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Determinantal Analysis of the Polynomial Integrability of Differential Systems

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Abstract. This work deals with the polynomial and formal (formal series) integrability of the polynomial differential systems around a singular point, namely the conditions which assure the start of the algorithmic process for computing the polynomial or the formal first integrals. When the linear part of the differential system is nonzero, we have established ([9]) the existence of the so called starting equations whose (integer) solutions are exactly the partition of the lower degree of the eventual formal first integrals.

In this work, we study some extensions of the starting equations to the case when the linear part is zero and, particularly, to the bidimensionnal homogeneous differential systems. The principal tool used here is the classical invariant theory.

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1 Introduction

Many works are devoted to the investigation of local (or global) formal first integrals of the differential system

$$\frac{dx^i}{dt} = P^i(x), \quad i = 1, 2, \ldots, n \tag{1}$$

where P^i are polynomials of degree m with coefficients in the field $\mathbb{C}$.

Definition 1. *A function $F \in \mathcal{C}^1(\mathcal{O})$ where $\mathcal{O}$ is an open set of $\mathbb{C}^n$, is a first integral of the differential system (1) if*

$$\forall\, x \in \mathcal{O}, \quad \Delta_P(F) = \sum_{j=1}^{n} \frac{\partial F}{\partial x^j}(x)\, P^j(x) = 0. \tag{2}$$

It is well known that around a regular point x_0, there are exactly $n-1$ functionally independent analytical first integrals. This result is just theoretic.

Around a singular point and for practical and computational reasons, many people [1, 3, 5] have oriented their investigations to special classes of integrals like the polynomial, rational, algebraic or exponential ones. It occurs that these different types of first integrals need the knowledge of the polynomial ones and so, the knowledge of the degree of the first terms.

The same problem interested other mathematicians ([6–8]) which studied the local or analytical integrability. In these works, the question of the resonance of the eigenvalues of the linear *nonzero* part plays a fundamental role.

In [9], when the lower degree of P is 1, was given a constructive method to get, for any dimension n, a so called starting equation satisfied by the lower degree of an eventual first integral F.

This work consists in three parts. In the first one (Subsection 2.4), we give an "extension" of the starting equation ([9]) to the general homogeneous polynomial differential systems. In the second part (Section 3), we study the bidimensional homogeneous systems : we explicitly calculated the matrices whose the kernel contain the polynomial first integrals and we have reduced the integrability problem (existence of a polynomial first integral) to the nullity of some determinant. Finally, in the third part, we use the classical invariant theory to present significiant simplifications for computing the above determinant.

2 Notations and matricial writing of the integrability problem

2.1 The total-lexicographic order

Let $\mathrm{Sym}(n\,,\,k)$ be the linear space of the homogeneous algebraic forms of degree k in n variables and $\mathrm{S}(n)$ the infinite dimensional space of the formal series (expansions) :

$$\mathrm{S}(n) \;=\; \bigoplus_{k=1}^{\infty} \mathrm{Sym}(n\,,\,k),$$

where ([14, p. 21])

$$\dim(\mathrm{Sym}(n\,,\,k)) \;=\; \frac{n(n+1)(n+2)\,\cdots\,(n+k-1)}{k!} \;=\; \frac{(n+k-1)!}{(n-1)!k!}.$$

We identify the set of the multi-degrees of the multivariate polynomials with $\mathbb{N}^n$. The total degree of the monomial $(x^1)^{i_1}(x^2)^{i_n}\cdots(x^n)^{i_n}$ is, by definition, equal to $|i| \;=\; i_1 \,+\, i_2 \,+\, \cdots \,+\, i_n$.

The n-fold cartesian set $\mathbb{N}^n$ is provided by the total-lexicographical order :

$$i \,\geq\, j \Longleftrightarrow \begin{cases} |i| \,>\, |j| \\ \text{or} \\ |i| \,=\, |j| \text{ and the left-most nonzero entry of } i-j \quad \text{is} \quad \text{positive,} \end{cases}$$

which induces a total order over the monomials

$$\{(x^1)^{i_1}(x^2)^{i_2}\cdots(x^n)^{i_n}; \quad |i| \;=\; k\}.$$

Any algebraic form of $\mathrm{Sym}(n\,,\,k)$, $\sum_{|i|=k} f_{i_1,i_2,\ldots,i_n}\,(x^1)^{i_1}(x^2)^{i_2}\cdots(x^n)^{i_n}$, can be written as follows :

$$[f_{k,0,0,\ldots,0},\, f_{k-1,1,0,\ldots,0},\, f_{k-1,0,1,\ldots,0},\ldots,\, f_{0,0,0,\ldots,k}] \begin{bmatrix} (x^1)^k \\ (x^1)^{k-1}(x^2)^1 \\ (x^1)^{k-1}(x^3)^1 \\ \vdots \\ (x^n)^k \end{bmatrix} = F_k\,X^k, \quad (3)$$

where F_k and X^k denote the corresponding row and column vectors. Using this notation, the formal series $\sum_{k=1}^{\infty}\sum_{|i|=k} f_{i_1,i_2,\ldots,i_n}\,(x^1)^{i_1}(x^2)^{i_2}\cdots(x^n)^{i_n}$ and the right side of (1) with vanishing linear part become respectively

$$F_1\,X^1 + F_2\,X^2 + \cdots + F_k\,X^k + \cdots\,, \qquad P_l\,X^l + P_{l+1}\,X^{l+1} + \cdots + P_m\,X^m, \quad (4)$$

where, this time, P_i $(i = l, l+1, \ldots, m)$ denotes a matrix with n rows and $\dfrac{(n+i-1)!}{(n-1)!i!}$ columns.

2.2 The integrability conditions

The formal series $F(x) = F_1\,X^1 + F_2\,X^2 + \cdots + F_k\,X^k + \cdots$ is a first integral of the system (1) if, by definition, $\Delta_P(F) = 0$ i.e.

$$\sum_{j=1}^{n} \frac{\partial(F_1\,X^1 + F_2\,X^2 + \cdots)}{\partial x^j}\Big[P_l^j X^l + P_{l+1}^j X^{l+1} + \cdots + P_m^j X^m\Big] = 0.$$

After collecting the terms w.r.t. the total degree, we obtain an infinite sequence of conditions :

$$\begin{aligned}
l:&\quad \sum_{j=1}^{n} \frac{\partial(F_1X^1)}{\partial x^j}\,P_l^j X^l = 0,\\
l{+}1:&\quad \sum_{j=1}^{n}\Big[\frac{\partial(F_1X^1)}{\partial x^j}\,P_{l+1}^j X^{l+1} + \frac{\partial(F_2X^2)}{\partial x^j}\,P_l^j X^l\Big] = 0,\\
l{+}2:&\quad \sum_{j=1}^{n}\Big[\frac{\partial(F_1X^1)}{\partial x^j}\,P_{l+2}^j X^{l+2} + \frac{\partial(F_2X^2)}{\partial x^j}\,P_{l+1}^j X^{l+1} + \frac{\partial(F_3X^3)}{\partial x^j}\,P_l^j X^l\Big] = 0,\\
&\vdots \qquad\qquad\qquad \vdots\\
m:&\quad \sum_{j=1}^{n}\Big[\frac{\partial(F_1X^1)}{\partial x^j}P_m^j X^m + \frac{\partial(F_2X^2)}{\partial x^j}P_{m-1}^j X^{m-1} + \cdots + \frac{\partial(F_{m-l+1}X^{m-l+1})}{\partial x^j}P_l^j X^l\Big] = 0,\\
m{+}1:&\quad \sum_{j=1}^{n}\Big[\frac{\partial(F_2X^2)}{\partial x^j}P_m^j X^m + \frac{\partial(F_3X^3)}{\partial x^j}P_{m-1}^j X^{m-1} + \cdots + \frac{\partial(F_{m-l+2}X^{m-l+2})}{\partial x^j}P_l^j X^l\Big] = 0,\\
&\vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

