BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(55), 2007, Pages 64–72
ISSN 1024–7696

About Quasiconformal Maps in Finsler Spaces

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Abstract. We consider a constant C which measures the deviation of the Finsler metric from a Riemannian metric and we prove that the problem of the existence of quasiconformal mappings between Finsler spaces can be reduced to the same problem between Riemann spaces.

Mathematics subject classification: 30C65.
Keywords and phrases: Characteristic function, quasiconformal map, Finsler metric, regular atlas.

1 Introduction

The quasiconformal mappings represent a generalization of the conformal transformations. It is known that there exist different equi- valent definitions for the conformal transformations, most of these using some conformal invariants (modulus of the ring or a family of arcs, angles, infinitesimal circles,...) or, as the solutions of a Cauchy-Riemann system.

The conformal transformations were used for the modelling, sometimes with approximation, of some phenomena. For example, in the hydrodynamic, where were considered "ideal fluids" (incompressible and not viscous) and their flow was without whirlpools.

The definitions of quasiconformal mappings appeared, naturally, from the corresponding definitions of the conformal transformations, for example, by substituting quasi-invariance for the invariance.

K. Suominen extends the study of the quasiconformality to the finite dimensional Riemannian manifolds [1], and P. Caraman to the Riemann-Wiener manifolds [2].

The study of quasiconformality was extended by us to the infinit dimensional Riemannian manifolds and to the Finsler spaces [3, 4].

In 1982 M. Nakai and H. Tanaka proved the existence of quasiconformal mappings between finite dimensional Riemannian manifolds [5].

In this paper we associate to a Finsler space a constant C, which measures the deviation of the Finslerian metric from a riemannian metric. By using this constant we establish an inequality between the Finslerian and Riemannian characteristic functions and we prove that the problem of the existence of quasiconformal mappings between finite dimensional Finsler spaces can be reduced to the same problem between finite dimensional Riemann manifolds. The main result is

Theorem A. *A homeomorphism f is Finslerian quasiconformal iff f is Riemannian quasiconformal.*

2 Regular atlases

Let us consider M a n-dimensional, connected, paracompact, orientable, C^{∞}-differentiable manifold and $L : TM \to \mathbb{R}$ a Finsler metric on M ($TM = \bigcup_{x \in M} T_x M$ denotes the tangent bundle of M and $T_x M$ the tangent space at $x \in M$).

The restriction of L to $T_x M$, $L(x,.) : T_x M \to \mathbb{R}$, is a norm, generally non-Hilbertian, denoted by $\|\cdot\|$ and

$$L^2(x,X) = a_{ij}(x,X) X^i X^j,$$

for every $X = X^i \frac{\partial}{\partial x^i} \in T_x M$, where

$$a_{ij}(x,X) = \frac{1}{2} \frac{\partial^2 L^2(x,X)}{\partial X^i \partial X^j}$$

are homogeneous functions of degree zero with respect to X. We have

$$\|X\| = L(x,X) = \sqrt{a_{ij}(x,X) X^i X^j}.$$

The manifold M is a metric space with the geodesic metric

$$d(x,y) = \inf \{ \ell(\gamma) \,/\, \gamma \in \Gamma \},$$

where Γ is the set of all differentiable arcs joining x with y and $\ell(\gamma) = \int_0^1 \left\| \frac{d\gamma}{dt} \right\| dt$.

The geodesics of M are the autoparalleles of nonlinear Cartan connection ∇ and their equation is $\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t) = 0$.

If $\gamma_X(t)$ is the geodesic with the initial condition (x,X), then $\gamma_X(t) = \gamma_{\alpha X}(\alpha^{-1} t)$ for every $\alpha \in \mathbb{R} \setminus \{0\}$ and the map $\exp_x : V_x \to M$, $\exp_x X = \gamma_X(1)$ satisfies $\|Y\| = d(x, \exp_x Y)$ for every $Y \in V_x$ (V_x is the maximum domain where $\exp_x$ is a diffeomorphism).

