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On mixed LCA groups with commutative rings of continuous endomorphisms

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Abstract. Let $\mathcal{L}$ be the class of locally compact abelian (LCA) groups. For $X \in \mathcal{L}$, let $E(X)$ denote the ring of continuous endomorphisms of X. In this paper, we determine for certain subclasses $\mathcal{S}$ of $\mathcal{L}$ the groups $X \in \mathcal{S}$ such that $E(X)$ is commutative. The main results concern the case of mixed LCA groups.

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1 Introduction

This paper is in continuation to the papers [14, 15] and [16] relating to LCA groups with commutative rings of continuous endomorphisms. We shall be mainly concerned with the case of mixed groups. The motivation for our work comes from a result of T. Szele and J. Szendrei. In [17], they have given among others a complete description of discrete mixed abelian groups without nonzero elements of infinite p-height for all relevant primes p, which have commutative endomorphism rings.

The main objective of the paper is to extend this result to the more general framework of all LCA groups. We also derive information about bounded order-by-discrete LCA groups with commutative rings of continuous endomorphisms.

2 Notation

In what follows we use the notation and terminology of [14, 15] and [16]. In addition, if $p \in \mathbb{P}$, $n \in \mathbb{N}_0$, and V is a closed subgroup of a group $X \in \mathcal{L}$, we let

$$p^{-n}V = \{x \in X | p^n x \in V\}.$$

For a subset S of $\mathbb{P}$, let

$$w_S(X) = \bigcap_{p \in S} \bigcap_{n \in \mathbb{N}} \overline{p^n X}.$$

Further, let $(X_i)_{i \in I}$ be a collection of topological groups. For $i \in I$, let U_i be an open subgroup of X_i. We denote by $\prod_{i \in I}^{loc}(X_i; U_i)$ the local product of $(X_i)_{i \in I}$

with respect to $(U_i)_{i\in I}$. Recall that, by definition, $\prod_{i\in I}^{loc}(X_i;U_i)$ is the cartesian product of the family $(X_i)_{i\in I}$, topologized by declaring all neighborhoods of zero in the topological group $\prod_{i\in I}U_i$ to be a fundamental system of neighborhoods of zero in $\prod_{i\in I}^{loc}(X_i;U_i)$ [3, Ch. III, §2, Exercice 26]. Clearly, the local direct product $\prod_{i\in I}(X_i;U_i)$ is open in $\prod_{i\in I}^{loc}(X_i;U_i)$. It is also clear that if each U_i is compact, then $\prod_{i\in I}^{loc}(X_i;U_i)$ is locally compact.

3 Groups with no elements of infinite topological S-height

In [17], T. Szele and J. Szendrei gave among other results a complete description of discrete, mixed, abelian groups with no elements of actually infinite height, which have commutative endomorphism rings. Their theorem reads:

Theorem 3.1 ([17], Theorem 2)**.** *Let X be a discrete mixed group in $\mathcal{L}$ with no elements of infinite $S(X)$-height, i. e. such that*

$$\bigcap_{p\in S(X)}\bigcap_{n\in\mathbb{N}}p^nX=\{0\}.$$

Then $E(X)$ is commutative if and only if X is isomorphic to an $S(X)$-pure subgroup of

$$\prod_{p\in S(X)}^{loc}\big(\mathbb{Z}(p^{n_p});\{0\}\big)$$

containing

$$\bigoplus_{p\in S(X)}\mathbb{Z}(p^{n_p}),$$

where $n_p\in\mathbb{N}_0$ for all $p\in S(X)$.

Our aim here is to extend this theorem to more general groups in $\mathcal{L}$. But first we use it to obtain the solution to our problem in the case of compact groups in $\mathcal{L}$ having nontrivial connected component and dense torsion subgroup.

Corollary 3.2. *Let X be a compact group in $\mathcal{L}$ with $X\neq c(X)\neq\{0\}$ and $\overline{\sum_{p\in S(X)}t_p(X)}=X$. The endomorphism ring $E(X)$ is commutative if and only if X is topologically isomorphic to a quotient group of*

$$\Big(\prod_{p\in S(X)}^{loc}\big(\mathbb{Z}(p^{n_p});\{0\}\big)\Big)^*$$

by a closed $S(X)$-pure subgroup contained in

$$c\Big(\Big(\prod_{p\in S(X)}^{loc}\big(\mathbb{Z}(p^{n_p});\{0\}\big)\Big)^*\Big),$$

where $n_p\in\mathbb{N}_0$ for all $p\in S(X)$.

Proof. Since X is compact with $X \neq c(X) \neq \{0\}$ and $A(X^*; c(X)) = t(X^*)$ [8, (24.24)], it follows that X^* is discrete and mixed. Also, since $\overline{\sum_{p\in S(X)} t_p(X)} = X$, we conclude by [4, Proposition 3.3.3] and [8, (24.22)] that

$$\begin{aligned}\bigcap_{p\in S(X)} \bigcap_{n\in\mathbb{N}} p^n X^* &= A(X^*; \overline{\sum_{p\in S(X)} \sum_{n\in\mathbb{N}} X[p^n]}) \\ &= A(X^*; \overline{\sum_{p\in S(X)} t_p(X)}) = \{0\},\end{aligned}$$

so that X^* has no elements of infinite $S(X)$-height.

Let $G = \prod_{p\in S(X)}^{loc} (\mathbb{Z}(p^{n_p}); \{0\})$, and let Γ be a closed subgroup of G^* For $p \in S(X)$ and $k \in \mathbb{N}$, we have

$$\begin{aligned}A(G; p^k\Gamma) &= \{x \in G \mid p^k\gamma(x) = 0 \quad \text{for all} \quad \gamma \in \Gamma\} \\ &= \{x \in G \mid \gamma(p^k x) = 0 \quad \text{for all} \quad \gamma \in \Gamma\} \\ &= \{x \in G \mid p^k x \in A(G; \Gamma)\} = p^{-k} A(G; \Gamma).\end{aligned}$$

Since

$$p^k G^* \cap \Gamma = A(G^*; G[p^k]) \cap A(G^*; A(G; \Gamma)) = A(G^*; G[p^k] + A(G; \Gamma)),$$

it then follows that $p^k G^* \cap \Gamma = p^k \Gamma$ if and only if

$$A(G^*; G[p^k] + A(G; \Gamma)) = A(G^*; A(G; p^k\Gamma)) = A(G^*; p^{-k} A(G; \Gamma)),$$

or equivalently if $G[p^k] + A(G; \Gamma) = p^{-k} A(G; \Gamma)$, which in its turn is equivalent to $p^k G \cap A(G; \Gamma) = p^k A(G; \Gamma)$. Consequently, Γ is $S(X)$-pure in G^* if and only if $A(G; \Gamma)$ is $S(X)$-pure in G. Finally, observing that a closed subgroup of G^* is contained in $c(G^*)$ if and only if its annihilator in G contains $t(G) = \bigoplus_{p\in S(X)} \mathbb{Z}(p^{n_p})$, the assertion follows from Theorem 3.1 and duality. □

Definition 3.3. *Let S be a nonempty subset of $\mathbb{P}$. A group $X \in \mathcal{L}$ is said to have no elements of infinite topological S-height in case $w_S(X) = \{0\}$.*

We can prove the following generalization of Theorem 3.1.