Putting $p = \min(k, m)$, the equation corresponding to the total degree k in x is :

$$\sum_{j=1}^{n} \Big[\frac{\partial (F_{k+1-p} X^{k+1-p})}{\partial x^j} P_p^j X^p + \frac{\partial (F_{k+2-p} X^{k+2-p})}{\partial x^j} P_{p-1}^j X^{p-1} + \\ + \cdots + \frac{\partial (F_{k-l+1} X^{k-l+1})}{\partial x^j} P_l^j X^l \Big] = 0. \qquad (5)$$

Since the equation (5) is homogeneous of degree k in the coordinates of x, there are $N = p - l + 1$ matrices, denoted $M_{[i,k]}$ $(i = k-p+1, \ldots, k-l+1)$, such that the previous equation can be rewritten in the form :

$$\Big[F_{k-p+1} M_{[k-p+1,k]} + F_{k-p+2} M_{[k-p+2,k]} + \cdots + F_{k-l+1} M_{[k-l+1,k]} \Big] X^k = 0. \qquad (6)$$

For the differential systems of lower degree l, $k = l,\ l+1,\ l+2, \ldots$, we get :

$$\begin{cases} F_1\, M_{[1,\,l]} = 0 \\ F_1\, M_{[1,\,l+1]} + F_2\, M_{[2,\,l+1]} = 0 \\ F_1\, M_{[1,\,l+2]} + F_2\, M_{[2,l+2]} + F_3\, M_{[3,l+2]} = 0 \\ \qquad\qquad\qquad\qquad \vdots \\ \qquad\qquad\qquad\qquad \vdots \end{cases} \qquad (7)$$

The matrix $M_{[i,k]}$ has exactely $\dfrac{(n+i-1)!}{(n-1)!i!}$ rows and $\dfrac{(n+k-1)!}{(n-1)!k!}$ columns.

We give some examples of the matrices $M_{[d,l+d-1]}$ when $n = 2$.

<u>l = 2</u>; $P^1(x) = \sum_{i=0}^{2} \binom{2}{i} a_i (x^1)^{2-i} (x^2)^i$, $P^2(x) = \sum_{i=0}^{2} \binom{2}{i} b_i (x^1)^{2-i} (x^2)^i$.

$$M_{[1,2]} = \begin{bmatrix} a_0 & 2a_1 & a_2 \\ b_0 & 2b_1 & b_2 \end{bmatrix}, \qquad M_{[2,3]} = 2 \begin{bmatrix} a_0 & 2a_1 & a_2 & 0 \\ b_0 & a_0 + 2b_1 & 2a_1 + b_2 & a_2 \\ 0 & b_0 & 2b_1 & b_2 \end{bmatrix},$$

$$M_{[3,4]} = 3 \begin{bmatrix} a_0 & 2\,a_1 & a_2 & 0 & 0 \\ b_0 & 2\,b_1 + 2\,a_0 & b_2 + 4\,a_1 & 2\,a_2 & 0 \\ 0 & 2\,b_0 & 4\,b_1 + a_0 & 2\,b_2 + 2\,a_1 & a_2 \\ 0 & 0 & b_0 & 2\,b_1 & b_2 \end{bmatrix},$$

$$M_{[4,5]} = 4 \begin{bmatrix} a_0 & 2\,a_1 & a_2 & 0 & 0 & 0 \\ b_0 & 3\,a_0 + 2\,b_1 & b_2 + 6\,a_1 & 3\,a_2 & 0 & 0 \\ 0 & 3\,b_0 & 6\,b_1 + 3\,a_0 & 3\,b_2 + 6\,a_1 & 3\,a_2 & 0 \\ 0 & 0 & 3\,b_0 & 6\,b_1 + a_0 & 2\,a_1 + 3\,b_2 & a_2 \\ 0 & 0 & 0 & b_0 & 2\,b_1 & b_2 \end{bmatrix}.$$

__l = 3__; $P^1(x) = \sum_{i=0}^{3} \binom{3}{i} a_i (x^1)^{3-i}(x^2)^i$, $P^3(x) = \sum_{i=0}^{3} \binom{3}{i} b_i (x^1)^{3-i}(x^2)^i$.

$$M_{[1,3]}=\begin{bmatrix} a_0 & 3\,a_1 & 3\,a_2 & a_3 \\ b_0 & 3\,b_1 & 3\,b_2 & b_3 \end{bmatrix}, M_{[2,4]}=2\begin{bmatrix} a_0 & 3a_1 & 3a_2 & a_3 & 0 \\ b_0 & 3b_1+a_0 & 3b_2+3a_1 & b_3+3a_2 & a_3 \\ 0 & b_0 & 3\,b_1 & 3b_2 & b_3 \end{bmatrix},$$

$$M_{[3,5]} = 3\begin{bmatrix} a_0 & 3\,a_1 & 3\,a_2 & a_3 & 0 & 0 \\ b_0 & 2\,a_0+3\,b_1 & 3\,b_2+6\,a_1 & b_3+6\,a_2 & 2\,a_3 & 0 \\ 0 & 2\,b_0 & 6\,b_1+a_0 & 6\,b_2+3\,a_1 & 3\,a_2+2\,b_3 & a_3 \\ 0 & 0 & b_0 & 3\,b_1 & 3\,b_2 & b_3 \end{bmatrix},$$

$$M_{[4,6]} = 4\begin{bmatrix} a_0 & 3\,a_1 & 3\,a_2 & a_3 & 0 & 0 & 0 \\ b_0 & 3\,a_0+3\,b_1 & 9\,a_1+3\,b_2 & b_3+9\,a_2 & 3\,a_3 & 0 & 0 \\ 0 & 3\,b_0 & 9\,b_1+3\,a_0 & 9\,b_2+9\,a_1 & 3\,b_3+9\,a_2 & 3\,a_3 & 0 \\ 0 & 0 & 3\,b_0 & 9\,b_1+a_0 & 3\,a_1+9\,b_2 & 3\,a_2+3\,b_3 & a_3 \\ 0 & 0 & 0 & b_0 & 3\,b_1 & 3\,b_2 & b_3 \end{bmatrix}.$$

Proposition 1. *If the differential system (1) has a formal first integral of lower degree d, then :*

$$rank\,(M_{[d,l+d-1]}) < \frac{(d+n-1)!}{(n-1)!\,d!}.$$

The existence of the formal first integral of lower degree d implies that the linear system $F_d\, M_{[d,l+d-1]} = 0$ admits a non-vanishing solution. If for any d, $rank\,(M_{[d,l+d-1]}) = \dfrac{(d+n-1)!}{(n-1)!\,d!}$, the differential system (1) hasn't a formal first integral.

2.3 The case of the lower degree $l = 1$ ([9])

When $l = 1$, the matrices $M_{[d,l+d-1]} = M_{[d,d]}$ are square.
Let $A = (A^i_j)_{1\le i,j\le n}$ be the matrix of the linear part of the differential system (1).

Proposition 2. *[9] The matrix $L = M_{[d,d]}$ is defined by :*

$$\begin{cases} L^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n} = 0 & \text{if } |i-j| > 2, \\ L^{i_1 i_2 \ldots i_n}_{i_1 \ldots (i_l-1)\ldots(i_q+1)\ldots i_n} = i_l A^l_q\,, & \\ L^{i_1 i_2 \ldots i_n}_{i_1 \ldots (i_l+1)\ldots(i_q-1)\ldots i_n} = i_q A^q_l\,, & \\ L^{i_1 i_2 \ldots i_n}_{i_1 i_2 \ldots i_n} = (i_1 A^1_1 + i_2 A^2_2 + \cdots + i_n A^n_n). & \end{cases} \tag{8}$$

Corollary 3. *[9] The matrix $L = M_{[d,d]}$ is diagonal (respectively lower triangular, upper triangular) for any $d = 1, 2, 3, \ldots$, if and only if the matrix A is diagonal (respectively lower triangular, upper triangular).*

Corollary 4. *[9] If the eigenvalues of the matrix A are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of the matrix $L = M_{[d,d]}$ have the form :*

$$i_1\lambda_1 + i_2\lambda_2 + \cdots + i_n\lambda_n,$$

where $i_1, i_2, \ldots i_n \in \mathbb{N}$ and $i_1 + i_2 + \ldots + i_n = d$.