Lemma 1. *If M is a Finsler space, then for every $\varepsilon \in (0,\infty)$ there exists $r_x = r(x,\varepsilon) \in (0,\infty)$ such that $\exp_x$ is a $(1+\varepsilon)$-isometry on the ball $B(0_x, r_x) \subset T_x M$, that is*

$$(1+\varepsilon)^{-1} \|Y - Z\| \le d(\exp_x Y, \exp_x Z) \le (1+\varepsilon) \|Y - Z\|, \tag{1}$$

for every $Y, Z \in B(0_x, r_x)$.

Proof. For every $\varepsilon \in (0,\infty)$ let us consider $r_x \in (0,\infty)$ such that the inequality $\|T_x \exp_x\| \le 1 + \varepsilon$ is satisfied, for every $X \in B(0_x, r_x)$. Let us consider $\gamma : [0,1] \to T_x M$, $\gamma(t) = tY + (1-t) Z$ and $\gamma_1(t) = (\exp_x \circ \gamma)(t)$. We have

$$\begin{aligned} d\left(\exp_x Y, \exp_x Z\right) &\leq \ell\left(\gamma_1\right) = \int_0^1 \left\| \frac{d\gamma_1}{dt} \right\| dt \leq \int_0^1 \left\| \frac{d\gamma}{dt} \right\| \left\| T_{\gamma(t)} \exp_x \right\| dt \leq \\ &\leq (1+\varepsilon)\, \ell\left(\gamma\right) = (1+\varepsilon) \left\| Y - Z \right\|. \end{aligned}$$

Analogously, results the left-hand side of (1).

Remark 1. If $Z = 0_x$ and $Y = \exp_x^{-1} y$, we obtain

$$d\left(x, y\right) = d\left(\exp_x 0_x, \exp_x Y\right) = \left\| Y \right\|$$

and hence $\exp_x\left(B\left(0_x, r_x\right)\right)$ is the geodesic ball $B\left(x, r_x\right)$, consequently $\exp_x^{-1}$ is a $(1+\varepsilon)$-isometry, too.

Let us consider the homeomorphism $\varphi_x : B\left(0, \alpha_x\right) \to B\left(x, r_x\right)$, $\alpha_x \in (0, \infty)$, such that $\varphi_x(0) = x$ ($B\left(0, \alpha_x\right) = B_x$ is the ball with center 0 and radius α_x in $\mathbb{R}^n$).

The pair $h_x = \left(B_x, \varphi_x\right)$ is called *φ-chart* at x and the set $A = \left\{h_x / \ x \in M\right\}$ is called *φ-atlas* on M.

Obviously, for every $\varepsilon \in (0, \infty)$, $h_x^\varepsilon = \left(B\left(0_x, r_x\right), \exp_x\right)$ is an exp-chart and $A^\varepsilon = \left\{h_x^\varepsilon / \ x \in M\right\}$ is an exp-atlas, called the *atlas of geodesic balls*.

To any φ-atlas we can associate the function $k_A : M \to [1, \infty]$

$$k_A(x) = \limsup_{\alpha \to 0} \frac{\sup\left\{d\left(x, y\right) / \ y \in \varphi_x\left(S\left(0, \alpha\right)\right)\right\}}{\inf\left\{d\left(x, y\right) / \ y \in \varphi_x\left(S\left(0, \alpha\right)\right)\right\}},\ \alpha \in \left(0, \alpha_x\right),$$

called the *parameter* of A; $k_A(x)$ is called the *parameter* of the φ-chart h_x (we shall sometimes omit the subscript A if the choice of the atlas is clear from context).

If $k(x) < \infty$ we say that the φ-chart h_x is *k-regular*. If all the φ-charts of A are k-regular, we say that the atlas A is *k-regular*.

If φ_x is a conformal homeomorphism we say that h_x is a *conformal chart*. The atlas A is said to be *conformal* if its charts are conformal. In this case we obtain $k(x) = 1$ and so, any conformal atlas has the parameter $k = 1$. Particularly, the atlas of geodesic balls A^ε has the parameter $k = 1$.