Theorem 3.4. *Let X be a mixed group in $\mathcal{L}$ with no elements of infinite topological S-height, where $S = S_0(X)$. The following statements are equivalent:*

(i) *The subgroups $p^n X$ with $p \in S$ and $n \in \mathbb{N}$ are open in X, and $E(X)$ is commutative.*

(ii) *The cyclic, pure, p-subgroups of X, where $p \in S$, split topologically from X, and $E(X)$ is commutative.*

(iii) *S is infinite and X is topologically isomorphic to an S-pure subgroup of*

$$\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)$$

containing

$$\prod_{p\in S}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big),$$

where $l_p, n_p \in \mathbb{N}$ and $n_p \neq 0$ for all $p \in S$.

Proof. First observe that since X has no elements of infinite topological S-height, the subgroups $t_p(X)$ are reduced for all $p \in S$, so that X contains nonzero, cyclic, pure, p-subgroups for all $p \in S$ [5, Corollary 27.3].

Assume X satisfies (i), and let A be a cyclic, pure, p-subgroup of X, where $p \in S$. Then $A \cong \mathbb{Z}(p^n)$ for some $n \in \mathbb{N}_0$. Moreover, A splits algebraically from X [5, Proposition 27.1], and hence we can write $X = A \dotplus G$ for some subgroup G of X. It follows that $p^nX = p^nG \subset G$. As p^nX is open in X, we deduce that G is open in X too, so $X = A \oplus G$ by [1, Corollary 6.8]. This proves that (i) implies (ii).

Now assume (ii) holds. Letting $p \in S$, choose an arbitrary nonzero, cyclic, pure, p-subgroup $B(p)$ of X. Then $B(p) \cong \mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0$. By hypothesis, there exists a closed subgroup $C(p)$ of X such that $X = B(p) \oplus C(p)$. We first show that $t_p(C(p)) = \{0\}$ and $\overline{pC(p)} = C(p)$. To do this, observe that since $E(X)$ is commutative, we must have by [14, Lemma 3.5]

$$H(B(p), C(p)) = \{0\} = H(C(p), B(p)).$$

Now, if $t_p(C(p))$ were nonzero, it would clearly follow that $H(B(p), C(p)) \neq \{0\}$, a contradiction. Thus $t_p(C(p)) = \{0\}$, and hence $t_p(X) = B(p)$. Suppose further that $\overline{pC(p)} \neq C(p)$ and pick an arbitrary element $a \in C(p) \setminus \overline{pC(p)}$. Then $\pi(a)$ is a nonzero element of $C(p)/\overline{pC(p)}$, where $\pi \in H\big(C(p), C(p)/\overline{pC(p)}\big)$ denotes the canonical projection. By [13, (3.8)], we can write $C(p)/\overline{pC(p)} = \langle \pi(a)\rangle \oplus \Gamma$ for some closed subgroup Γ of $C(p)/\overline{pC(p)}$. Let φ denote the canonical projection of $C(p)/\overline{pC(p)}$ onto $\langle \pi(a)\rangle$. Since $\langle \pi(a)\rangle$ is a nonzero cyclic p-group, $H(\langle \pi(a)\rangle, B(p)) \neq \{0\}$. Choosing an arbitrary nonzero $h \in H(\langle \pi(a)\rangle, B(p))$, it is clear that $h\circ\varphi\circ\pi$ is a nonzero element of $H(C(p), B(p))$, a contradiction. This shows that $\overline{pC(p)} = C(p)$, and hence for all $n \in \mathbb{N}$, $\overline{p^nC(p)} = C(p)$. As $\overline{p^{n_p}X} = \overline{p^{n_p}C(p)}$, it follows in particular that $\bigcap_{n\in\mathbb{N}} \overline{p^nX} = C(p)$. We next proceed to establish the topological isomorphism whose existence is asserted in (iii). For every $p \in S$, fix an arbitrary isomorphism f_p from $B(p)$ onto $\mathbb{Z}(p^{n_p})$, and let $g_p \in H(X, B(p))$ denote the canonical projection of X onto $B(p)$ with kernel $C(p)$. Also pick an arbitrary compact open subgroup U of X. Clearly, we have $f_p(g_p(U)) = \mathbb{Z}(p^{n_p})[p^{l_p}]$ for some $l_p \in \mathbb{N}$. Define

$$\alpha : X \to \prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)$$

by setting $\alpha(x) = (f_p g_p(x))_{p\in S}$ for all $x \in X$. Then α is a group homomorphism and $\alpha(U) \subset \prod_{p\in S} \mathbb{Z}(p^{n_p})[p^{l_p}]$. Moreover, α is injective because

$$\ker(\alpha) = \bigcap_{p\in S} \ker(f_p g_p) = \bigcap_{p\in S} C(p) = \bigcap_{p\in S} \bigcap_{n\in\mathbb{N}} \overline{p^n X} = \{0\}.$$

Further, since every $f_p g_p$ is continuous, it follows that the homomorphism $x \to (f_p g_p(x))_{p\in S}$ from U to $\prod_{p\in S} \mathbb{Z}(p^{n_p})[p^{l_p}]$ is continuous [2, Ch. I, §4, Proposition 1]. As U is open in X, it then follows that α is continuous as well [3, Ch. III, §2, Proposition 23]. In particular, $\alpha(U)$ is compact and hence closed in $\prod_{p\in S} \mathbb{Z}(p^{n_p})[p^{l_p}]$. Taking into account that $\bigoplus_{p\in S} \mathbb{Z}(p^{n_p})[p^{l_p}]$ is dense in $\prod_{p\in S} \mathbb{Z}(p^{n_p})[p^{l_p}]$ [3, Ch. III, §2, Proposition 25] and contained in $\alpha(U)$, we conclude that $\alpha(U) = \prod_{p\in S} \mathbb{Z}(p^{n_p})[p^{l_p}]$. This implies that α is open because U is compact in X [2, Ch. I, §9, Théorème 2, Corollaire 2] and $\prod_{p\in S} \mathbb{Z}(p^{n_p})[p^{l_p}]$ is open in $\prod^{loc}_{p\in S}\big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big)$. Consequently, α establishes a topological isomorphism from X onto $\alpha(X)$. Also, since $\bigoplus_{p\in S} \mathbb{Z}(p^{n_p}) \subset \alpha(X)$ and

$$\overline{\bigoplus_{p\in S} \mathbb{Z}(p^{n_p})} = \prod_{p\in S} \big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big)$$

[3, Ch. III, §2, Exercice 26], we have

$$\prod_{p\in S} \big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big) \subset \alpha(X).$$