From Corollary 2, it follows

$$det(M_{[d,d]}) = \prod_{i_1+i_2+\cdots+i_n=d} \Big(i_1\lambda_1 + i_2\lambda_2 + \cdots + i_n\lambda_n\Big) \tag{9}$$

It is clear that the existence of a formal first integral (of lower degree d) of (1) with $A \neq 0$, implies the existence of a non-negative integer d such that $F_d\, M_{[d,d]} = 0$ has a nonzero solution F_d, i.e. $det(M_{[d,d]}) = 0$.

The factors $\Big(i_1\lambda_1 + i_2\lambda_2 + \cdots + i_n\lambda_n\Big)$ can be regrouped in orbits $\mathcal{O}(i_1, i_2, \ldots, i_n)$ with respect to the action of the symmetric group over the multidegrees. These orbits are represented by the partitions of d in not more than n parts:

$$det(M_{[d,d]}) = \prod_{\substack{i_1+i_2+\cdots+i_n=d \\ i_1 \geq i_2 \geq \cdots \geq i_n \geq 0}} \left[\prod_{(j_1,j_2,\ldots,j_n)\in\mathcal{O}(i_1,i_2,\ldots,i_n)} \Big(j_1\lambda_1 + j_2\lambda_2 + \cdots + j_n\lambda_n\Big)\right].$$

As a symmetric function, the polynomial

$$\mathcal{R} = \prod_{(j_1,j_2,\ldots,j_n)\in\mathcal{O}(i_1,i_2,\ldots,i_n)} \Big(j_1\lambda_1 + j_2\lambda_2 + \cdots + j_n\lambda_n\Big)$$

belongs to the ring $\mathbb{Z}[\sigma_1, \sigma_2, \ldots, \sigma_n]$ where $\sigma_i = \sum_{1\leq j_1 \leq j_2 \cdots \leq j_n \leq n} \lambda_{j_1} \ldots \lambda_{j_i}$.

The equation $\mathcal{R} = 0$ is called the starting equation of the existence of the formal first integrals. Its integer solutions give the lower degree of the eventual first integral.

Some generic examples

1. The partition $(d, 0, \ldots, 0)$ corresponds to the factor $\det(A)$.
2. When $d = kn$, the partition $(k, k, \ldots, k)$ represents a one element orbit. So, the factor that corresponds to this partition is $k^n \mathrm{trace}(A)$.
3. When $n = 2$, the starting equation is

   $$\Big[d_1 d_2 \Big(\mathrm{trace}(A)\Big)^2 + (d_1 - d_2)^2 \det(A)\Big] = 0.$$

   In [9], the case of the dimension 3 is also detailed and for other dimensions, a procedure for obtaining the starting equation is given.

Remark 1. The above Diophantine equation (9) can be found under various aspects in many works ([4,7,8]). For example, in [7], when the linear part A is not zero, the author wrote : "*If system (1) has nontrivial integrals analytic in a neighbourhood of a trivial solution $x = 0$, then eigenvalues of the matrix A have to satisfy certain resonant conditions*".

The starting equation gives namely an achieved form of these resonant conditions.

2.4 A consequence (strong condition) for the homogeneous polynomial systems

Let's return to the systems (1) which we suppose homogeneous of degree m :

$$\frac{dx^i}{dt} = P^i(x), \quad i = 1, 2, \ldots, n. \tag{10}$$

Denote by $Jac(x)$ the Jacobian matrix of P and by $J(x)$ and $T(x)$ respectively the determinant and the trace of the matrix $Jac(x)$.
A polynomial first integral $F_d X^d$ satisfies the relation

$$\sum_{j=1}^{n} \frac{\partial(F_d X^d)}{\partial x^j} \Big[P^j(x)\Big] = \sum_{j=1}^{n} \sum_{|i|=k} i_j f_{i_1,\ldots,i_n} (x^1)^{i_1} \cdots (x^j)^{i_j-1} \ldots (x^n)^{i_n} = 0. \tag{11}$$

Using the Euler's formulae

$$P^j(x) = \frac{1}{m} \sum_{k=1}^{n} x^k \frac{\partial(P^j(x))}{\partial x^k} = \frac{1}{m} \sum_{k=1}^{n} [Jac(x)]^j_k \, x^k,$$

the relation (11) becomes :

$$F_d L_d(x) X^d = 0$$

where the matrix $L_d(x)$ has the same structure as the matrix L when $l = 1$ (see Proposition 2). This is due to the substitution $A^j_k := [Jac(x)]^j_k$ which leads to the matrix $M_{[d,d]}(x)$ and thus, to the starting equation $\mathcal{R}(x)$, depending on x.

Proposition 5. *Suppose that the starting equation $\mathcal{R}(x) = 0$ admits an integer n-tuple solution $(d_1, d_2, \ldots, d_n)$ and suppose that the linear equation $F_d \cdot L_d(x) = 0$ has a constant solution F (not depending on x). Then, the system (10) has a polynomial first integral of total degree d.*

Corollary 6. *When $n = 2$, the strong condition is given by the equation*

$$\Big[d_1 d_2 \Big(T(x)\Big)^2 + (d_1 - d_2)^2 \, J(x)\Big] = 0. \tag{12}$$

There are two particular cases of the above Corollary 6.

First case: $n = 2$ and $T(x) = 0$. Hence, $d_1 = d_2$.
We see later that in this case, the polynomial $x^2 P^1(x) - x^1 P^2(x)$ is a first integral.

This case can be extended to the $2n$-dimensional differential systems

$$\frac{dx^i}{dt} = P^i(x,y), \quad \frac{dy^i}{dt} = Q^i(x,y), \qquad i = 1,2,\ldots,n$$

satisfying the conditions

$$\frac{\partial P^i(x,y)}{\partial x^i} + \frac{\partial Q^i(x,y)}{\partial y^i} = 0, \quad (i = 1,2,\ldots n).$$

The polynomial function $\sum_{i=1}^{n}\Big(y^i P^i(x,y) - x^i Q^i(x,y)\Big)$ is a first integral. Among these systems we find (of course) the homogeneous Hamiltonian ones

$$\frac{dx^i}{dt} = -\frac{\partial H(x,y)}{\partial y^i}, \quad \frac{dy^i}{dt} = \frac{\partial H(x,y)}{\partial x^i}.$$

Remark 2. In general (see Proposition 1.16 from [2]), the condition $J(x) = 0$ implies the algebraic dependence $W(P^1, P^2, \ldots, P^n) = 0$ where W is a multivariate polynomial with coefficients in $\mathbb{C}$.

Furthemore, if the polynomials $P^1, P^2, \ldots, P^n$ are homogeneous of the same degree, the polynomial W is necessary homogeneous.

Second case: $n = 2$ and $J(x) = 0$.

From the above remark and the homogeneity of W, we have : $W(x) = \prod_{i=1}^{k}(\alpha_i x^1 + \beta_i x^2)$ and so,

$$W(P^1, P^2) = 0 \Longrightarrow \exists\, \alpha, \beta \in \mathbb{C};\ \alpha P^1(x) + \beta P^2(x) = 0.$$

Thus, the linear form $\alpha x^1 + \beta x^2$ is a first integral.