Let $f : D \to \tilde{D}$ be a homeomorphism, where $D,\ \tilde{D}$ are domains in M.

If A is a φ-atlas on D, we can consider the $\tilde{\varphi}$-atlas $\tilde{A} = \left\{\tilde{h}_{\tilde{x}} \ / \ \tilde{x} \in \tilde{M}\right\}$, on $\tilde{D}$, with $\tilde{h}_{\tilde{x}} = \left(B_{\tilde{x}}, \tilde{\varphi}_{\tilde{x}}\right)$, $B_{\tilde{x}} = B\left(0, \alpha_{\tilde{x}}\right)$, $\tilde{\varphi}_{\tilde{x}} = f \circ \varphi_x$, $\tilde{x} = f(x)$ and $\alpha_{\tilde{x}}$ chosen such that $\tilde{\varphi}_{\tilde{x}}\left(B_{\tilde{x}}\right) \subset B\left(\tilde{x}, r_{\tilde{x}}\right)$, ($B\left(\tilde{x}, r_{\tilde{x}}\right)$ is the geodesic ball in $\tilde{D}$ where $\exp_{\tilde{x}}^{-1}$ is $(1+\varepsilon)$-isometry).

$\tilde{A}$ and $\tilde{h}_{\tilde{x}}$ are called, respectively, *$\tilde{\varphi}$-atlas* and *$\tilde{\varphi}$-chart induced by f*.

The parameter of $\tilde{A}$ will be

$$\tilde{k}_{\tilde{A}}(\tilde{x}) = \limsup_{\alpha \to 0} \frac{\sup\left\{\tilde{d}\left(\tilde{x}, \tilde{y}\right) / \ \tilde{y} \in \tilde{\varphi}_{\tilde{x}}\left(S\left(0, \alpha\right)\right)\right\}}{\inf\left\{\tilde{d}\left(\tilde{x}, \tilde{y}\right) / \ \tilde{y} \in \tilde{\varphi}_{\tilde{x}}\left(S\left(0, \alpha\right)\right)\right\}},\ \alpha \in \left(0, \alpha_{\tilde{x}}\right).$$

Generally, if A is k-regular, it does not result that $\tilde{A}$ is $\tilde{k}$-regular.

The homeomorphism f is called $k\tilde{k}-$*regular* if there exists a $\varphi-$atlas A, $k-$regular, on D such that the $\tilde{\varphi}-$atlas, $\tilde{A}$, induced by f is $\tilde{k}-$regular on $\tilde{D}$.

The function

$$q_f : D \to [1, \infty] , \ q_f(x) = \inf \left\{ k(x) \cdot \tilde{k}(\tilde{x}) \right\} ,$$

where infimum is taken over all $k-$regular $\varphi-$atlases on D, is called the *Finslerian characteristic function* of f.

It follows that f is $k\tilde{k}-$regular if $q_f(x) < \infty$, for every $x \in D$.

Let us consider a $f-$isomorphism of vector bundles $T : TD \to T\tilde{D}$. The restriction, T_x, of T to T_xD, $T_x : T_xD \to T_{\tilde{x}}\tilde{D}$, $\tilde{x} = f(x)$, is an isomorphism of linear spaces, hence the image by T_x of $B(0_x, \alpha_x)$ is an ellipsoid $\tilde{E}_0(T_x) \subset B(0_{\tilde{x}}, r_{\tilde{x}}) \subset T_{\tilde{x}}\tilde{D}$, where $r_{\tilde{x}} = \alpha_x \|T_x\|$. We can consider α_x such that $\exp_{\tilde{x}}$ is $(1+\varepsilon)$-isometry on $B(0_{\tilde{x}}, r_{\tilde{x}})$.

