$ by

$$G(x,\lambda) = \frac{(1-h)F(x) + \lambda(-x)}{\|(1-h)F(x) + \lambda(-x)\|}, \ 0 \leq \lambda \leq 1, \ \forall x \in S^{m-1}.$$

G is an homotopic mapping on S^{m-1} of g with the antipodal mapping $\mathbf{a} : S^{m-1} \to S^{m-1}$ (defined by $\mathbf{a}(x) = -x$). Consequently, these mappings must have the same degree, what is not possible because the degree of g is an even number, while the degree of $\mathbf{a}$ is 1.

Corollary 2. *The origin* $0 \in E$ *is not asymptotically stable for (2) if* n *is an even number.*

Theorem 4. *Let* (2), *given on a finite dimensional real vector space* E, *be such that its associated algebra* $E([.,...,.])$ *has a symmetric positive definite bilinear form* $H : E \times E \to \mathbb{R}$. *If* H *satisfies*

$$H(X, X^n) = 0, \quad \forall X \in E,$$

(*or* $H(X, X^n) \leq 0$) *then the origin* $0 \in E$ *is a stable point.*

Proof. The function $V : E \to \mathbb{R}$ defined by $V(x) = H(x,x)$ is a LIAPUNOV function. Indeed, V is a positive definite quadratic form and its derivative $\dot{V}(X(t))$ vanishes identically along any trajectory $X(t)$ of (2).

Consequently, the existence of H implies the nonexistence of an idempotent.

Similar arguments as for Theorem 3.10 [2] allow us to prove the following result.

Theorem 5. *Let (2) be a nHDE and $E([\cdot,...,\cdot])$ be its associated n-ary algebra.*

(1) *The trajectory through $P \in E$ does not pass through aP for any $a \leq 0$. If P lies on a periodic trajectory, the trajectory through P does not pass through aP for any $a \neq 1$.*

(2) *If $\gamma \subset E$ is a periodic orbit with the least period τ, then $a\gamma = \{aP | P \in \gamma\}$ is a periodic trajectory with the least period $\tau/|a|$ for $a \neq 0$. Thus, scalar multiples of periodic orbits are periodic, and solutions of any period exist, provided that one periodic orbit exists.*

(3) *The periodic trajectories lie on cones.*

Theorem 6. *Let (2) be a nHDE and $E([\cdot,...,\cdot])$ be its associated n-ary algebra. Then no periodic orbit is an attractor.*

Proof. If $\gamma(t)$, $\gamma(t_0) = P$ is a periodic solution and $\mathcal{U}$ is an open neighborhood of it, then there exists $a \in \mathbb{R}$ such that $aP \in \mathcal{U}$. Then $a\gamma(t)$ is also a periodic solution contained in $\mathcal{U}$. Consequently, $\lim\limits_{n\to\infty} \|a\gamma - \gamma\| \neq 0$ and γ is not an attractor.

3) Let E_0 be a closed ideal of $E([\cdot,...,\cdot])$ and E_1 – a closed vector subspace which is its complement in E, i.e. $E = E_0 \oplus E_1$. We denote by $p_i : E \to E_i$ $(i = 0,1)$ the two projectors associated with the direct sum decomposition of E, $X_i = p_i(X)$, $i = 0,1$, then (2) becomes

$$\begin{cases} \dfrac{dX_0}{dt} = F(X_0) + \sum\limits_{i=1}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} G(\underbrace{X_0,...,X_0}_{i\ times}, \underbrace{X_1,...,X_1}_{n-i\ times}) + (p_0 \circ F)(X_1) \\ \dfrac{dX_1}{dt} = (p_1 \circ F)(X_1). \end{cases}$$

In the particular case when E_1 is also a closed ideal for $E([.,.,.])$, then

$$\begin{cases} \dfrac{dX_0}{dt} = F(X_0) \\ \dfrac{dX_1}{dt} = F(X_1). \end{cases}$$

Further, if E_1 is only a subalgebra of $E([\cdot,...,\cdot])$, then

$$\begin{cases} \dfrac{dX_0}{dt} = F(X_0) + \sum\limits_{i=1}^{n-1} \begin{pmatrix} n \\ i \end{pmatrix} G(\underbrace{X_0,...,X_0}_{i\ times}, \underbrace{X_1,...,X_1}_{n-i\ times}) \\ \dfrac{dX_1}{dt} = (p_1 \circ F)(X_1). \end{cases}$$