3 The bidimensional homogeneous systems

We have seen that the starting equations of (1) depend only on the homogeneous part of lower degree of (1). Let's consider the homogeneous differential systems of (total) degree l.

$$\frac{dx}{dt} = A(x,y) = \sum_{i=0}^{l}\binom{l}{i} a_i x^{l-i} y^i\,, \quad \frac{dy}{dt} = B(x,y) = \sum_{i=0}^{l}\binom{l}{i} b_i x^{l-i} y^i, \quad (13)$$

where $A(x,y)\,,\ B(x,y) \in \mathbb{C}_l[x\,,\,y]$.
Because of the homogeneity of the polynomials A and B, a formal first integral is necessarily a polynomial one which satisfies the equation $F_d M_{[d,d+l-1]} X^{l+l-1} = 0$.

By the proposition 1, a necessairy and sufficient condition for finding a nontrivial solution F_d is :

$$rank\Big(M_{[d,l+d-1]}\Big) < d+1.$$

In the following section we compute concretely the matrices $M_{[d,l+d-1]}$, we establish the equivalence between the rank condition and the nullity of some determinant and finally, by using the classical theory, we reduce the computation of this determinant to that of its leading term.

3.1 Some basic facts about the integrals of homogeneous planar differential systems

The following results are wellknown.

Lemma 7. *Let $H(x,y) = H_1(x,y)H_2(x,y)$ be a factorisation of the polynomial H into two coprime polynomials H_1 and H_2. The polynomial H is a partial integral of (13) if and only if H_1 and H_2 are also partial integrals of (13).*

Proof. [4] (Lemma 2.2, p. 8).

Proposition 8. *Let $K(x,y)$ be the polynomial $yA(x,y) - xB(x,y)$. Then*

$$\frac{\partial K(x,y)}{\partial x}A(x,y) + \frac{\partial K(x,y)}{\partial y}B(x,y) = Div(A,B)\,K(x,y)$$

where $Div(A,B)$ is the divergence of the vector field (A,B).

Proof. Putting $A_x = \frac{\partial A}{\partial x}$, $A_y = \frac{\partial A}{\partial y}$, $B_x = \frac{\partial B}{\partial x}$, $B_y = \frac{\partial B}{\partial y}$ and using the Euler's formulae, we get

$$\frac{\partial K(x,y)}{\partial x}A(x,y) + \frac{\partial K(x,y)}{\partial y}B(x,y) = (yA_x - xB_x - B)A + (yA_y + A - xB_y)B$$

$$= (A_x + B_y)K(x,y) + B(xA_x + yA_y) - A(xB_x + yB_y) = Div(A,B)K(x,y).$$

Remark 3. Each factor of $K(x,y)$ is a partial integral.

Proposition 9. *The line $\alpha\, x + \beta y = 0$ is an invariant curve for the system (13) if and only if $K(\beta,-\alpha) = 0$.*

Proof. The sufficient condition follows immediately from the fact that

$$K(x,y) = \prod_{i=1}^{m+1}(\alpha_i x + \beta_i y)$$

and the previous results.
Let $\alpha\, x + \beta y = 0$ be the equation of the line. The point $(\beta,-\alpha)$ belongs to the line and so, $\alpha A(\beta,-\alpha) + \beta B(\beta,-\alpha) = 0$.

Corollary 10. *If $Div(A,B) = 0$, then $K(x,y)$ is a first integral,*

Proposition 11. *If $K(x,y) \neq 0$, then $\frac{1}{K(x,y)}$ is an integrating factor for the system (13).*

Proof. Directly from the definition of the integrating factor.

3.2 Computation of the matrices $M_{[d,l+d-1]}$

Proposition 12. *Let $M_{[d,l+d-1]}$ be the matrix defined in Section 2.2. With the assumption $a_{-1} = b_{l+1} = 0$ and $\binom{i}{k} = 0$ for any $j = 0,1,2,\dots d$, $r = 0,1,2,\dots,l+d-1$ such that $i < k$, we have:*

$$\left[M_{[d,l+d-1]}\right]_r^j = d\left[\binom{d-1}{j}\binom{l}{r-j}a_{r-j} + \binom{d-1}{j-1}\binom{l}{r+1-j}b_{r+1-j}\right]. \quad (14)$$

Proof. The polynomial $F(x,y) = \sum_{j=0}^{d}\binom{d}{j} f_j x^{d-j}y^j$ is a first integral if and only if

$$F_d M_{[d,l+d-1]} X^{l+d-1} = \frac{\partial F(x,y)}{\partial x} A(x,y) + \frac{\partial F(x,y)}{\partial y} B(x,y) =$$

$$= d\sum_{i=0}^{l}\binom{l}{i}\left[a_i\sum_{j=0}^{d-1}\binom{d-1}{j}x^{l+d-i-j-1}f_j\,y^j + b_i\sum_{j=1}^{d}\binom{d-1}{j-1}x^{l+d-i-j}f_j y^{j-1}\right]y^i =$$

$$= dx^{d-1}f_0\sum_{r=0}^{l}\binom{l}{r}a_r x^{l-r}y^r \; + d\sum_{j=1}^{d-1}f_j\Big[\sum_{r=j}^{l+j}\binom{d-1}{j}\binom{l}{r-j}a_{r-j}\,x^{l+d-r-1}y^r$$

$$+\sum_{r=j-1}^{l+j-1}\binom{d-1}{j-1}\binom{l}{r-j+1}b_{r-j+1}\,x^{l+d-r-1}y^r\Big] + d\,y^{d-1}\,f_d\sum_{r=0}^{l}\binom{l}{r}b_r x^{l-r}y^r = 0.$$

The element $\left[M_{[d,l+d-1]}\right]_r^j$ is the coefficient of $f_j\,y^r$ ($j = 0,1,2,\dots d$ and $r = 0,1,2,\dots,l+d-1$) in the last expression. More precisely :

$$\begin{aligned}
\left[M_{[d,l+d-1]}\right]_r^0 &= d\binom{l}{r}a_r \text{ if } 0 \le r \le l,\\
\left[M_{[d,l+d-1]}\right]_{j-1}^j &= d\binom{d-1}{j-1}b_0 \text{ if } 1 \le j \le d-1,\\
\left[M_{[d,l+d-1]}\right]_r^j &= d\left[\binom{d-1}{j}\binom{l}{r-j}a_{r-j} + \binom{d-1}{j-1}\binom{l}{r+1-j}b_{r+1-j}\right]\\
&\qquad \text{if } 1 \le j \le d-1 \text{ and } j \le r \le l+j-1,\\
\left[M_{[d,l+d-1]}\right]_{l+j}^j &= d\binom{d-1}{j}a_l \text{ if } 1 \le j \le d-1,\\
\left[M_{[d,l+d-1]}\right]_r^d &= d\binom{l}{r}b_r \text{ if } d-1 \le r \le l+d-1,\\
\left[M_{[d,l+d-1]}\right]_r^j &= 0 \quad \text{elsewhere}
\end{aligned}$$

where $j = 0, 1, 2, \ldots d$ and $r = 0, 1, 2, \ldots, l+d-1$. These expressions can be rewritten using the copact form given in the proposition.

Corollary 13. *For all* $d \in \{1, 2, \ldots, \}$, $i \in \{1, 2, \ldots, d\}$, *and* $j \in \{1, 2, \ldots, l + d - 1\}$, *we have :*

$$d \Big[M_{[d+1,l+d]} \Big]_j^i = (d+1)\Big(\Big[M_{[d+1,l+d]} \Big]_j^i + \Big[M_{[d+1,l+d]} \Big]_{j-1}^{i-1} \Big).$$

By Proposition 1, the condition on the rank requires the computation of $\begin{pmatrix} d+l \\ d+1 \end{pmatrix}$ minors. The aim of the following subsection is to reduce the computation of these minors to that of one and only one determinant of some matrix.