Finally, it is clear that for each $p \in S$ the multiplication by p is an open mapping on $\prod^{loc}_{p\in S}\big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big)$, and hence on X. To show that $\alpha(X)$ is S-pure in $\prod^{loc}_{p\in S}\big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big)$, pick any $q \in S$ and $n \in \mathbb{N}$, and let $x \in X$ be such that

$$\alpha(x) \in q^n \prod^{loc}_{p\in S} \big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big).$$

Letting $\alpha(x) = q^n (y_p)_{p\in S}$ with $(y_p)_{p\in S} \in \prod^{loc}_{p\in S}\big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big)$, we set

$$y'_p = \begin{cases} y_q, & \text{if } p = q, \\ 0, & \text{if } p \neq q, \end{cases} \qquad \text{and} \qquad y''_p = \begin{cases} 0, & \text{if } p = q, \\ y_p, & \text{if } p \neq q. \end{cases}$$

Clearly $\alpha(x) = q^n (y'_p)_{p\in S} + q^n (y''_p)_{p\in S}$. As $X = B(q) \oplus C(q)$, we can write $x = b_q + c_q$ for some $b_q \in B(q)$ and $c_q \in C(q)$. Since for $p \neq q$ we have $f_p g_p(b_q) = 0$ (because $H(\mathbb{Z}(q^{n_q}), \mathbb{Z}(p^{n_p})) = \{0\}$), and since $f_q g_q(c_q) = 0$ (because $c_q \in \ker(g_q)$), we conclude that $\alpha(b_q) = q^n (y'_p)_{p\in S}$ and $\alpha(c_q) = q^n (y''_p)_{p\in S}$. Remembering that $f_q : B(q) \to \mathbb{Z}(q^{n_q})$ is an isomorphism, choose $b'_q \in B(q)$ such that $f_q(b'_q) = y_q$. As $b_q - q^n b'_q \in \ker(\alpha)$, we have $b_q = q^n b'_q$. Also, since the multiplication by q is an open map and $C(q)$ is an open subgroup, we have $qC_q = \overline{qC_q} = C_q$, so that $q^n C_q = C_q$. Hence there exists $c'_q \in C_q$ such that $q^n c'_q = c_q$. It follows that

$$\alpha(x) = \alpha(b_q) + \alpha(c_q) = q^n(\alpha(b'_q) + \alpha(c'_q)),$$

so that

$$\alpha(X) \cap q^n \prod_{p \in S}^{loc} \big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big) \subset q^n \alpha(X).$$

As the converse inclusion clearly holds, we have

$$\alpha(X) \cap q^n \prod_{p \in S}^{loc} \big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big) = q^n \alpha(X),$$

so $\alpha(X)$ is S-pure in $\prod_{p \in S}^{loc} \big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big)$. Consequently, (ii) implies (iii).

Next assume (iii) holds. We already mentioned that the multiplication by $p \in S$ is open on $\prod_{p \in S}^{loc} \big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big)$, and hence on X. Let $X(d)$ denote the group X taken discrete. It then follows from our hypotheses that $X(d)$ is isomorphic to an S-pure subgroup of $\prod_{p \in S(X)}^{loc} (\mathbb{Z}(p^{n_p}); \{0\})$ containing $\bigoplus_{p \in S(X)} \mathbb{Z}(p^{n_p})$, so that $E(X(d))$ is commutative by Theorem 3.1. As $E(X) \subset E(X(d))$, this proves that (iii) implies (i). $\square$

To state the dual analog of Theorem 3.4, a few definitions are in order. In the first one, we reconsider the notion of comixed LCA group, introduced in [16, Definition 6.5]. The reason for this modification is that we want an LCA group to be comixed if and only if its dual is mixed.

Definition 3.5. *A group $X \in \mathcal{L}$ is said to be comixed if either* (1) $\bigcap_{n \in \mathbb{N}_0} \overline{nX}$ *is a nontrivial subgroup of X, i. e. $\{0\} \neq \bigcap_{n \in \mathbb{N}_0} \overline{nX} \neq X$,* or (2) $\bigcap_{n \in \mathbb{N}_0} \overline{nX} = \{0\}$ *and X has no compact subgroups of the form $\overline{mX}$, where $m \in \mathbb{N}_0$.*

Definition 3.6. *Let $p \in \mathbb{P}$. A closed subgroup G of a group $X \in \mathcal{L}$ is said to be p-copure if, for each $n \in \mathbb{N}$, one has $p^{-n}G = \overline{G + X[p^n]}$. Given a nonempty subset S of $\mathbb{P}$, we say G is S-copure in case it is p-copure for all $p \in S$. G is called copure if it is $\mathbb{P}$-copure.*

As is easy to see, p-purity and p-copurity coincide for discrete and for compact groups.

Definition 3.7. *Let $p \in \mathbb{P}$. A subgroup G of an abelian group X is said to be p-submaximal if X/G is a cyclic p-group.*

Our next definition is inspired by one in [1, (4.34)].

Definition 3.8. *Let S be a nonempty subset of $\mathbb{P}$. A group $X \in \mathcal{L}$ is said to be S-power-proper if for each $p \in S$ and $n \in \mathbb{N}$ the multiplication by p^n is a proper map, i. e. for each open subset U of X, p^nU is open in p^nX, taken with its topology induced from X.*

We have

Corollary 3.9. *Let X be a comixed group in $\mathcal{L}$, and let $S = \{p \in \mathbb{P} \mid \overline{pX} \neq X\}$. If $\overline{\sum_{p \in S} t_p(X)} = X$, the following statements are equivalent:*

(i) *X is an S-power-proper group with commutative ring $E(X)$, and the subgroups $X[p^n]$ are compact for all $p \in S$ and $n \in \mathbb{N}$.*

(ii) *The closed, copure, p-submaximal subgroups of X, where $p \in S$, split topologically from X, and $E(X)$ is commutative.*

(iii) *S is infinite and X is topologically isomorphic to a quotient group of*

$$\Big(\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)\Big)^*$$

by a closed S-copure subgroup, contained in

$$c\Big(\Big(\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)\Big)^*\Big),$$

where $n_p, l_p \in \mathbb{N}$ and $n_p \neq 0$ for all $p \in S$.

Proof. Since X is a comixed group, X^* is mixed. It is also easy to see that $S = S_0(X^*)$, and $\overline{\sum_{p\in S} t_p(X)} = X$ if and only if X^* has no elements of infinite topological S-height.

Assume (i). Since X is S-power-proper, X^* is S-power-proper too [1, P.23(d)]. It follows that, for any $p \in S$ and $n \in \mathbb{N}$, the subgroup $p^n X^*$ is closed and hence open in X^* (because $X[p^n]$ is compact).

