BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(56), 2008, Pages 125–130
ISSN 1024–7696

Orthogonal Solutions for a Hyperbolic System

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Abstract. We consider the hyperbolic system

$$\begin{cases} u_t = a\nabla v + f_1(u,v) \\ v_t = a\nabla u + f_2(u,v) \\ u(0,x) = \xi(x) \\ v(0,x) = \eta(x), \end{cases}$$

and we are looking for necessary and sufficient conditions on the forcing terms f_i, $i = 1, 2$, in order that the semigroup solutions, u and v, starting from orthogonal data $\xi, \eta \in L^2(\mathbb{R}^n)$, remain orthogonal on $\mathbb{R}_+$.

Mathematics subject classification: 35L45, 47D03, 47J35.
Keywords and phrases: First-order hyperbolic systems, orthogonal solutions, viability, tangency condition.

1 The main result

Let us consider the hyperbolic system

$$\begin{cases} u_t = a\nabla v + f_1(u,v) \\ v_t = a\nabla u + f_2(u,v) \\ u(0,x) = \xi(x) \\ v(0,x) = \eta(x), \end{cases} \tag{1}$$

where $a \in \mathbb{R}^n$, $f_1 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $f_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\xi, \eta \in L^2(\mathbb{R}^n)$. We are looking for necessary and sufficient conditions on the forcing terms f_i, $i = 1, 2$, in order that the mild solutions, u and v, of (1), starting from orthogonal data $\xi, \eta \in L^2(\mathbb{R}^n)$, remain orthogonal on $\mathbb{R}_+$, i.e.,

$$\langle u(t,\cdot), v(t,\cdot)\rangle = 0 \tag{2}$$

for each $t \in \mathbb{R}_+$, whenever $\xi, \eta \in L^2(\mathbb{R}^n)$ satisfy

$$\langle \xi, \eta\rangle = 0. \tag{3}$$

The main result of this paper concerning the problem above is

Theorem 1. *Let us assume that* $f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, *are globally Lipschitz. Then, a necessary and sufficient condition in order that for each* $\xi, \eta \in L^2(\mathbb{R}^n)$, *satisfying* (3), *to exist a unique mild solution* $(u, v) : \mathbb{R}_+ \to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ *of* (1), *satisfying* (2) *for each* $t \in \mathbb{R}_+$, *is*

$$\langle \xi, f_2(\xi, \eta) \rangle + \langle \eta, f_1(\xi, \eta) \rangle = 0, \tag{4}$$

for each $\xi, \eta \in L^2(\mathbb{R}^n)$ *satisfying* (3).

The proof of Theorem 1 is based on a combination of C_0-semigroup techniques developed in Vrabie [7] and viability arguments which we recall in the next section.

2 Introduction to mild viability

Let X be a Banach space, let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \geq 0\}$, and let $f : K \to X$ be a continuous function. Let us consider the Cauchy problem

$$\begin{cases} u'(t) \in Au(t) + f(u(t)) \\ u(0) = \xi. \end{cases} \tag{5}$$

Definition 1. We say that K is *mild viable with respect to* $A + f$ if for each $\xi \in K$ there exist $T > 0$ and a continuous function $u : [\,0, T\,] \to K$ satisfying

$$u(t) = S(t)\xi + \int_0^t S(t-s) f(u(s)) ds$$

for each $t \in [\,0, T\,]$.

In order to get a necessary and sufficient condition for mild viability, some preliminaries are needed.