3.3 Reduction to a square matrix

Proposition 14. *The polynomial* $F_d M_{[d,l+d-1]} X^{l+d-1} \in \mathbb{C}[x, y]$ *vanishes identically if and only if the polynomial*

$$\sum_{k=0}^{l-1} \begin{pmatrix} l-1-k \\ k \end{pmatrix} \frac{\partial^{l-1} F_d \, M_{[d,l+d-1]} \, X^{l+d-1}}{\partial x^{l-1-k} \, \partial y^k} \, u^{l-1-k} v^k \tag{15}$$

vanishes in $\mathbb{C}[x, y, u, v]$.

Proof. With the help of the Euler's formulae,

$$\frac{\partial Q(x,y)}{\partial x} x + \frac{\partial Q(x,y)}{\partial y} y = (l + d - 1) \, Q(x, y)$$

where $Q = F_d \, M_{[d,l+d-1]} \, X^{l+d-1}$, we remark that the homogeneous polynomial Q vanishes if and only if $\dfrac{\partial Q(x,y)}{\partial x} = \dfrac{\partial Q(x,y)}{\partial y} = 0$. By the same way, we claim that the polynomial Q vanishes if and only if all its derivatives of order $l - 1$ vanish.

In the following, we denote by $S_{d,k}$ the $(d+1) \times (d+1)$-matrix such that

$$F_d \, S_{d,k} \, X^d = \frac{\partial^{l-1} \Big(F_d \, M_{[d,l+d-1]} \, X^{l+d-1} \Big)}{\partial x^{l-1-k} \, \partial y^k}$$

and by $S_{[d]}(u\, , \, v)$ the matrix $\sum_{k=0}^{l-1} \begin{pmatrix} l-1-k \\ k \end{pmatrix} S_{d,k} \, u^{l-1-k} v^k$.

Proposition 15. *The differential system (13) has a polynomial first integral of degree d if and only if*

$$\det S_{[d]}(u, v) = 0$$

Examples of matrices $S_{[d]}(u,v)$.

<u>**l = 2 :**</u>

$$S_{[1]}(u,v)=\begin{bmatrix}2a_0 & 2a_1\\ 2b_0 & 2b_1\end{bmatrix}u+\begin{bmatrix}2a_1 & 2a_2\\ 2b_1 & 2b_2\end{bmatrix}v,$$

$$S_{[2]}(u,v)=\begin{bmatrix}3a_0 & 4a_1 & a_2\\ 3b_0 & 2a_0+4b_1 & 2a_1+b_2\\ 0 & 2\,b_0 & 2b_1\end{bmatrix}u+\begin{bmatrix}2\,a_1 & 2a_2 & 0\\ a_0+2b_1 & 4a_1+2b_2 & 3a_2\\ b_0 & 4b_1 & 3b_2\end{bmatrix}v,$$

$S_{[3]}(u,v)=$

$$\begin{bmatrix}4a_0 & 6a_1 & 2a_2 & 0\\ 4b_0 & 6b_1+6a_0 & 8a_1+2b_2 & 2a_2\\ 0 & 6b_0 & 2a_0+8b_1 & 2b_2+2a_1\\ 0 & 0 & 2\,b_0 & 2b_1\end{bmatrix}u+\begin{bmatrix}2a_1 & 2a_2 & 0 & 0\\ 2b_1+2a_0 & 8a_1+2b_2 & 6a_2 & 0\\ 2b_0 & 2a_0+8b_1 & 6b_2+6a_1 & 4a_2\\ 0 & 2b_0 & 6b_1 & 4b_2\end{bmatrix}v.$$

<u>**l = 3 :**</u>

$$S_{[1]}(u,v)\begin{bmatrix}3\,a_0 & 3\,a_1\\ 3\,b_0 & 3\,b_1\end{bmatrix}u^2+\begin{bmatrix}6\,a_1 & 6\,a_2\\ 6\,b_1 & 6\,b_2\end{bmatrix}uv+\begin{bmatrix}3\,a_2 & 3\,a_3\\ 3\,b_2 & 3\,b_3\end{bmatrix}v^2,$$

$S_{[2]}(u,v)=$

$$\begin{bmatrix}6\,a_0 & 9\,a_1 & 3\,a_2\\ 6\,b_0 & 3\,a_0+9\,b_1 & 3\,b_2+3\,a_1\\ 0 & 3\,b_0 & 3\,b_1\end{bmatrix}u^2+\begin{bmatrix}9\,a_1 & 12\,a_2 & 3\,a_3\\ 3\,a_0+9\,b_1 & 12\,b_2+12\,a_1 & 9\,a_2+3\,b_3\\ 3\,b_0 & 12\,b_1 & 9\,b_2\end{bmatrix}uv+$$

$$+\begin{bmatrix}3\,a_2 & 3\,a_3 & 0\\ 3\,b_2+3\,a_1 & 9\,a_2+3\,b_3 & 6\,a_3\\ 3\,b_1 & 9\,b_2 & 6\,b_3\end{bmatrix}v^2.$$

3.4 Computation of the matrices $S_{d,k}$

Starting from the polynomial

$$F_d\,M_{[d,l+d-1]}\;X^{l+d-1}\;=\sum_{i=0}^{d}\sum_{j=0}^{l+d-1}f_i\Big(M_{[d,l+d-1]}\Big)_j^i x^{l+d-1-j}y^j,$$

we get

$$\frac{\partial^{l-1}F_d\,M_{[d,l+d-1]}X^{l+d-1}}{\partial^{l-1-k}x\partial^k y}=\sum_{i=0}^{d}\sum_{j=k}^{k+d}f_i\Big(M_{[d,l+d-1]}\Big)_j^i\frac{(l+d-1-j)!j!}{(d+k-j)!(j-k)!}x^{d+k-j}y^{j-k}$$

$$=\sum_{i=0}^{d}\sum_{p=0}^{d}f_i\Big(M_{[d,l+d-1]}\Big)_{k+p}^i\frac{(l+d-1-k-p)!}{(d-p)!}\,\frac{(k+p)!}{p!}\,x^{d-p}y^p.$$

Proposition 16. *The* $(d+1)\times(d+1)$*-matrix* $S_{d,k}(u\,,\,v)$ *(for* $k\in\{0,1,2,...,l-1\}$*) is defined by :*

$$\Big(S_{d,k}\Big)^i_p = \Big(M_{[d,l+d-1]}\Big)^i_{k+p}\,\frac{(l+d-1-k-p)!}{(d-p)!}\,\frac{(k+p)!}{p!}. \tag{16}$$

Proof. The coefficient of $f_i\,y^p$ corresponds to the coefficient $(S_{d,k})^i_p$.

Consequently, by Proposition 15, the differential system (13) admits a polynomial first integral if and only if there exists a positive integer d such that

$$det(S_{[d]}(u,v)){=}s^d_0u^{(l-1)(d+1)}{+}s^d_1u^{(l-1)(d+1)-1}v{+}\cdots{+}s^d_{(l-1)(d+1)}v^{(l-1)(d+1)}=0. \tag{17}$$

For verifying the condition $rank\Big(M_{[d,l+d-1]}\Big) < d+1$, we must compute $N_1 = \binom{d+l}{d+1}$ minors, but for verifying the condition $det(S_{[d]}(u,v)) = 0$, we need the computation of $N_2=(d+1)(l-1)+1$ expressions. The next table shows how the difference N_1-N_2 increases w.r.t. the degrees l and d.

l \ d	1	2	3	4	5	6	7	8	9	10
2	0	0	0	0	0	0	0	0	0	0
3	3	6	10	15	21	28	36	45	55	66
4	7	16	30	50	77	112	156	210	275	352
5	12	31	65	120	203	322	486	705	990	1353
6	18	52	121	246	455	784	1278	1992	2992	4356
7	25	82	205	456	917	1708	2994	4995	7997	12364
8	33	116	325	786	1709	3424	6424	11430	19437	31812

Remark 4.