Pick any $p \in S$, and let G be a closed, copure, p-submaximal subgroup of X. Since $A(X^*, G) \cong (X/G)^*$, we see that $A(X^*, G)$ is a cyclic, p-subgroup of X^*. Moreover, since G is p-copure in X, we also have $p^{-n}G = \overline{G + X[p^n]}$. Passing to annihilators, we obtain

$$p^n A(X^*, G) = A(X^*, G) \cap p^n X^*,$$

so that $A(X^*, G)$ is p-pure and thus pure in X^* [5, p. 114, (g)]. It then follows from Theorem 3.4 that $A(X^*, G)$ splits topologically from X^*, and hence G splits topologically from X [1, Corollary 6.10]. Thus (i) implies (ii).

Now assume (ii), and pick any $p \in S$ and any cyclic, pure, p-subgroup Γ of X^*. It is easy to see that $A(X, \Gamma)$ is a closed, copure, p-submaximal subgroup of X. By hypothesis, $A(X, \Gamma)$ splits topologically from X, so that Γ splits topologically from X^*. Consequently, X^* satisfies condition (ii) and hence (iii) of Theorem 3.4. Observing that

$$k\Big(\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)\Big) = \prod_{p\in S}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big),$$

and passing to duals, we deduce that (ii) implies (iii).

Assume (iii). It follows that X^* satisfies condition (iii) of Theorem 3.4, so that $p1_{X^*}$ is an open mapping on X^* for all $p \in S$. By using duality, it is then easy to see that (i) holds. □

We recall from [7] the following definition.

Definition 3.10. *Let X be a discrete, torsionfree group in $\mathcal{L}$. An independent subset M of X is said to be quasi-pure independent if $\langle M\rangle_*$ is the internal direct sum of subgroups $\langle x\rangle_*$ with $x \in M$, and $\langle x\rangle = \langle x\rangle_*$ whenever $\langle x\rangle_*$ is cyclic and $x \in M$.*

By Zorn's lemma, any quasi-pure independent subset of a discrete, torsionfree group $X \in \mathcal{L}$ is contained in a maximal quasi-pure independent subset of X [7, Proposition 123].

We now state and prove the main theorem of this section, which extends Theorem 3.4.

Theorem 3.11. *Let X be a group in $\mathcal{L}$ such that $t(X/c(X)) \neq \{0\}$, and let $S = S_0(X/c(X))$. Suppose, in addition, the following conditions hold:*

(i) *$w_S(X/c(X))$ is densely divisible and contains no compact elements;*

(ii) *The cyclic, pure, p-subgroups of X, where $p \in S$, and the compact, connected subgroups of X split topologically from X.*

Then $E(X)$ is commutative if and only if for each $p \in S$ there exist $n_p, l_p \in \mathbb{N}$ with $n_p \neq 0$ such that X is topologically isomorphic either to an S-pure subgroup of

$$\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big)$$

containing

$$\prod_{p\in S}\big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big),$$

or to a group of the form

$$D \times \prod_{p\in S}\big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big),$$

where D is topologically isomorphic to either $\mathbb{R}$, $\mathbb{Q}$, or an S-torsionfree quotient of $\mathbb{Q}^*$ by a closed subgroup.

Proof. Assume $E(X)$ is commutative. By [16, Theorem 4.6], there are two cases to consider:

(a) X is residual;

(b) $X \cong D \times Y$, where D is topologically isomorphic with either $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Q}^*$, and Y is a topological torsion group with $t(Y) \neq \{0\}$.

Assume (a) holds. If $c(X) = \{0\}$, we deduce from (i) that $w_S(X)$ is densely divisible and contains no compact elements. As $d(X) \subset k(X)$, it follows that $w_S(X) = \{0\}$. Consequently, if X is mixed, we have by Theorem 3.4 that S is infinite and X is topologically isomorphic to an S-pure subgroup of $\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big)$ containing $\prod_{p\in S}\big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big)$, where $l_p, n_p \in \mathbb{N}$ and $n_p \neq 0$ for all $p \in S$.

In case X is torsion, we deduce from [14, Corollary 5.7] that $X \cong \bigoplus_{p\in S}\mathbb{Z}(p^{n_p})$. It remains to observe that $\bigoplus_{p\in S}\mathbb{Z}(p^{n_p})$ is S-pure in $\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p});\{0\}\big)$.

Next suppose that $C = c(X)$ is nonzero. Since X is residual, C is compact [8, (24.24)], and hence, in view of (ii), we can write $X = C \oplus Z$ for some closed subgroup Z of X. In particular, $E(C)$ and $E(Z)$ are commutative [14, Lemma 3.2]. We also must have $C = \overline{t(C)}$. For if not, it would follow from [1, Proposition 6.12] that $C \cong (\mathbb{Q}^*)^\nu \times \overline{t(C)}$ for some cardinal number $\nu \geq 1$, contradicting the fact that X is residual. Thus $C = \overline{t(C)}$. Next we shall show that C is topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed, necessarily nonzero subgroup. To see this, it is enough to show that C^* is topologically isomorphic to a subgroup of $\mathbb{Q}$. First observe that, being the character group of a compact and connected group, C^* is discrete and torsionfree [8, (23.17) and (24.25)]. Moreover, C^* is reduced because otherwise it would contain a direct summand isomorphic to $\mathbb{Q}$, and hence C would contain a topological direct summand topologically isomorphic to $\mathbb{Q}^*$, in contradiction with the fact that $C = \overline{t(C)}$. Now, if A is a closed, pure subgroup of C, then A is compact and connected [12, Corollary, p. 369]. Consequently, we can write $X = A \oplus B$ for some closed subgroup B of X. It is then clear that $C = A \oplus (B \cap C)$ [1, Proposition 6.5]. Since a subgroup L of the discrete group C^* is pure in C^* if and only if $A(C, L)$ is pure in C [1, Corollary 7.6], we deduce from [1, Corollary 6.10] that every pure subgroup of C^* splits from C^*. Now, let M be a maximal quasi-pure independent subset of C^*, and hence

$$\langle M\rangle_* \cong \bigoplus_{x\in M}\langle x\rangle_*.$$

Since $\langle M\rangle_*$ splits from C^*, we conclude by the maximality of M that $C^* = \langle M\rangle_*$, so C^* is completely decomposable. Further, since $E(C^*)$ is commutative, it follows from [10, Theorem 3] that the groups $\langle x\rangle_*$, where $x \in M$, have incomparable types. Assume by way of contradiction that $|M| > 1$, and pick any distinct elements $a, b \in M$. Then

$$G = \langle a\rangle_* \oplus \langle b\rangle_* \tag{3.1}$$

is pure in C^*, has rank two, and is completely decomposable. For $g \in G$, let $\tau(g)$ denote the type of g. We have $\tau(a+b) = \inf\big(\tau(a), \tau(b)\big)$ [6, §85, C)]. As $\tau(a)$ and $\tau(b)$ are incomparable, we also have $\tau(a+b) < \tau(a)$ and $\tau(a+b) < \tau(b)$. Further, since $\langle a+b\rangle_*$ splits from C^*, we clearly have

$$G = \langle a+b\rangle_* \oplus \Gamma \tag{3.2}$$

for some subgroup of rank one Γ of G [5, §16, Exercise 3(d)]. Since the number of summands of a given type in some decomposition of a discrete, completely decomposable group as a direct sum of groups of rank one is an invariant of that group [6, Proposition 86.1], (3.1) and (3.2) lead to a contradiction. Therefore we must have $|M| = 1$, so that C^* is isomorphic to a subgroup of $\mathbb{Q}$, and hence C is topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup. On the other hand, since $X/c(X) \cong Z$, it is clear from (i) that $w_S(Z) = \{0\}$. Therefore, in case