It follows that $\tilde{h}_{\tilde{x}} = (B(0_x, \alpha_x), T_x)$ is a T-chart at $0_{\tilde{x}} \in T_{\tilde{x}}\tilde{D}$ and so, we can consider a $\tilde{T}$-chart on $\tilde{D}$, induced by $\exp_{\tilde{x}}$, $\tilde{H}_{\tilde{x}} = \left(B(0_x, \alpha_x), \tilde{T}_{\tilde{x}}\right)$, $\tilde{T}_{\tilde{x}} = \exp_{\tilde{x}} \circ T_x$, and, in such a way, we obtain a $\tilde{T}$-atlas $\tilde{A} = \left\{\tilde{H}_{\tilde{x}} \ / \ \tilde{x} \in \tilde{D}\right\}$, called *atlas of geodesic ellipsoids.*

The geodesic ellipsoid $E_0(T_x) = \exp_{\tilde{x}}\left(\tilde{E}_0(T_x)\right)$ has the same extreme semiaxes as $\tilde{E}_0(T_x)$ ($\exp_{\tilde{x}}$ behaves as an isometry for the distances measured from $0_{\tilde{x}}$).

Let us consider $\tilde{E}_\alpha(T_x) = T_x(S(0_x, \alpha))$, $\alpha \in (0, \alpha_x)$, and

$$P_\alpha = \left\{ \tilde{d}\left(0_{\tilde{x}}, \tilde{Y}\right) \ / \ \tilde{Y} \in \tilde{E}_\alpha(T_x) \right\} = \left\{ \left\|\tilde{Y}\right\| \ / \ \|Y\| = \alpha \right\}.$$

The extreme semiaxes of $\tilde{E}_\alpha(T_x)$ are given by

$$\tilde{a}_0(\alpha, \tilde{x}) = \inf P_\alpha = \alpha \left\|T_x^{-1}\right\|^{-1},$$

$$\tilde{a}_1(\alpha, \tilde{x}) = \sup P_\alpha = \alpha \|T_x\|.$$

The function

$$p_T : \tilde{M} \to \mathbb{R}, \ p_T(\tilde{x}) = \frac{\tilde{a}_1(\alpha, \tilde{x})}{\tilde{a}_0(\alpha, \tilde{x})} = \|T_x\| \left\|T_x^{-1}\right\|$$

is called *the principal characteristic parameter* of the atlas of geodesic ellipsoids.

Arguing as above for f^{-1} and T^{-1}, we obtain

$$p_{T^{-1}}(x) = \|T_x\| \left\|T_x^{-1}\right\| = p_T(\tilde{x}).$$

The parameter of the atlas of geodesic ellipsoids is

$$\tilde{k}(\tilde{x}) = \limsup_{\alpha \to 0} \frac{\tilde{a}_1(\alpha, \tilde{x})}{\tilde{a}_0(\alpha, \tilde{x})} = p_T(\tilde{x}).$$

Lemma 2. *If $f : D \to \tilde{D}$ is a differentiable homeomorphism at $x \in D$ with $T_x f$ bijective, then:*

a) f is $k\tilde{k}$-regular on D iff for every $x \in D$, $F_x = \exp_{\tilde{x}}^{-1} \circ f \circ \exp_x$ is $k\tilde{k}$-regular at $0_x \in T_x D$;

b) F_x is $k\tilde{k}$-regular at 0_x iff $T_x f$ is $k\tilde{k}$-regular at 0_x;

c) $q_f(x) = q_{T_x f}(0_x) = q_{F_x}(0_x)$.

Proof. a) Let A be a φ-atlas k-regular on D, such that the induced $\tilde{\varphi}$-atlas $\tilde{A}$ is $\tilde{k}$-regular on $\tilde{D}$. We consider $h_x = (B_x, \varphi_x) \in A$ and $\tilde{h}_{\tilde{x}} = (B_{\tilde{x}}, \tilde{\varphi}_{\tilde{x}}) \in \tilde{A}$, where $\tilde{\varphi}_{\tilde{x}} = f \circ \varphi_x$. It follows that $H_x = (B_x, \phi_x)$, $\phi_x = \exp_x^{-1} \circ \varphi_x$ is k-regular and $\tilde{H}_{\tilde{x}} = \left(B_{\tilde{x}}, \tilde{\phi}_{\tilde{x}}\right)$, $\tilde{\phi}_{\tilde{x}} = \exp_{\tilde{x}}^{-1} \circ \tilde{\varphi}_{\tilde{x}}$ is $\tilde{k}$-regular.