1. When $l=2$, the coefficients of the determinant (17) correspond to the minors of the matrix $M_{[d,d+1]} = [C_1,C_2,\ldots,C_{d+2}]$, rewritten with the columns $C_i\,;\,i=1,2,\ldots,d+2$:

   $$s^d_k=\lambda_i\,det([C_1,C_2,\ldots,C_{d+1-k},\check{C}_{d+2-k},C_{d+3-k},\ldots,C_{d+2}])$$

   for $k=0,\ldots,d+1$ and $\lambda_i\in\mathbb{N}$.
   Here, the symbol "ˇ" means the removing of the corresponding column.

2. When $l>2$, we have :

   $$s^d_0=\lambda_0\,det([C_1,C_2,\ldots,C_{d+1}])$$
   $$s^d_1=\lambda_1 det([C_1,C_2,\ldots,\hat{C}_{d+1},C_{d+2}])$$

It is obvious that when the degree d increases, the computation of the determinant $\det(S_{[d]}(u,v))$ becomes more and more complicated. However, by using the classical invariant theory, we will show that from the knowledge of the leading term s^d_0, we can deduce that of the other terms s^d_1, $s^d_2,\ldots,s^d_{d+1}$.

4 The computation of the determinant $\det(S_{[d]}(u, v))$ by using the classical invariant theory

4.1 Introduction to the classical invariant theory ([10, 12, 14])

Let $(\mathbb{C}^n)^*$ be the dual of the vector space $\mathbb{C}^n$. The linear space of the differential systems (13) can be looked upon as the tensorial product $S(n\,,\,m) \otimes (\mathbb{C}^n)^*$ denoted by $\mathcal{S}^1_m$. For example, $\mathcal{S}^1_1$ is the linear differential systems. In tensorial language, $\mathcal{S}^1_m$ is the space of tensors once contravariant and m times covariant which are symmetric with respect to the lower indices.

Let G be the linear group acting rationally on a finite-dimensional vector space $\mathcal{W}$, $GL(\mathcal{W})$ the group of automorphisms of $\mathcal{W}$ and

$$\rho \ : G \longmapsto GL(\mathcal{W})$$

the corresponding rational representation. Let $\mathbb{C}[\mathcal{W}]$ be the algebra of polynomials whose indeterminates are the coordinates of a generic vector of $\mathcal{W}$.

Definition 2. *A polynomial function $K \in \mathbb{C}[\mathcal{W}]$ is said to be a G-invariant of $\mathcal{W}$ if there exists a character of the group G, denoted λ, such that*

$$\forall g \in G, \quad K \circ \rho(g) = \lambda(g).K.$$

Here, the character of the group G is a rational (commutative) morphism of group G into $\mathbb{C}_m$ where $\mathbb{C}_m$ is the multiplicative group of $\mathbb{C}$.
If $\lambda(g) \equiv 1$, the invariant is said absolute. Otherwise, it is relative.

Definition 3. *A $GL(n, \mathbb{C})$-invariant of $\mathcal{S}^1_m$ is a $GL(n, \mathbb{C})$-invariant of the linear space*
$\mathcal{W} = \mathcal{S}^1_m$.

A $GL(n, \mathbb{C})$-covariant of $\mathcal{S}^1_m$ is a $GL(n, \mathbb{C})$-invariant of the linear space $\mathcal{W} = \mathcal{S}^1_m \times (\mathbb{C}^n)$.

When G is a subgroup of $GL(n, \mathbb{C})$, $\lambda(g) = (\det\, g)^{-\kappa}$ with the so called weight $\kappa \in \mathbb{Z}$.

Remark 5. If $G = SL(n, \mathbb{C})$, all the covariants are absolute.

Remark 6. A polynomial $K \in \mathbb{C}[\mathcal{S}^1_m]$ is a $GL(n, \mathbb{C})$ covariant if and only if it is a $SL(n, \mathbb{C})$ covariant.

Examples

1. Concerning the $GL(n, \mathbb{C})$-invariants, take for example the trace and the determinant of the linear part of (1).
2. The divergence and the jacobian determinant of the vector field P are the simplest $GL(n, \mathbb{C})$-covariants of the space of the differential systems (1).

For more details, see [10–12, 14, 15].

The following theorem gives a procedure for calculating the generators of the the $GL(n, \mathbb{C})$-covariants, step by step increasing the degree.

Theorem 1 (Fundamental Theorem of the classical invariant theory)**.** *The expressions obtained with the help of succesive alternations and complete contraction over the tensorial products*

$$(\mathcal{S}_m^1)^{\otimes p} \otimes (\mathcal{V})^{\otimes r}$$

form a system of generators of the algebra of $GL(2, \mathbb{C})$-covariants of $\mathcal{S}_m^1$.
Such polynomials are called basic covariants.

From now on, we will be ineterested in the bidimensional case.
A $GL(2, \mathbb{C})$-covariant K of degree k (with respect to (x, y)) is a polynomial

$$C_0 x^k + \binom{k}{1} C_1 x^{k-1} y + \cdots + \binom{k}{i} C_i x^{k-i} y^i + \cdots + C_k y^k$$

where the coefficients C_i are homogeneous polynomial functions of degree d depending on coefficients a_j and b_j. The integers k and d satisfy the relation

$$d(m - 1) - k = 2\,\kappa$$

where κ is the weight of the covariant K.

To apply previous Theorem 1, it is useful to introduce the tensorial writing of the algebraic forms and the polynomial differential systems.

Putting $x = x^1$, $y = x^2$ and using Einstein's notation : $\gamma_i \delta^i = \sum_{i=1}^{2} \gamma_i \delta^i$ the polynomials F, A and B, once symmetrised ([10]), become :

$$F(x, y) = \varphi(x) = \varphi_{i_1 i_2 \ldots i_d}\, x^{i_1} x^{i_2} \cdots x^{i_d},$$

$$A(x, y) = \Lambda^1(x) = \alpha^1_{i_1 i_2 \ldots i_d}\, x^{i_1} x^{i_2} \cdots x^{i_m},$$

$$B(x, y) = \Lambda^2(x) = \alpha^2_{i_1 i_2 \ldots i_d}\, x^{i_1} x^{i_2} \cdots x^{i_m},$$

where $\alpha^1_{11..122..2} = a_i$ and $\alpha^2_{11..122..2} = b_i$ (with i "2") and $i_1, i_2, \ldots \in \{1, 2\}$.
Consequently,

$$\begin{aligned} D_{A,B} F(x, y) &= \frac{\partial \varphi(x)}{\partial x^1} \Lambda^1(x) + \frac{\partial \varphi(x)}{\partial x^2} \Lambda^2(x) \\ &= d \varphi_{i_1 i_2 \ldots i_d} \alpha^{i_d}_{j_1 j_2 \ldots j_m} x^{i_1} x^{i_2} \cdots x^{i_{d-1}} x^{j_1} x^{j_2} \cdots x^{j_m} \end{aligned}$$

is a $GL(2, \mathbb{C})$-covariant because it is a total contraction (see Theorem 1).

Proposition 17. *The polynomial $F_d S_{[d]}(u, v) X^d$ is a $GL(2, \mathbb{C})$-covariant.*

Proof. It is wellknown ([14]) that for any covariant $K(x,y)$, the polynomial

$$\frac{\partial K(x,y)}{\partial x}u + \frac{\partial K(x,y)}{\partial y}v$$

is a covariant. This is the polarization process. Repeating this process k times, we obtain again a covariant :

$$\sum_{i=0}^{k}\binom{k}{i}\frac{\partial^k K(x,y)}{\partial x^{k-i}\partial y^i}u^{k-i}v^i.$$

Consequently, since $D_{A,B}F$ is a covariant, the polynomial

$$F_d S_{[d]}(u,v)X^d = \sum_{i=0}^{l-1}\binom{l-1}{i}\frac{\partial^k D_{A,B}F}{\partial x^{k-i}\partial y^i}u^{k-i}v^i$$

is also a covariant.