Z is mixed, we deduce from Theorem 3.4 that Z is topologically isomorphic to an S-pure subgroup of $\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)$ containing $\prod_{p\in S}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)$. As $k(Z)=Z$ by [16, Lemma 4.4] and

$$k\Big(\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)\Big)=\prod_{p\in S}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big),$$

it then follows that $Z\cong\prod_{p\in S}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)$. In the case when Z is torsion, we have $X\cong\bigoplus_{p\in S}\mathbb{Z}(p^{n_p})$ [14, Corollary 5.7]. Finally, if C were not S-torsionfree, we would clearly have $H(\mathbb{Z}(p^{n_p}),C)\neq\{0\}$ for some $p\in S$. Then, combining the canonical projection of Z onto Z_p with an arbitrary isomorphism from Z_p onto $\mathbb{Z}(p^{n_p})$ and with any nonzero $h\in H(\mathbb{Z}(p^{n_p}),C)$, we would obtain a nonzero element of $H(Z,C)$, in contradiction with [14, Lemma 3.5]. Thus C must be S-torsionfree.

Now assume (b) holds. If D is topologically isomorphic with either $\mathbb{R}$ or $\mathbb{Q}^*$, we must have $c(Y)=\{0\}$ since otherwise it would follow from [8, (25.20)] respectively [1, Corollary 4.10] that $H(D,Y)\neq\{0\}$, which is in contradiction with [14, Lemma 3.5]. As $k(Y)=Y$, we then see from (i) that $w_S(Y)=\{0\}$. In case $D\cong\mathbb{Q}$, we deduce by using as above [14, Lemma 3.5] that $d(Y)=\{0\}$. It follows that $w_S(X)\cong D$, and hence again $w_S(Y)=\{0\}$. Since $E(Y)$ is commutative [14, Lemma 3.2], we conclude as for Z in the case when $X\cong C\times Z$ that

$$Y\cong\prod_{p\in S}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big),$$

where $n_p,l_p\in\mathbb{N}$ and $n_p\neq 0$ for all $p\in S$.

The converse is clear. □

By use of duality, we obtain the following

Corollary 3.12. *Let X be a group in $\mathcal{L}$ such that $k(X)$ is not densely divisible, and let $S=\{p\in\mathbb{P}|\overline{p\cdot k(X)}\neq k(X)\}$. Suppose, in addition, the following conditions hold:*

(i) *$k(X)/\overline{\sum_{p\in S}t_p(X)}$ is torsionfree and connected;*

(ii) *The closed, copure, p-submaximal subgroups of X, where $p\in S$, and the open subgroups of X relative to which X has torsionfree quotients split topologically from X.*

Then $E(X)$ is commutative if and only if for each $p\in S$ there exist $n_p,l_p\in\mathbb{N}$ with $n_p\neq 0$ such that X is topologically isomorphic either to a quotient of

$$\Big(\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)\Big)^*$$

by a closed, S-copure subgroup contained in

$$c\Big(\Big(\prod_{p\in S}^{loc}\big(\mathbb{Z}(p^{n_p});\mathbb{Z}(p^{n_p})[p^{l_p}]\big)\Big)^*\Big),$$

or to a group of the form

$$D \times \prod_{p\in S}\big(\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{l_p}]\big),$$

where D is topologically isomorphic to either $\mathbb{R}$, $\mathbb{Q}^*$, or an S-divisible subgroup of $\mathbb{Q}$.

Proof. As is easy to see, $k(X)$ is not densely divisible if and only if $t(X^*/c(X^*)) \neq \{0\}$ [1, Theorem 4.15]. It is also clear that $S_0(X^*/c(X^*)) = S$. Let $\Gamma = X^*/c(X^*)$. If $W = w_S(\Gamma)$, then

$$A(\Gamma^*; W) = \overline{\sum_{p\in S} t_p(\Gamma^*)},$$

so that

$$\begin{aligned} W^* \cong \Gamma^*/A(\Gamma^*; W) &\cong k(X)/\overline{\sum_{p\in S} t_p(k(X))} \\ &= k(X)/\overline{\sum_{p\in S} t_p(X)}. \end{aligned}$$

It follows that X satisfies condition (i) if and only if X^* satisfies condition (i) of Theorem 3.11. Similarly, X satisfies condition (ii) if and only if X^* satisfies condition (ii) of Theorem 3.11. The assertion follows from Theorem 3.11 and duality. □

4 Bounded order-by-discrete groups and their duals

In this section we will be dealing with bounded order-by-discrete groups and compact-by-bounded order groups, which were introduced in [14]. We begin with a characterization of bounded order-by-discrete groups.

Theorem 4.1. *A group $X \in \mathcal{L}$ is bounded order-by-discrete if and only if $c(X) = \{0\}$ and $k(X) = t(X)$.*

Proof. Assume $X \in \mathcal{L}$ is bounded order-by-discrete, and pick an arbitrary closed subgroup of bounded order B of X such that X/B is discrete. Since B is then open in X [8, (5.6)] and $t(X) \supset B$, it follows that $t(X)$ is open in X too. In particular, $t(X)$ is locally compact and $c(X) \subset t(X)$. As every torsion group in $\mathcal{L}$ is totally disconnected [1, Theorem 3.5], we must have $c(X) = \{0\}$, so that $k(X)$ is a topological torsion group. Letting $x \in k(X)$ be arbitrary, we then have $\lim_{n\to\infty}(n!x) = 0$, so $n!x \in t(X)$ for sufficiently large $n \in \mathbb{N}$, and hence $x \in t(X)$. It follows that $k(X) = t(X)$.

For the converse, observe that since $c(X) = \{0\}$, $k(X)$ and hence $t(X)$ is open in X [4, Proposition 3.3.6]. It follows that $t(X)$ is locally compact. Since $t(X) = \bigcup_{n\in\mathbb{N}_0} X[n]$, it then follows by Baire Category Theorem [8, (5.28)] that there is an

$n_0 \in \mathbb{N}_0$ such that $X[n_0]$ has nonempty interior, so that $X[n_0]$ is open in $t(X)$ and hence in X. Consequently, X is bounded order-by-discrete. □

Dualizing Theorem 4.1 gives the following characterization of compact-by-bounded order groups.