4) If only a finite number of the powers of x_0 are linearly independent, then the subalgebra $K(x_0)$ spanned by all these powers will be a finite-dimensional one. Let $(x_0, x_1, ..., x_s)$ be the basis of $K(x_0)$ consisting of the first smallest independent powers of x_0. In this case, the solution of (2) gets the form

$$X(t) = f_0(t)x_0 + f_1(t)x_1 + ... + f_s(t)x_s$$

where the mappings $f_i(t)$, $i = 0, 1, ..., s$, satisfy an n-homogeneous differential system of the form

$$\frac{df_i}{dt} = \sum_{j_1,j_2,...,j_n=0}^{s} C_{ij_1j_2...j_n} f_{j_1} f_{j_2} ... f_{j_n}, \quad i, j_1, j_2, ..., j_n = 0, 1, ..., s$$

with the initial conditions $f_0(t_0) = 1, f_1(t_0) = 0, ..., f_s(t_0) = 0$; here, $C_{ij_1j_2...j_n}$ are the structure constants of the subalgebra $K(x_0)$ in basis $(x_0, x_1, ..., x_s)$ defined by

$$[x_{j_1}, x_{j_2}, ..., x_{j_n}] = \sum_{i=0}^{s} C_{ij_1j_2...j_n} x_i, \quad i, j_1, j_2, ..., j_n = 0, 1, ..., s.$$

This situation is usually met in the case of n-homogeneous differential systems on finite-dimensional spaces. It warns us of the necessity to pay a special attention to the algebras with a single generator.

8 nHDEs on real finite dimensional spaces

If $E = \mathbb{R}^p$, $\mathcal{B} = (e_1, e_2, ..., e_p) \subset \mathbb{R}^p$ is its natural basis and $X = X^i e_i$, then (2) becomes

$$\frac{dX^i}{dt} = C^i_{j_1j_2...j_n} X^{j_1} X^{j_2} ... X^{j_n}, \quad i, j_1, j_2, ..., j_n = 1, ..., p \tag{16}$$

where the EINSTEIN's convention on summation is used.

The associated n-ary algebra $E([\cdot, ..., \cdot])$ has $C^i_{j_1j_2...j_n}$ as its structure constants in basis $\mathcal{B}$. It is suitable to denote the left multiplication $L_{\underbrace{x_0, ..., x_0}_{n-1\, times}}$ by G_{x_0}, for any $x_0 \in E$ (i.e., $G_{x_0} = L_{\underbrace{x_0, ..., x_0}_{n-1\, times}}$). G_{x_0} is an endomorphism of E having, in basis $\mathcal{B}$, the matrix

$$[G_{x_0}]_{\mathcal{B}} = [C^i_{j_1j_2...j_{n-1}j} x_0^{j_1} x_0^{j_2} ... x_0^{j_{n-1}}].$$

The solution of the CAUCHY problem (10) is an analytical function; more exactly, there exists an analytical function f such that its solution has the form:

$$X(t) = f((t - t_0) \cdot G_{x_0})(x_0).$$

For example, recall that in the case when $E([\cdot, ..., \cdot])$ is a power-associative algebra we have

$$f(\lambda) = (1 - (n-1)\lambda)^{-\frac{1}{n-1}}$$

and the solution of the CAUCHY problem for (10) is

$$[X(t)]_{\mathcal{B}} = f((t-t_0)[G_{x_0}]_{\mathcal{B}})[x_0]_{\mathcal{B}}. \tag{17}$$

For every analytical function h we can obtain $h(G_{x_0})$ using the JORDAN (e.g., upper) form for G_{x_0}. Indeed, there exists a basis $\mathcal{B}_J$ in E such that the matrix of G_{x_0} is

$$[G_{x_0}]_J = diag \ \ (J_1, J_2, ..., J_s, R_1, R_2, ..., R_q),$$

where J_i are superior Jordan cells corresponding to the real eigenvalues of G_{x_0}, while R_i are the superior Jordan blocs corresponding to the complex eigenvalues of G_{x_0} (see, [3]). If S denotes the transformation matrix from $\mathcal{B}$ to $\mathcal{B}_J$ one gets

$$h([G_{x_0}]_{\mathcal{B}}) = S \cdot h([G_{x_0}]_J) \cdot S^{-1},$$

where $h([G_{x_0}]_J) = diag \ \ (h(J_1), h(J_2), ..., h(J_s), h(R_1), h(R_2), ..., h(R_q))$.