Corollary 18. *The polynomial $det(S_{[d]}(u,v))$ is a $GL(2,\mathbb{C})$-covariant.*

Proof. Let $Sym(n,d)^*$ be the dual of the linear space $Sym(n,d)$ and $GL(Sym(n,d))$ (resp. $GL(Sym(n,d))^*$) the group of the automorphisms of $Sym(n,d)$ (resp. $GL(Sym(n,d))^*$). We denote the elements of $Sym(n,d)$ by F_d and those of $Sym(n,d)^*$ by X^d.

The change of coordinates $(x,y)^T \leftrightarrow (\overline{x},\overline{y}) = [g^{-1}(x,y)]^T$, where $(x,y)^T$ is the transpose vector of (x,y), induces the linear representations :

$$\Phi\ : GL(2,\mathbb{C}) \to GL(Sym(n,d)) \quad \Psi\ : GL(2,\mathbb{C}) \to GL((Sym(n,d))^*).$$

Since the polynomial F_dX^d is an absolute $GL(2,\mathbb{C})$-covariant, we have :

$$\forall\, g \in GL(2,\mathbb{C}), \quad \Phi(g)\Psi(g) = 1.$$

Hence, the matrix $S_{[d]}(u,v)$ is transformed into $\overline{S}_{[d,d]}(\overline{u},\overline{v}) = \Phi(g)S_{[d]}(u,v)\Phi(g)^{-1}$. Consequently,

$$det(\overline{S}_{[d,d]}(\overline{u},\overline{v})) = det(S_{[d]}(u,v)).$$

4.2 Differential operators

Let's consider the connected component of the identity of the subgroups H_l, H_u and H_d of the lower-triangular, upper-triangular, diagonal matrices parametrized by $\tau \in \mathbb{C}$:

$$H_l = \left\{\begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix}, \tau \in \mathbb{C}\right\}, H_u = \left\{\begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \tau \in \mathbb{C}\right\}, H_d = \left\{\begin{pmatrix} e^{\tau} & 0 \\ 0 & e^{-\tau} \end{pmatrix}, \tau \in \mathbb{C}\right\}.$$

The family H_l, H_u, H_d generates the unimodular group $SL(2,\mathbb{C})$. H_l (resp H_u, H_d) is homeomorphic to the additive group $(\mathbb{C}\,,+)$. Each element of H_l (resp H_u, H_d) can be identified with $\tau \in \mathbb{C}$. These groups induce linear actions over the space $(\mathcal{S}_m^1)$

$$\Phi\ : H_l \to Aut(\mathcal{S}_m), \qquad \Psi\ : H_u \to Aut(\mathcal{S}_m), \qquad \Gamma\ : H_d \to Aut(\mathcal{S}_m)$$

defined by $\Phi(t)(a,b)=(\underline{a}(t),\underline{b}(t))$, $\Psi(t)(a,b)=(\overline{a}(t),\overline{b}(t))$ and $\Gamma(t)(a,b)=(\hat{a}(t),\hat{b}(t))$ where

$$\begin{array}{ll} \underline{a}_k(\tau) = \displaystyle\sum_{i=k}^{m} \binom{m-k}{i-k} \tau^{i-k} a_i & \underline{b}_k(\tau) = \sum_{i=k}^{m} \binom{m-k}{i-k} \tau^{i-k} (b_i - \tau\, a_i) \\ \overline{a}_k(\tau) = \displaystyle\sum_{i=0}^{k} \binom{k}{i} \tau^{k-i} (a_i - \tau\, b_i) & \overline{b}_k(\tau) = \sum_{i=0}^{k} \binom{k}{i} \tau^{k-i} b_i \\ \hat{a}_k(\tau) = e^{\tau(m-2k-1)} a_k & \hat{b}_k(\tau) = e^{\tau(m-2k+1)} b_k. \end{array} \tag{18}$$

To each subgroup H_l, H_u, H_d, we associate a differential operator acting over the algebra of the polynomials $\mathbb{C}[a\,,\,b]$

$$\Omega_l \;=\; \sum_{k=1}^{m} (m-k+1)\Big(a_k \,\frac{\partial}{\partial a_{k-1}} \;+\; b_k\, \frac{\partial}{\partial b_{k-1}} \Big) \;-\; \sum_{k=0}^{m} a_k\, \frac{\partial}{\partial b_k}, \tag{19}$$

$$\Omega_u \;=\; \sum_{k=1}^{m} (k)\Big(a_{k-1} \,\frac{\partial}{\partial a_k} \;+\; b_{k-1}\, \frac{\partial}{\partial b_k} \Big) \;-\; \sum_{k=0}^{m} b_k\, \frac{\partial}{\partial a_k}, \tag{20}$$

and

$$\Omega_d \;=\; \sum_{k=0}^{m} \Big(\,(m-2k-1)\, a_k \,\frac{\partial}{\partial a_k} \;+\; (m-2k+1)\, b_k\, \frac{\partial}{\partial b_k} \Big). \tag{21}$$

They are obtained by derivating respectively the expressions $\Phi(\tau)(a,b)$, $\Psi(\tau)(a,b)$ and $\Gamma(\tau)(a,b)$ with respect to τ and setting $\tau = 0$.

These operators play an important role in the description of the algebra of the $GL(2,\mathbb{C})$-covariants. Indeed, the next result gives a relation between any covariant

$$U{=}A_0 u^p{+}A_1\binom{p}{1}u^{p-1}v{+}\,\cdots\,{+}A_i\binom{p}{i}u^{p-i}v^i{+}\,\cdots\,{+}A_{p-1}\binom{p}{p-1}uv^{p-1}{+}A_p\, v^p. \tag{22}$$

and its leading term A_0.

Definition 4. *The weight of the coefficient a_k (resp. b_k) is the number k (resp. $k-1$). The weight of any monomial $\lambda a_0^{i_0}\cdots a_l^{i_l} b_0^{j_0}\cdots b_l^{j_l}$ is the number $\sum_{k=0}^{l}(ki_k + (k-1)j_k)$. A polynomial $K \in \mathbb{C}[a_0,\ldots,a_l,b_0\ldots b_l]$ is isobaric if all its monomials have the same weight.*

Proposition 19. *For any $GL(2,\mathbb{C})$-covariant (22), with coefficients $A_0\,, A_1,\,\ldots, A_p$, homogeneous polynomial functions of $a_0,\, a_1,\,\ldots\,, a_m\,,\, b_0,\, b_1,\,\ldots\,, b_m$, we have :*

$$\binom{p}{k} A_k \;=\; \frac{1}{k!}\Omega_l^{(k)}(A_0) \qquad \forall k = 0,\, 1,\, 2,\, \ldots, p$$

Corollary 20. *If the homogeneous polynomial I depending on $a_0, a_1, \ldots, a_m$ and $b_0, b_1, \ldots, b_m$, is a $GL(2,\mathbb{C})$-invariant , then $\Omega_l(I) = 0$.*

Proposition 21. *For any $GL(2,\mathbb{C})$-covariant (22), where coefficients $A_0, A_1, \ldots, A_p$ are homogeneous polynomial functions, depending on $a_0, a_1, \ldots, a_m$, $b_0, b_1, \ldots, b_m$, we have :*

$$\Omega_u(A_0) = 0.$$

Theorem 2 ([15])**.** *An algebraic form*

$$U = A_0 u^p + A_1 \binom{p}{1} u^{p-1} v + \cdots + A_{p-1} \binom{p}{k-1} u v^{p-1} + A_p v^p$$

with an isobaric polynomial $A_0 \in \mathbb{C}[a_0, a_1, \ldots, a_l, b_0, b_1, \ldots, b_l]$, is a $GL(2,\mathbb{C})$-covariant if and only if $\Omega_u(A_0) = 0$ and

$$U = \sum_{k=0}^{d} \frac{1}{k!} \Omega_l^k(A_0)\, u^{d-k}\, v^k.$$

Let's return to the determinant of the matrix $S_{[d]}(u,v)$ which is a covariant (18).