Because

$$\tilde{\phi}_{\tilde{x}} = \exp_{\tilde{x}}^{-1} \circ \tilde{\varphi}_{\tilde{x}} = \exp_{\tilde{x}}^{-1} \circ f \circ \exp_x \circ \exp_x^{-1} \circ \varphi_x = F_x \circ \phi_x,$$

it follows that $\tilde{H}_{\tilde{x}}$ is the $\tilde{\phi}_{\tilde{x}}$-chart induced by F_x and so, F_x is $k\tilde{k}$-regular.

For $f = \exp_{\tilde{x}} \circ F_x \circ \exp_x^{-1}$, arguing as above, we obtain the sufficiency. In addition, we obtain

$$q_f(x) = q_{F_x}(0_x). \tag{2}$$

b) Let us consider the k-regular chart $H_x = (B_x, \phi_x)$ with the parameter

$$k(x) = \limsup_{\alpha \to 0} \frac{\sup P_\alpha}{\inf P_\alpha} < \infty, \ P_\alpha = \{\|X\| \ / \ X \in \phi_x(S(0,\alpha))\} \tag{3}$$

and the chart, $\bar{H}_{\tilde{x}} = \left(B_x, \bar{\phi}_{\tilde{x}}\right)$, $\bar{\phi}_{\tilde{x}} = T_x f \circ \phi_x$, induced by $T_x f$, for which

$$\bar{k}(\tilde{x}) = \limsup_{\alpha \to 0} \frac{\sup P'_\alpha}{\inf P'_\alpha}, \ P'_\alpha = \{\|(T_x f)(X)\| \ / \ X \in \phi_x(S(0,\alpha))\}. \tag{4}$$

For the $\tilde{\phi}$-chart, $\tilde{H}_{\tilde{x}} = \left(B_x, \tilde{\phi}_{\tilde{x}}\right)$, $\tilde{\phi}_{\tilde{x}} = F_x \circ \phi_x$, induced by F_x, we have

$$\tilde{k}(\tilde{x}) = \limsup_{\alpha \to 0} \frac{\sup P''_\alpha}{\inf P''_\alpha}, \ P''_\alpha = \{\|F_x(X)\| \ / \ X \in \phi_x(S(0,\alpha))\}. \tag{5}$$

Since $T_x f = DF_x(0_x)$, it follows that

$$F_x(X) = (T_x f)(X) + \varepsilon_x(X)\|X\|, \ \varepsilon_x : T_x M \to T_x M, \ \lim_{X \to 0_x} \|\varepsilon_x(X)\| = 0.$$

We have

$$\limsup_{\alpha \to 0} \left(\sup \bar{P}_\alpha\right) = 0, \ \bar{P}_\alpha = \{\|\varepsilon_x(X)\| \ / \ X \in \phi_x(S(0,\alpha))\}. \tag{6}$$

Since $T_x f$ is an isomorphism of topological vector spaces, we get

$$\|(T_x f)(X)\| \geq \frac{\|X\|}{\left\|(T_x f)^{-1}\right\|}.$$

It follows that

$$\sup P'_\alpha \geq \frac{\sup P_\alpha}{\left\|(T_x f)^{-1}\right\|};\ \ \inf P'_\alpha \geq \frac{\inf P_\alpha}{\left\|(T_x f)^{-1}\right\|}. \tag{7}$$