9 Example

Let us consider, in $\mathbb{R}^4$ with the natural basis (e_1, e_2, e_3, e_4), the cubic homogeneous differential system

$$\begin{cases} \dfrac{dx^1}{dt} = (x^1)^3 - 3x^1(x^2)^2 \\ \dfrac{dx^2}{dt} = 3(x^1)^2x^2 - (x^2)^3 \\ \dfrac{dx^3}{dt} = 3\left[(x^1)^2x^3 - (x^2)^2x^3 - 2x^1x^2x^4\right] \\ \dfrac{dx^4}{dt} = 3\left[(x^1)^2x^4 - (x^2)^2x^4 + 2x^1x^2x^3\right]. \end{cases}$$

The left multiplication $L_{X,X}$ has the matrix

$$L_{X,X} = \begin{bmatrix} (x^1)^2 - (x^2)^2 & -2x^1x^2 & 0 & 0 \\ 2x^1x^2 & (x^1)^2 - (x^2)^2 & 0 & 0 \\ 2(x^1x^3 - x^2x^4) & -2(x^1x^4 + x^2x^3) & (x^1)^2 - (x^2)^2 & -2x^1x^2 \\ 2(x^1x^4 + x^2x^3) & 2(x^1x^3 - x^2x^4) & 2x^1x^2 & (x^1)^2 - (x^2)^2 \end{bmatrix}.$$

The ternary algebra associated with this system is associative. If $x_0 = ae_1 + be_2 + ce_3 + de_4$, then L_{x_0,x_0} has the eigenvalues $\lambda_1, \lambda_2 = a^2 - b^2 \pm 2iab = (a \pm ib)^2$ with multiplicity 2. Thus, the solution $X(t)$ with $X(t_0) = x_0$ is

$$X(t) = (f(\alpha + i\beta)A_1 + f(\alpha - i\beta)A_2 + f'(\alpha + i\beta)A_3 + f'(\alpha - i\beta)A_4)(x_0),$$

where $\lambda_1 = \alpha + i\beta$, $f(\lambda) = (1 - 2(t-t_0)\lambda)^{-\frac{1}{2}}$ and A_1, A_2, A_3, A_4 can be defined by the identities

$$\begin{cases} A_1 + A_2 = I_4, \\ 2i\beta(A_3 - A_4) = A^2 - 2\alpha A + (\alpha^2 + \beta^2) I_4, \\ i\beta(A_1 - A_2) + (A_3 + A_4) = A - \alpha I_2, \\ i\beta(3\alpha^2 - \beta^2)(A_1 - A_2) + 3(\alpha^2 - \beta^2)(A_3 + A_4) + 6i\alpha\beta(A_3 - A_4) = \\ \qquad = A^3 - \alpha(\alpha^2 - 3\beta^2) I_4, \end{cases}$$

where A denotes the matrix of the left multiplication L_{x_0,x_0}.

The solution of the CAUCHY problem with $X(t_0) = x_0$ is

$$\begin{cases} x^1 = \Re e \dfrac{\rho}{\sqrt{cos\ 2\alpha - isin\ 2\alpha - 2\rho^2(t - t_0)}}, \\ x^2 = \Im m \dfrac{\rho}{\sqrt{cos\ 2\alpha - isin\ 2\alpha - 2\rho^2(t - t_0)}}, \\ x^3 = \Re e \dfrac{\kappa\rho}{\sqrt{cos\ 2\alpha - isin\ 2\alpha - 2\rho^2(t - t_0)}}, \\ x^4 = \Im m \dfrac{\kappa\rho}{\sqrt{cos\ 2\alpha - isin\ 2\alpha - 2\rho^2(t - t_0)}}, \end{cases}$$

where $a + ib = \rho(cos\ \alpha + i\ sin\ \alpha)$, $\kappa = \dfrac{c + id}{a + ib}$.

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