Definition 2. We say that $\eta \in X$ is *A-tangent to K at* $\xi \in K$ if

$$\liminf_{h \downarrow 0} \frac{1}{h} \mathrm{dist}\,(S(h)\xi + h\eta; K) = 0.$$

In other words, $\eta \in X$ is A-tangent to K at $\xi \in K$ if for each $\delta > 0$ and each neighborhood V of 0 there exist $h \in (0, \delta)$ and $p \in V$ such that

$$S(h)\xi + h(\eta + p) \in K. \tag{6}$$

The set of all A-tangent elements to K at $\xi \in K$ is denoted by $\mathcal{T}_K^A(\xi)$. We notice that if $A \equiv 0$, then $\mathcal{T}_K^A(\xi)$ is the *contingent cone* at $\xi \in K$ in the sense of Bouligand [1] and Severi [6], i.e.

$$\mathcal{T}_K^0(\xi) = \mathcal{T}_K(\xi).$$

Proposition 1. *If $\eta \in \mathcal{T}_K^A(\xi)$ then, for every function $h \mapsto \eta_h$ from $(0,1)$ to X satisfying $\lim_{h\downarrow 0} \eta_h = \eta$, we have*

$$\liminf_{h\downarrow 0} \frac{1}{h} \mathrm{dist}\,(S(h)\xi + h\eta_h; K) = 0. \tag{7}$$

If there exists a function $h \mapsto \eta_h$ from $(0,1)$ to X satisfying both

$$\lim_{h\downarrow 0} \eta_h = \eta$$

and (7), then $\eta \in \mathcal{T}_K^A(\xi)$.

The next result is a necessary and sufficient condition for mild viability due to Cârjă and Motreanu [3]. For a more general theorem extending both Nagumo's [4] and Pavel's [5] main viability results, see Burlică and Roşu [2].

Theorem 2. *Let $A : D(A) \subseteq X \to X$ be the infinitesimal generator of a C_0-semigroup, $K \subseteq X$ a nonempty and locally closed subset in X and let $f : K \to X$ be a locally Lipschitz function. Then, a necessary and sufficient condition in order that K be mild viable with respect to $A + f$ is the generalized tangency condition*

$$f(\xi) \in \mathcal{T}_K^A(\xi) \tag{8}$$

for each $\xi \in K$.

3 The abstract Banach space setting

Let K be a nonempty subset in X, invariant with respect to A, in the sense that $S(t)K \subseteq K$ for each $t \in \mathbb{R}_+$, and let $f : K \to X$ be a continuous function. Next, we prove some appropriate sufficient conditions on f in order that K be mild viable with respect to $A + f$.

Lemma 1. *Let X be a Banach space, $A : D(A) \subseteq X \to X$ the infinitesimal generator of a C_0-semigroup, $\{S(t) : X \to X;\ t \geq 0\}$, and K a nonempty subset in X. Assume that K is invariant with respect to A, i.e., $S(t)K \subseteq K$ for each $t \in \mathbb{R}_+$. Then $\mathcal{T}_K(\xi) \subseteq \mathcal{T}_K^A(\xi)$ for each $\xi \in K$. If, instead of a C_0-semigroup, A generates a C_0-group of isometries, $\{G(t) : X \to X;\ t \in \mathbb{R}\}$, satisfying $G(t)K \subseteq K$ (or, equivalently, $G(t)K = K$) for each $t \in \mathbb{R}$, then $\mathcal{T}_K(\xi) = \mathcal{T}_K^A(\xi)$ for each $\xi \in K$.*

Proof. Let $\eta \in \mathcal{T}_K(\xi)$. By Proposition 1, it suffices to check that

$$\liminf_{h\downarrow 0} \frac{1}{h} \mathrm{dist}\,(S(h)\xi + hS(h)\eta; K) = 0.$$

Let $M \geq 1$ and $a \in \mathbb{R}$ be such that $\|S(t)\| \leq Me^{at}$ for each $t \geq 0$. Since $S(t)K \subseteq K$ for each $t \geq 0$, we have

$$\mathrm{dist}\,(S(h)\xi + hS(h)\eta; K) \leq$$
$$\leq \mathrm{dist}\,(S(h)\xi + hS(h)\eta; S(h)K) \leq Me^{ah}\mathrm{dist}\,(\xi + h\eta; K).$$