We have

$$\|T_x f(X)\| - \|\varepsilon_x\| \|X\| \leq \|F_x(X)\| \leq \|T_x f(X)\| + \|\varepsilon_x\| \|X\|$$

and so

$$\begin{aligned} \sup P'_\alpha - \sup P_\alpha \sup \bar{P}_\alpha &\leq \sup P''_\alpha \leq \sup P'_\alpha + \sup P_\alpha \sup \bar{P}_\alpha, \\ \inf P'_\alpha - \sup P_\alpha \sup \bar{P}_\alpha &\leq \inf P''_\alpha \leq \inf P'_\alpha + \sup P_\alpha \sup \bar{P}_\alpha. \end{aligned}$$

We obtain

$$\begin{cases} \bar{k}(\tilde{x}) \limsup\limits_{\alpha \to 0} \dfrac{1 - \left\|(T_x f)^{-1}\right\| \sup \bar{P}_\alpha}{1 + \dfrac{\sup P_\alpha}{\inf P_\alpha} \dfrac{\inf P_\alpha}{\inf P'_\alpha} \sup \bar{P}_\alpha} \leq \tilde{k}(\tilde{x}), \\ \tilde{k}(\tilde{x}) \leq \bar{k}(\tilde{x}) \limsup\limits_{\alpha \to 0} \dfrac{1 + \left\|(T_x f)^{-1}\right\| \sup \bar{P}_\alpha}{1 - \dfrac{\sup P_\alpha}{\inf P_\alpha} \dfrac{\inf P_\alpha}{\inf P'_\alpha} \sup \bar{P}_\alpha}. \end{cases} \tag{8}$$

From (3), (6), (7) and (8) it follows that $\tilde{k}(\tilde{x}) = \bar{k}(\tilde{x})$, which proves the assertion b) and we have

$$q_{F_x}(0_x) = q_{T_x f}(0_x). \tag{9}$$

c) It results from (2) and (9).

Lemma 3. *If $T : V \to \tilde{V}$ is an isomorphism of n-dimensional normed vector spaces, then T is $k\tilde{k}$-regular with $k(X) = \|T\| \left\|T^{-1}\right\|$, $\tilde{k}\left(\tilde{X}\right) = 1$ and $q_T(X) = p_{T^{-1}}(X)$, for every $X \in V$.*

Proof. Let us consider the $\tilde{k}$-regular $\tilde{\phi}$-chart, $\tilde{H}_{\tilde{X}} = \left(B\left(0_{\tilde{X}}, 1\right), \tilde{\phi}_{\tilde{X}}\right)$, $\tilde{\phi}_{\tilde{X}}\tilde{Y} = \tilde{X} + \tilde{Y}$, $\tilde{X} = TX$. It follows that $\tilde{k}\left(\tilde{X}\right) = 1$. The map T^{-1} induces a ϕ-chart $H_X = \left(B\left(0_{\tilde{X}}, 1\right), \phi_X\right)$, $\phi_X = T^{-1} \circ \tilde{\phi}_{\tilde{X}}$, with $k(X) = \|T\| \left\|T^{-1}\right\| < \infty$ and so H_X is k-regular. Thus, we obtain that T is $k\tilde{k}$-regular with $k(X) = \|T\| \left\|T^{-1}\right\|$, $\tilde{k}\left(\tilde{X}\right) = 1$. We have $p_{T^{-1}}(X) = k(X)\,\tilde{k}\left(\tilde{X}\right) = \|T\| \left\|T^{-1}\right\|$ and so

$$q_T(X) \le p_{T^{-1}}(X). \tag{10}$$

Let us consider a k-regular φ-chart, $h_X = (B(0,1), \varphi_X)$ and the $\tilde{\varphi}$-chart induced by T, $\tilde{h}_{\tilde{X}} = \left(B(0,1), \tilde{\varphi}_{\tilde{X}}\right)$, $\tilde{\varphi}_{\tilde{X}} = T \circ \varphi_X$, with the parameter $\tilde{k}\left(\tilde{X}\right)$. We have two cases:

1) $k(X) \ge p_{T^{-1}}(X)$, which implies that $k(X)\tilde{k}\left(\tilde{X}\right) \ge p_{T^{-1}}(X)$. It follows that $q_T(X) \ge p_{T^{-1}}(X)$ and from (10) we obtain $q_T(X) = p_{T^{-1}}(X)$.