Corollary 4.2. *A group $X \in \mathcal{L}$ is compact-by-bounded order if and only if $c(X) = \bigcap_{n \in \mathbb{N}_0} \overline{nX}$ and $k(X) = X$.*

Proof. It is easy to see that X is compact-by-bounded order if and only if X^* is bounded order-by-discrete. The assertion follows then from Theorem 4.1 and duality. □

The following lemma considers a special case of bounded order-by-discrete groups having commutative rings of continuous endomorphisms.

Lemma 4.3. *Let $X \in \mathcal{L}$ be a bounded order-by-discrete, reduced group with primary components of bounded order. If $E(X)$ is commutative, then the following conditions hold:*

(i) *X is discrete;*

(ii) *$t(X) \cong \bigoplus_{p \in S(X)} \mathbb{Z}(p^{n_p})$, where the n_p's are positive integers;*

(iii) *$X/t(X)$ is $S(X)$-divisible;*

(iv) *$\bigcap_{p \in S(X)} p^{n_p} X$ is $S(X)$-divisible and torsionfree, and $X/\bigcap_{p \in S(X)} p^{n_p} X$ is isomorphic to an $S(X)$-pure subgroup of $\prod^{loc}_{p \in S(X)} \big(\mathbb{Z}(p^{n_p}); \{0\}\big)$ containing $\bigoplus_{p \in S(X)} \mathbb{Z}(p^{n_p})$.*

Proof. As we saw in Theorem 4.1, $c(X) = \{0\}$ and $k(X) = t(X)$, so $t(X)$ is a topological torsion group. Moreover, $t(X)$ is open in X. Fix any $p \in S(X)$ and any compact open subgroup U of X such that $U \subset t(X)$. By [1, Theorem 3.13], we have $t(X) \cong \prod_{q \in S(X)} (t_q(X); t_q(U))$, so that

$$t(X) = t_p(X) \oplus t_p(X)^{\#}, \tag{4.1}$$

where $t_p(X)^{\#} = \overline{\sum_{q \in S(X) \setminus \{p\}} t_q(X)}$. Since $t_p(X)$ is of bounded order, we also have

$$X = t_p(X) \dotplus C(p) \tag{4.2}$$

for some subgroup $C(p)$ of X [5, Corollary 27.4]. Our first task is to show that the last direct sum is topological. For $q \in S(X)$, let

$$q^{n_q} = \max\big\{o(x) \mid x \in t_q(X)\big\}.$$

It follows from decomposition (4.1) that $p^{n_p} t(X) \subset t_p(X)^{\#}$. In a similar way, writing $U = t_p(U) \oplus t_p(U)^{\#}$, where $t_p(U)^{\#} = \overline{\sum_{q \in S(X) \setminus \{p\}} t_q(U)}$, we obtain

$p^{n_p}U = p^{n_p}t(U) \subset t_p(U)^{\#}$. On the other hand, letting $q \in S(X) \setminus \{p\}$, we can choose $a(q), b(q) \in \mathbb{Z}$ such that $a(q)p^{n_p} + b(q)q^{n_q} = 1$. For $x \in t_q(X)$, we then have

$$x = a(q)p^{n_p}x + b(q)q^{n_q}x = p^{n_p}a(q)x \in p^{n_p}t(X),$$

so that $t_q(X) \subset p^{n_p}t(X)$. In a similar way, for $x \in t_q(U)$ we have $x \in p^{n_p}U$, and hence $t_q(U) \subset p^{n_p}U$. It follows that

$$t_p(X)^{\#} = \overline{\sum_{q \in S(X)\setminus\{p\}} t_q(X)} \subset \overline{p^{n_p}t(X)}$$

and

$$t_p(U)^{\#} = \overline{\sum_{q \in S(X)\setminus\{p\}} t_q(U)} \subset \overline{p^{n_p}U} = p^{n_p}U,$$

so that $t_p(X)^{\#} = \overline{p^{n_p}t(X)}$ and $t_p(U)^{\#} = p^{n_p}U$. As $t_p(U)^{\#} = U \cap t_p(X)^{\#}$, $p^{n_p}U$ is open in $t_p(X)^{\#}$, so $p^{n_p}t(X)$ is open in $t_p(X)^{\#}$ too, and hence $t_p(X)^{\#} = p^{n_p}t(X)$. Let $\varphi_p \in E(t(X))$ be the canonical projection of $t(X)$ onto $t_p(X)$ given by (4.1), and $\psi_p : X \to X$ be the canonical projection of X onto $t_p(X)$ given by (4.2). Since

$$t_p(X)^{\#} = p^{n_p}t(X) \subset p^{n_p}X \subset C(p),$$

it is clear that $\psi_p|_{t(X)} = \eta \circ \varphi_p$, where η is the canonical injection of $t(X)$ into X. Further, since $t(X)$ is open in X, it follows that ψ_p is continuous on X [3, Ch. III, §2, Proposition 23], and thus $X = t_p(X) \oplus C(p)$ by [3, Ch.III, §6, Proposition 2].

Now, taking account of [14, Lemma 3.2], we conclude that $E(t_p(X))$ is commutative, and so $t_p(X) \cong \mathbb{Z}(p^{n_p})$ by [14, Theorem 5.2]. Since in view of [14, Lemma 3.5] we must have $H(C(p), t_p(X)) = \{0\}$, it can be shown as in the proof of Theorem 3.4 that $\overline{pC(p)} = C(p)$.

Finally, since $p \in S(X)$ was arbitrarily chosen, we conclude that $t(X)$ is countable, and hence discrete [11, Ch. I, Theorem 2, Corollary]. But $t(X)$ is open in X, so X is discrete too. In particular, $\overline{qC(q)} = qC(q)$ for all $q \in S(X)$, and so, for all $q \in S(X)$, $X/t_q(X)$ is q-divisible as an isomorphic copy of $C(q)$. Since

$$X/t(X) \cong (X/t_q(X))/(t(X)/t_q(X))$$

for all $q \in S(X)$, it follows that $X/t(X)$ is $S(X)$-divisible. Thus X satisfies (i), (ii) and (iii).

To establish the first part of (iv), let $X_\infty = \bigcap_{p \in S(X)} p^{n_p}X$, and pick any $s \in S(X)$ and $x \in X_\infty$. Since $s^{n_s}X$ is s-divisible, there exists $y \in s^{n_s}X$ such that $x = sy$. Letting $r \in S(X) \setminus \{s\}$, choose $a(r), b(r) \in \mathbb{Z}$ such that $a(r)s + b(r)r^{n_r} = 1$. We have

$$y = a(r)sy + b(r)r^{n_r}y = a(r)x + b(r)r^{n_r}y \in X_\infty + r^{n_r}X \subset r^{n_r}X,$$