Thus

$$\liminf_{h\downarrow 0}\frac{1}{h}\mathrm{dist}\,(S(h)\xi+hS(h)\eta;K)\leq\liminf_{h\downarrow 0}\frac{1}{h}Me^{ah}\mathrm{dist}\,(\xi+h\eta;K)=0.$$

Since the conclusion in the case of a C_0-group of isometries follows from the preceding one, this completes the proof. □

Theorem 3. *Let X be a Banach space, $A:D(A)\subseteq X\to X$ the infinitesimal generator of a C_0-semigroup, $\{S(t):X\to X;\ t\geq 0\}$, K a nonempty and locally closed subset in X, and $f:K\to X$ a locally Lipschitz function. If $S(t)K\subseteq K$ for each $t\geq 0$ and*

$$f(\xi)\in\mathcal{T}_K(\xi)\tag{9}$$

for each $\xi\in K$, then K is mild viable with respect to $A+f$.

Proof. The conclusion follows from Lemma 1 and Theorem 2. □

Theorem 4. *Let X be a Banach space, $A:D(A)\subseteq X\to X$ the infinitesimal generator of a C_0-group of isometries, $\{G(t):X\to X;\ t\in\mathbb{R}\}$, K a nonempty and locally closed subset in X, and $f:K\to X$ a locally Lipschitz function. If $G(t)K\subseteq K$ (or, equivalently, $G(t)K=K$) for each $t\in\mathbb{R}$, then a necessary and sufficient condition in order that K be mild viable with respect to $A+f$ is* (9).

Proof. The conclusion follows from Lemma 1 and Theorem 2. □

4 Proof of the main result

We can now pass to the proof of the main result which rests heavily on Theorem 4.
Proof. First, let us observe that the problem (1) can be rewritten as an abstract evolution equation of the form (5), where $X=L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, $A:D(A)\subseteq X\to X$ is defined by

$$\begin{cases} D(A)=\{(u,v)\in X;\ (a\nabla v,a\nabla u)\in X\}\\ A(u,v)=(a\nabla v,a\nabla u)\ \text{ for all }\ (u,v)\in D(A),\end{cases}\tag{10}$$

and $f:X\to X$ is given by

$$f(u,v)(x)=(f_1(u(x),v(x)),f_2(u(x),v(x))),\tag{11}$$

for each $(u,v)\in X$ and a.e. for $x\in\mathbb{R}^n$.

On X we consider the usual Hilbert space norm

$$\|(u,v)\|=\sqrt{\langle u,u\rangle+\langle v,v\rangle},$$

for each $(u,v)\in X$, where $\langle\cdot,\cdot\rangle$ is the usual inner product on $L^2(\mathbb{R}^n)$.

It is well-known that the linear operator A, defined by (10), generates a C_0-group of isometries, $\{G(t):X\to X;\ t\in\mathbb{R}\}$, given by

$$[G(t)(u,v)]\,(x)=\frac{1}{2}\begin{pmatrix} u(x+ta)+u(x-ta)+v(x+ta)-v(x-ta)\\ u(x+ta)-u(x-ta)+v(x+ta)+v(x-ta)\end{pmatrix}^{\mathcal{T}},$$

where $B^{\mathcal{T}}$ denotes the transpose of the matrix B. See Vrabie [7]. Second, since f_i, $i = 1, 2$, are globally Lipschitz, the function $f : X \to X$, given by (11), is well-defined and globally Lipschitz on X.