Corollary 22. *The covariant (17) is determined by its leading term s_0^d :*

$$det(S_{[d]}(u,v) = \sum_{k=0}^{d} \frac{1}{k!} \Omega_l^k(s_0^d)\, u^{d-k}\, v^k.$$

4.3 The computation of the leading term s_0^d: an open question

Following the previous subsection, the computation of $det(S_{[d,d]})$ can be reduced to that of its leading term, s_0^d. This is somewhat difficult.

Examples of s_0^d: for any $i, j \in \{0,1,2,\ldots,l\}$ such that $i < j$ and $k \in \{0,1,2,\ldots,l-1\}$, we put

$$\delta_{i,j} = \binom{l}{i}\binom{l}{j}\Big(a_i b_j - a_j b_i\Big), \qquad \tau_k = \binom{l}{k} a_k + \binom{l}{k+1} b_{k+1}.$$

Then, with the help of Groebner basis [13], we get :
<u>for $l = 2$</u> :

$$\begin{aligned}
s_0^1 &:= \delta_{0,1};\\
s_0^2 &:= -b_0\delta_{0,2} + \delta_{0,1}\tau_0\\
s_0^3 &:= \delta_{0,1}\left(2\,{\tau_0}^2 + \delta_{0,1}\right) - 4\,b_0\delta_{0,2}\tau_0 + 2\,{b_0}^2\delta_{1,2}\\
s_0^4 &:= 3\,\delta_{0,1}\tau_0\left(3\,{\tau_0}^2 + 4\,\delta_{0,1}\right) + 9\,\left(-\delta_{0,1}a_2 + 2\,\tau_0\delta_{1,2} + \tau_1\delta_{0,2}\right){b_0}^2\\
&\quad -3\,\delta_{0,2}\left(4\,\delta_{0,1} + 9\,{\tau_0}^2\right) b_0\\
s_0^5 &:= 4\,\delta_{0,1}\left(\delta_{0,1} + 6\,{\tau_0}^2\right)\left(9\,\delta_{0,1} + 4\,{\tau_0}^2\right) + 8\left(-36\,a_2\delta_{0,1}\tau_0 + 9\,\delta_{1,2}\delta_{0,1} + 36\,\delta_{0,2}\tau_0\tau_1\right.\\
&\quad \left.+36\,\delta_{1,2}{\tau_0}^2 + 20\,{\delta_{0,2}}^2\right){b_0}^2 - 96\,\left(-a_2\delta_{0,2} + \tau_1\delta_{1,2}\right){b_0}^3 - 16\,\delta_{0,2}\tau_0(24\,{\tau_0}^2\\
&\quad +29\,\delta_{0,1})b_0.
\end{aligned}$$

<u>for $l = 3$</u> :

$$\begin{array}{rcl}
s_0^1 &:=& \delta_{0,1}; \\
s_0^2 &:=& -\delta_{0,2}b_0 + \delta_{0,1}\tau_0; \\
s_0^3 &:=& 2\,(\delta_{0,3} + \delta_{1,2}){b_0}^2 - 4\,\delta_{0,2}\tau_0 b_0 + \delta_{0,1}(2\,{\tau_0}^2 + \delta_{0,1}) \\
s_0^4 &:=& -18\,\delta_{1,3}{b_0}^3 + 9\,(3\,\delta_{0,3}\tau_0 - \tau_2\delta_{0,1} + 2\,\delta_{1,2}\tau_0 + \delta_{0,2}\tau_1){b_0}^2 - 3\,\delta_{0,2}(4\,\delta_{0,1} \\
&& +9\,{\tau_0}^2)b_0 + 3\,\delta_{0,1}\tau_0(3\,{\tau_0}^2 + 4\,\delta_{0,1}); \\
s_0^5 &:=& 192\,\delta_{2,3}{b_0}^4 + 96\,(2\,\delta_{0,1}a_3 + \tau_2\delta_{0,2} - 3\,\tau_1\delta_{0,3} - \tau_1\delta_{1,2} - 5\,\tau_0\delta_{1,3}){b_0}^3 \\
&& +8\,(9\,\delta_{0,1}\delta_{1,2} + 72\,\delta_{0,3}{\tau_0}^2 + 36\,\delta_{1,2}{\tau_0}^2 - 36\,\delta_{0,1}\tau_2\tau_0 + 36\,\tau_1\delta_{0,2}\tau_0 + 9\,\delta_{0,1}\delta_{0,3} \\
&& +20\,{\delta_{0,2}}^2){b_0}^2 - 16\tau_0\delta_{0,2}(29\delta_{0,1} + 24{\tau_0}^2)b_0 + 4\delta_{0,1}(\delta_{0,1} + 6\,{\tau_0}^2)(9\delta_{0,1} + 4{\tau_0}^2).
\end{array}$$

We remark that if $b_0 = 0$ and for $l = 2, 3$,

$$\begin{array}{rcl}
Lt_1 &=& s_0^1 \;=\; \delta_{0,1} \\
Lt_2 &=& s_0^2 \;=\; \delta_{0,1}\tau_0 \\
Lt_3 &=& s_0^3 \;=\; \delta_{0,1}(\delta_{0,1} + 2{\tau_0}^2) \\
Lt_4 &=& s_0^4 \;=\; 3\,\tau_0\delta_{0,1}(3\,{\tau_0}^2 + 4\,\delta_{0,1}) \\
Lt_5 &=& s_0^5 \;=\; 4\,\delta_{0,1}(\delta_{0,1} + 6\,{\tau_0}^2)(9\,\delta_{0,1} + 4\,{\tau_0}^2).
\end{array}$$

It is easy to recognize the type of the factors that are present in these expressions. Indeed, it coincides with the starting equation , when the linear part is nonzero. In fact, this result is more general.

Proposition 23. *When $b_0 = 0$, the leading term of the polynomial $det(S_{[d]}(u, v)$ is defined, up to a numeric constant, by :*

$$Lt_d = c_d(\tau_0)^{(d)} \prod_{\substack{d_1 + d_2 = d \\ d_1 > d_2}} \Big[d_1 d_2\, \tau_0^2 + (d_1 - d_2)^2\, \delta_{0,1}\Big], \tag{23}$$

where c_d is a numerical coefficient, $(d) = 0$ if d is odd and $(d) = 1$ if d is even.

Proof. It is obvious that $s_0^d = det(S_{[d]}(1, 0))$. Since $b_0 = 0$, from (16) and (14), the matrix $S_{[d]}(1, 0) = S_{d,0}$ is upper triangular and so, its determinant is equal to the product of the diagonal elements which are regrouped two by two (the j^{th} term with the $(d + 1 - j)^{th}$ term, for $j = 1, \dots, d + 1)$:

1. when $d_1 > d_2$, we get $(d_1 a_0 + d_2 l b_1)(d_2 a_0 + d_1 l b_1) = (d_1^2 + d_2^2)\tau_0^2 + (d_1 - d_2)^2\delta_{0,1}$,
2. when $d_1 = d_2 = \frac{d}{2}$, the coefficient $(S_{d,0})_{d_1}^{d_1} = \lambda(a_0 + l b_1) = \lambda\tau_0$.

Remark 7. By analogy, when $a_m = 0$, the relation

$$Mt_d = c_d(\tau_{l-1})^{(d)} \prod_{\substack{d_1 + d_2 = d \\ d_1 > d_2}} \Big[d_1 d_2\, \tau_{l-1}^2 + (d_1 - d_2)^2\, \delta_{l-1,l}\Big]$$

is verified.

At this step, we arrive at the question: how to deduce the leading coefficient s_0^d from the above expression (23)? Does there exist some operator which transforms Lt_d to the leading term s_0^d? Up to now, this question is open.

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