2) $k(X) < p_{T^{-1}}(X)$. We denote by $\sigma_\alpha = \varphi_X(S(0,\alpha))$ and

$$r_0 = \inf\{\|Y - X\| \ / \ Y \in \sigma_\alpha\}, r_1 = \sup\{\|Y - X\| \ / \ Y \in \sigma_\alpha\};$$

it follows that $\sigma_\alpha \subset \bar{B}(X, r_1) - B(X, r_0)$. Taking $t_0 = r_0 \|T\|$ and $t_1 = r_1 \left\|T^{-1}\right\|^{-1}$, it follows that the ellipsoid $E_{t_0}\left(T^{-1}\right) = T^{-1}\left(S\left(\tilde{X}, t_0\right)\right)$ has the minimum semiaxis $a_0(t_0, X) = r_0 = t_0 \|T\|^{-1}$ and $E_{t_1}\left(T^{-1}\right) = T^{-1}\left(S\left(\tilde{X}, t_1\right)\right)$ has the maximum semiaxis $a_1(t_1, X) = r_1 = t_1 \left\|T^{-1}\right\|$. We obtain

$$k(X) = r_1 r_0^{-1} = t_1 t_0^{-1} \|T\| \left\|T^{-1}\right\| = t_1 t_0^{-1} p_{T^{-1}}(X)$$

and since $k(X) < p_{T^{-1}}(X)$ it follows that $t_1 < t_0$. We have

$$\begin{aligned} t_0 &\le \sup\left\{\left\|TY - \tilde{X}\right\| \ / \ Y \in \sigma_\alpha\right\}, \\ t_1 &\ge \inf\left\{\left\|TY - \tilde{X}\right\| \ / \ Y \in \sigma_\alpha\right\}, \end{aligned}$$

hence $\tilde{k}\left(\tilde{X}\right) \ge t_0 t_1^{-1} = (k(X))^{-1} p_{T^{-1}}(X)$ and then $k(X)\tilde{k}\left(\tilde{X}\right) \ge p_{T^{-1}}(X)$, which implies $q_T(X) \ge p_{T^{-1}}(X)$. By using (10) we obtain $q_T(X) = p_{T^{-1}}(X)$.

Remark 2. From Lemmas 2 and 3, it follows that if $f : D \to \tilde{D}$ is a differentiable homeomorphism at x with $J_f(x) \ne 0$, then f is $k\tilde{k}$-regular at x and $q_f(x) = \|T_x f\| \left\|(T_x f)^{-1}\right\|$.

3 The proof of main result

We consider $\mathcal{X}(M)$ the Lie algebra of the tangent fields on M and $\mathcal{X}_0(M) = \{V \ / \ V \in \mathcal{X}(M), \ \|V(x)\| = 1, \ \forall x \in M\}$.

The matrix $a_V = [a_{ij}(x, V)]$, for a fixed $V \in \mathcal{X}_0(M)$, is a Riemannian metric on M and the map

$$\|\cdot\|_V : T_x M \to \mathbb{R}, \ \|X\|_V = \sqrt{a_{ij}(x, V) X^i X^j}$$

is an Euclidean norm in $T_x M$.