so that $y \in X_\infty$. As $x \in X_\infty$ and $s \in S(X)$ were arbitrary, it follows that X_∞ is $S(X)$-divisible. Moreover, since $X_\infty \cap t(X) = \{0\}$, X_∞ is also torsionfree. Now we proceed to establish the second part of (iv). For each $p \in S(X)$, let $g_p \in H(X, t_p(X))$ denote the canonical projection of X onto $t_p(X)$ with kernel $C(p)$, and f_p an isomorphism from $t_p(X)$ onto $\mathbb{Z}(p^{n_p})$. The mapping $\alpha : X \to \prod_{p\in S(X)}^{loc}\big(\mathbb{Z}(p^{n_p}); \{0\}\big)$, given by $\alpha(x) = (f_p g_p(x))_{p\in S(X)}$ for all $x \in X$, is then a group homomorphism with kernel X_∞, so that X/X_∞ is isomorphic with $\alpha(X)$. It is also clear that, for all $q \in S(X)$, α maps $t_q(X)$ onto the subgroup of $\prod_{p\in S(X)}^{loc}\big(\mathbb{Z}(p^{n_p}); \{0\}\big)$ consisting of all elements with zero p-components for $p \neq q$, whence we deduce that

$$\bigoplus_{p\in S(X)} \mathbb{Z}(p^{n_p}) \subset \alpha(X).$$

Finally, it can be seen, following the same way as in the proof of Theorem 3.4, that $\alpha(X)$ is $S(X)$-pure in $\prod_{p\in S(X)}^{loc}\big(\mathbb{Z}(p^{n_p}); \{0\}\big)$.

The proof is complete. □

Let us recall from [1] the following definition.

Definition 4.4. *Let $p \in \mathbb{P}$. A group $X \in \mathcal{L}$ is called p-thetic in case there exists $h \in H(\mathbb{Z}(p^\infty), X)$ such that $h(\mathbb{Z}(p^\infty))$ is dense in X.*

We are now ready to prove the main theorem of this section.

Theorem 4.5. *Let X be a bounded order-by-discrete group in $\mathcal{L}$. If $E(X)$ is commutative, then X is discrete and satisfies exactly one of the following conditions:*

(i) *X is isomorphic with either*

$$\bigoplus_{p\in S_1} \mathbb{Z}(p^\infty) \times \bigoplus_{p\in S_2} \mathbb{Z}(p^{n_p}) \qquad \textit{or} \qquad \mathbb{Q} \times \bigoplus_{p\in S(X)} \mathbb{Z}(p^{n_p}),$$

where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and the n_p's are positive integers.

(ii) *$S(X)$ is finite and $X = t(X) \oplus W$, where W is a reduced, $S(X)$-divisible subgroup of X, and $t(X) \cong \bigoplus_{p\in S(X)} \mathbb{Z}(p^{n_p})$ for some positive integers n_p.*

(iii) *X is reduced, $S(X)$ is infinite, $X/t(X)$ is $S(X)$-divisible, and there exist positive integers n_p, one for each $p \in S(X)$, such that the following conditions hold:*

1) $t(X) \cong \bigoplus_{p\in S(X)} \mathbb{Z}(p^{n_p})$;
2) $\bigcap_{p\in S(X)} p^{n_p}X$ *is $S(X)$-divisible and torsionfree;*
3) $X/\bigcap_{p\in S(X)} p^{n_p}X$ *is isomorphic to an $S(X)$-pure subgroup of* $\prod_{p\in S(X)}^{loc}\big(\mathbb{Z}(p^{n_p}); \{0\}\big)$ *containing* $\bigoplus_{p\in S(X)} \mathbb{Z}(p^{n_p})$.

Proof. First assume X contains a subgroup D algebraically isomorphic to $\mathbb{Z}(p^\infty)$ for some $p \in S(X)$. Since $\overline{D}$ is then p-thetic, it follows from [1, Proposition 5.20 and Proposition 5.21] that either $\overline{D} \cong \mathbb{Z}(p^\infty)$ or else $\overline{D}$ is compact and connected. But X is totally disconnected by Theorem 4.1, so that the latter cannot occur, and hence $\overline{D} \cong \mathbb{Z}(p^\infty)$. Now, since $\mathbb{Z}(p^\infty)$ is splitting in the class of totally disconnected LCA groups [1, Proposition 6.21], we can write $X = \overline{D} \oplus A$ for some closed subgroup A of X. If A were not a torsion group, it would follow by Theorem 4.1 that $t(A)$ is open in A, so $A/t(A)$ is nonzero, discrete and torsionfree. Hence we would have $H(A/t(A), \overline{D}) \neq \{0\}$, whence $H(A, \overline{D}) \neq \{0\}$, contradicting by [14, Lemma 3.5] the commutativity of $E(X)$. Consequently, A must be torsion. In particular, X is torsion as a direct sum of two torsion groups. It then follows from [14, Corollary 5.7] that

$$X \cong \bigoplus_{p \in S_1} \mathbb{Z}(p^\infty) \times \bigoplus_{p \in S_2} \mathbb{Z}(p^{n_p}),$$

where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and the n_p's are positive integers.

Next assume $d(t(X)) = \{0\}$ but still $d(X) \neq \{0\}$, and pick a subgroup V of X algebraically isomorphic to $\mathbb{Q}$. Since $t(X)$ is open in X and $V \cap t(X) = \{0\}$, it follows that V is discrete and hence closed in X [8, (5.10)]. We can write $X = V \oplus B$ for some closed subgroup B of X, because $\mathbb{Q}$ is splitting in the class of totally disconnected LCA groups [1, Proposition 6.21]. As above, we make use of [14, Lemma 3.5] to deduce that $H(B, V) = \{0\} = H(V, B)$, which implies $B = t(B)$ and $d(B) = \{0\}$. Since $E(B)$ is clearly commutative, it follows from [14, Corollary 5.7] that $B \cong \bigoplus_{p \in S(X)} \mathbb{Z}(p^{n_p})$, where the n_p's are positive integers.

Next assume X is reduced. If $t(X)$ is of bounded order, it follows that $t(X)$ splits algebraically from X [5, Theorem 27.5], and since by Theorem 4.1 $t(X)$ is open in X, this splitting is topological [1, Corollary 6.8], i. e. $X = t(X) \oplus W$ for some discrete subgroup W of X. As $E(t(X))$ must be commutative, we conclude from [14, Corollary 5.7] that $t(X)$ is discrete and isomorphic to $\bigoplus_{p \in S(X)} \mathbb{Z}(p^{n_p})$, where the n_p's are positive integers. It follows, in particular, that X is discrete. Moreover, since $t(X)$ is of bounded order, $S(X)$ must be finite. Finally, by [16, Theorem 6.1], we must also have $pW = W$ for all $p \in S(X)$.