Next, let us define

$$K = \{(\xi, \eta) \in X;\ \xi \text{ and } \eta \text{ satisfy (3), i.e., } \langle \xi, \eta \rangle = 0\}$$

which is nonempty and closed in X. Let us observe that $G(t)K \subseteq K$ for each $t \in \mathbb{R}$. Indeed, let $(\xi, \eta) \in K$ and let us denote by

$$G(t)(\xi, \eta) = (G_1(t)(\xi, \eta), G_2(t)(\xi, \eta)),$$

where

$$G_1(t)(\xi, \eta) = \frac{1}{2}\left(\xi(\cdot + ta) + \xi(\cdot - ta) + \eta(\cdot + ta) - \eta(\cdot - ta)\right)$$

$$G_2(t)(\xi, \eta) = \frac{1}{2}\left(\xi(\cdot + ta) - \xi(\cdot - ta) + \eta(\cdot + ta) + \eta(\cdot - ta)\right)$$

for each $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

We have

$$\begin{gathered}
\langle G_1(t)(\xi, \eta), G_2(t)(\xi, \eta) \rangle = \|\xi(\cdot + ta)\|_{L^2(\mathbb{R}^n)} - \langle \xi(\cdot + ta), \xi(\cdot - ta) \rangle + \\
+ \langle \xi(\cdot + ta), \eta(\cdot + ta) \rangle + \langle \xi(\cdot + ta), \eta(\cdot - ta) \rangle + \langle \xi(\cdot - ta), \xi(\cdot + ta) \rangle - \\
- \|\xi(\cdot - ta)\|_{L^2(\mathbb{R}^n)} + \langle \xi(\cdot - ta), \eta(\cdot + ta) \rangle + \langle \xi(\cdot - ta), \eta(\cdot - ta) \rangle + \\
+ \langle \eta(\cdot + ta), \xi(\cdot + ta) \rangle - \langle \eta(\cdot + ta), \xi(\cdot - ta) \rangle + \|\eta(\cdot + ta)\|_{L^2(\mathbb{R}^n)} + \\
+ \langle \eta(\cdot + ta), \eta(\cdot - ta) \rangle - \langle \eta(\cdot - ta), \xi(\cdot + ta) \rangle + \langle \eta(\cdot - ta), \xi(\cdot - ta) \rangle - \\
- \langle \eta(\cdot + ta), \eta(\cdot - ta) \rangle - \|\eta(\cdot - ta)\|_{L^2(\mathbb{R}^n)}.
\end{gathered}$$

Since the Lebesgue measure on $\mathbb{R}^n$ is translation invariant, we deduce that the right hand side vanishes which proves that $G(t)K \subseteq K$.

Thanks to Theorem 4, K is mild viable with respect to $A + f$ if and only if $f(\xi, \eta) \in \mathcal{T}_K(\xi, \eta)$ for each $(\xi, \eta) \in K$. The last condition is equivalent to the existence of two sequences, $(h_n)_n$ in $\mathbb{R}_+$ and $((p_n, q_n))_n$ in X, with $h_n \downarrow 0$, $\lim_n (p_n, q_n) = (0, 0)$ and such that

$$(\xi, \eta) + h_n(f_1(\xi, \eta), f_2(\xi, \eta)) + h_n(p_n, q_n) \in K$$

for $n = 1, 2, \ldots$. Equivalently,

$$\langle \xi + h_n f_1(\xi, \eta) + h_n p_n, \eta + h_n f_2(\xi, \eta) + h_n q_n \rangle = 0$$

for $n = 1, 2, \ldots$. A simple calculation using the fact that $\langle \xi, \eta \rangle = 0$, $h_n \downarrow 0$ and $\lim_n p_n = \lim_n q_n = 0$, shows that the last relation is equivalent to (4), and this shows that K is mild viable with respect to $A + f$.

Finally, since f_i, $i = 1, 2$, are globally Lipschitz, it follows that f inherits the very same property and thus it has linear growth. A classical argument involving Gronwall's Lemma and the fact that K is closed and mild viable with respect to $A + f$, shows that each mild solution of (5) can be continued to a global one and this completes the proof. □

Acknowledgements. This paper was supported by CNCSIS Grant A 1159/2006 and we thank the organizers for their kind invitation to deliver this lecture at the *International Conference "Algebraic Systems and Their Applications in Differential Equations and Other Domains of Mathematics"*, August 21-23, 2007, Chişinău, Republic of Moldova.

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Received December 3, 2007

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