Because the norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent, there exists the map $C_V : M \to [1, \infty)$ such that

$$C_V^{-1}(x)\|X(x)\|_V \le \|X(x)\| \le C_V(x)\|X(x)\|_V\,,\ \forall X \in \mathcal{X}(M)\,.$$

For every $V \in \mathcal{X}_0(M)$ we consider

$$\begin{aligned} P(x,V) = \ & \{C_V(x)\ /\ C_V^{-1}(x)\|X(x)\|_V \le \|X(x)\| \le C_V(x)\|X(x)\|_V\,, \\ & \forall X \in \mathcal{X}(M)\} \end{aligned}$$

and the map

$$C : M \to [1,\infty),\ C(x) = \inf\{Px,V)/V \in \mathcal{X}_0(M)\}\,.$$

It follows that for every $\varepsilon > 0$, there exists $V_\varepsilon \in \mathcal{X}_0(M)$ such that $C(x) \le C_{V_\varepsilon}(x) < C(x) + \varepsilon$ and so, we can find $V_0 \in \mathcal{X}_0(M)$ which satisfies

$$C^{-1}(x)\|X(x)\|_{V_0} \le \|X(x)\| \le C(x)\|X(x)\|_{V_0}\,,\ \forall X \in \mathcal{X}(M)\,.$$

If we consider $C = \sup\{C(x)\ /\ x \in M\} \in [1,\infty]$, we have

$$C^{-1}\|X(x)\|_{V_0} \le \|X(x)\| \le C\|X(x)\|_{V_0}\,,\ \forall X \in \mathcal{X}(M)\,,\ \forall x \in M$$

hence, if $C = 1$ then L is a Riemannian metric and if $C > 1$, it is a Finsler metric, that is C measures the deviation of the Finsler metric from a riemannian metric.

In the following we suppose that $C \in (1,\infty)$ and we denote by $\|X(x)\|_0$ the norm $\|X(x)\|_{V_0}$.

If $f : D \to \tilde{D}$ is a non-degenerate differentiable homeomorphism at $x \in M$, then between the Riemannian characteristic function $q_f^0(x) = \|T_x f\|_0 \left\|(T_x f)^{-1}\right\|_0$ and the Finslerian characteristic function $q_f(x)$ we have the relation

$$C^{-4}q_f^0(x) \le q_f(x) \le C^4 q_f^0(x)\,. \tag{11}$$

Lemma 4. *If $f : D \to \tilde{D}$ is a homeomorphism with q_f bounded in D, then f is almost everywhere (a.e.) differentiable (with respect to the Lebesgue measure) and $J_f(x) \neq 0$ a.e. in D.*

Proof. Let us consider the atlas of geodesic balls A^ε on D and $F_x = \exp_{\tilde{x}}^{-1} \circ f \circ \exp_x : B(0_x, r_x) \to T_{\tilde{x}}\tilde{D}$, $\tilde{x} = f(x)$. It results that $q_{F_x}(Y) \le (1+\varepsilon)^4 q_f(y)$ for every $Y \in B(0_x, r_x)$, $y = \exp_x Y$. We obtain that q_{F_x} is bounded on $B(0_x, r_x)$, hence it is differentiable a.e. with $J_{F_x} \neq 0$ a.e.(see [6]). It follows that f is differentiable a.e. on $B(x, r_x)$ with $J_f \neq 0$ a.e. Since M is paracompact the assertion of theorem follows.

Definition. A homeomorphism $f : D \to \tilde{D}$ is called $K-$*Finslerian quasiconformal* in D, $(K - FQC), 1 \le K < \infty$, if q_f is bounded in D and $q_f(x) \le K$ a.e. in D.

If the Finsler metric on M is a Riemannian metric, we say that f is $K-$Riemannian quasiconformal in D, $(K - RQC)$.

From (11) we obtain:

If f is $K - FQC$ in D, then $C^{-4}q_f^0(x) \leq q_f(x) \leq K$ a.e. in D and hence $q_f^0(x) \leq C^4 K$. We obtain that f is $K_0 - RQC$ in D, with $K_0 = C^4 K$.

Analogously, we obtain that if f is $K - RQC$ in D, then f is $K_0 - FQC$ in D, with $K_0 = C^4 K$, hence the *Theorem A* is proved.

Remark 3. From *Theorem A* it follows that the existence of the quasiconformal mappings in Finsler spaces can be reduced to the existence of the quasiconformal mappings in Riemann spaces.

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Received September 11, 2006