It remains to consider the case when $t(X)$ is not of bounded order. We shall show that then X has primary components of bounded order. Since X is bounded order-by-discrete, there is $n \in \mathbb{N}_0$ such that $X/X[n]$ is discrete. Pick any $p \in S(X)$, and write $n = p^{k_p} n'$, where $k_p \in \mathbb{N}$ and $p \nmid n'$. To see that $t_p(X)$ is of bounded order, it is enough to show that $t_p(X/X[n])$ is of bounded order. Suppose not. Then either $X/X[n]$ has a direct summand isomorphic to $\mathbb{Z}(p^\infty)$ [5, Theorem 21.2], or $t_p(X/X[n])$ is reduced and has direct summands of arbitrarily high orders [5, §27, Exercise 1]. Since $t_p(X/X[n])$ is pure in $X/X[n]$, we deduce from [5, Lemma 26.1 and Theorem 27.5] that in the second case $X/X[n]$ has as direct summands cyclic p-subgroups of arbitrarily high orders. Hence we can write

$$X/X[n] = T \oplus G, \tag{4.3}$$

where T is isomorphic to either $\mathbb{Z}(p^\infty)$, or $\mathbb{Z}(p^{l_p})$ with $l_p > k_p$. Here we must have $G \neq \{0\}$. This is clear in case $T \cong \mathbb{Z}(p^{l_p})$ because otherwise $t_p(X/X[n])$ would be of bounded order, contrary to our assumption. On the other hand, if we had $G = \{0\}$ and $T \cong \mathbb{Z}(p^\infty)$, then X would be torsion. As $E(X)$ is commutative, we would conclude from [14, Corollary 5.7] that X is also discrete and, for each $q \in S(X)$, $t_q(X)$ is isomorphic to either $\mathbb{Z}(q^\infty)$ or $\mathbb{Z}(q^{n_q})$ for some $n_q \in \mathbb{N}$. In particular, by [9, Corollary 8.11(ii)] we would have

$$X/X[n] \cong \bigoplus_{q \in S(X)} \Big(t_q(X)/t_q(X[n])\Big).$$

Since in the considered case $X/X[n] \cong \mathbb{Z}(p^\infty)$, this would imply $t_p(X) \cong \mathbb{Z}(p^\infty)$, contrary to the assumption that X is reduced. Thus $G \neq \{0\}$. Now, passing to duals in (4.3), we deduce that $\overline{nX^*} = T' \oplus G'$, where $T' \cong T^*$ and $G' \cong G^*$ [8, (23.18)]. As by [14, Lemma 3.1] $E(X^*)$ is commutative, we must have

$$H(G', T') = H(G', T')[n] \quad \text{and} \quad H(T', G' = H(T', G')[n],$$

since otherwise an application of [14, Lemma 3.5] with $\omega = n1_{X^*}$ and any $h \in H(G', T') \cup H(T', G')$ satisfying $nh \neq 0$ would produce a contradiction. Since for any $L, M \in \mathcal{L}$, $H(M^*, L^*) \cong H(L, M)$ [12, Corollary 2, p. 377], it follows that

$$H(T, G) = H(T, G)[n] \quad \text{and} \quad H(G, T) = H(G, T)[n].$$

Now we can show that either of the cases $T \cong \mathbb{Z}(p^\infty)$ or $T \cong \mathbb{Z}(p^{l_p})$ leads to a contradiction. Suppose first $T \cong \mathbb{Z}(p^\infty)$. We must have $G = t(G)$. For, if G contained an element, say a, of infinite order, then, choosing any $b \in T$ with $o(b) > p^{k_p}$, we could define $f \in H(\langle a \rangle, T)$ by the rule $f(a) = b$. Since T is divisible, there would exist $f_0 \in H(G, T)$ such that $f_0|_{\langle a \rangle} = f$, and hence $nf_0 \neq 0$, a contradiction. Thus $G = t(G)$, so $X/X[n] = t(X/X[n])$, and hence $X = t(X)$. Since by the assumption X is reduced, it follows from [14, Corollary 5.7] that $X \cong \bigoplus_{q \in S(X)} \mathbb{Z}(q^{n_q})$, where the n_q's are positive integers. But then $X/X[n]$ is reduced, contrary to the assumption that $T \cong \mathbb{Z}(p^\infty)$. Next suppose $T \cong \mathbb{Z}(p^{l_p})$. If there existed $c \in t_p(G)$ with $o(c) > p^{k_p}$, then we could find $c' \in t_p(G)$ such that $p^{k_p} < o(c') \leq p^{l_p}$. It would follow that there exists $g \in H(\mathbb{Z}(p^{l_p}), G)$ given by $g(1 + p^{l_p}\mathbb{Z}) = c'$ such that $ng \neq 0$. Since this would imply $H(T, G) \neq H(T, G)[n]$, we arrive at a contradiction. Hence we must have $p^{k_p} t_p(G) = \{0\}$, which implies $t_p(X/X[n])$ is of bounded order, a contradiction.

Consequently, our assumption that $t_p(X/X[n])$ is not of bounded order leads to a contradiction, so $t_p(X/X[n])$ must be of bounded order, whence we deduce that $t_p(X)$ is of bounded order too. As $p \in S(X)$ was arbitrary, it follows that X has primary components of bounded order. Moreover, since $t(X)$ is not of bounded order, $S(X)$ has to be infinite. Then, an application of Lemma 4.3 gives us (iii).

The proof is complete. □

We conclude this section by stating the dual analog of Theorem 4.5.

Corollary 4.6. *Let X be a compact-by-bounded order group in $\mathcal{L}$. If $E(X)$ is commutative, then X is compact and satisfies exactly one of the following conditions:*

(i) *X is topologically isomorphic with either*

$$\prod_{p\in S_1}\mathbb{Z}_p\times\prod_{p\in S_2}\mathbb{Z}(p^{n_p})\qquad \text{or}\qquad \mathbb{Q}^*\times\prod_{p\in S(X)}\mathbb{Z}(p^{n_p}),$$

where $S_1\cup S_2=S(X)$, $S_1\cap S_2=\varnothing$, and the n_p's are positive integers.

(ii) *$S(X)$ is finite and $X=c(X)\oplus M$, where $c(X)$ is $S(X)$-torsionfree with $m(c(X))=c(X)$, and $M\cong\prod_{p\in S(X)}\mathbb{Z}(p^{n_p})$ for some positive integers n_p.*

(iii) *$X=m(X)$, $S(X)$ is infinite, $c(X)$ is $S(X)$-torsionfree, and there exist positive integers n_p, one for each $p\in S(X)$, such that the following conditions hold:*

1) *$X/c(X)\cong\prod_{p\in S(X)}\mathbb{Z}(p^{n_p})$;*

2) *$X/\overline{\sum_{p\in S(X)}X[p^{n_p}]}$ is densely divisible and $S(X)$-torsionfree;*

3) *$\overline{\sum_{p\in S(X)}X[p^{n_p}]}$ is topologically isomorphic to a quotient group of $\left(\prod^{loc}_{p\in S}\big(\mathbb{Z}(p^{n_p});\{0\}\big)\right)^*$ by a closed, $S(X)$-pure subgroup contained in $c\left(\left(\prod^{loc}_{p\in S}\big(\mathbb{Z}(p^{n_p});\{0\}\big)\right)^*\right)$.*

Proof. Since a group $X\in\mathcal{L}$ is compact-by-bounded order if and only if X^* is bounded order-by-compact, the assertion follows from Theorem 4.5 and duality. □

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