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Modelling of explosive magnetorotational phenomena: from 2D to 3D

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Abstract. In the paper we describe the results of mathematical modelling of magnetorotational(MR) supernova explosion in 1D and 2D approach and formulate the problems and features for the numerical approach to simulations of the MR supernovae in 3D case.

Mathematics subject classification: 85A30.
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1 Introduction

Explosions of supernovae are very spectacular event in the Universe. Explanation of mechanism of core collapse supernova explosion is one of the most interesting and complicated problems of modern astrophysics. At the initial stage of core collapse supernova research the mechanisms of explosion had been connected with neutrino deposition, and bounce shock propagation. Spherically symmetrical numerical simulations have shown that the bounce shock appears at the distance 10-30km from the center, then it moves to the radius of about 100-200 km, and stalls, not giving an explosion. Farther investigation of this problem was an extension of the same mechanism to 2D and 3D cases. Numerical simulations of 2D and 3D models have an additional feature connected with a development of neutrino driven convection deep inside, and behind the shock. The extensive calculations have shown that this mechanism does not give supernova explosions either with a sufficient level of confidence. Recently improved models of the core collapse, where the neutrino transport was simulated by solving the Bolzmann equation, also do not explode [12].

The MR mechanism for core collapse supernova explosion was suggested by Bisnovatyi-Kogan in 1970 [9], see also [10]. The main idea of the MR mechanism is to transform part of the rotational energy of presupernova into the radial kinetic energy (explosion energy). During collapse the star rotates differentially. Differential rotation leads to the appearing and amplification of the toroidal component of the magnetic field. Growth of the magnetic field means amplification of the magnetic pressure with time. A compression wave appears near the region of the extremum of the magnetic field. This compression wave moves outwards along steeply decreasing density profile. In a short time it transforms to the fast MHD shock wave. When the shock reaches the surface of the collapsing star it ejects part of the matter and

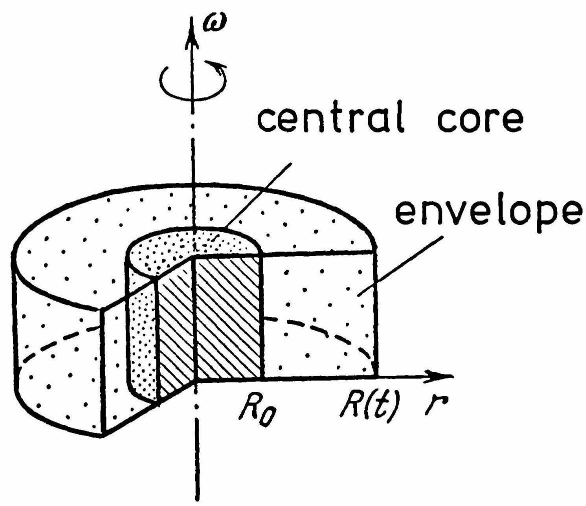

Figure 1. Model of MR presupernova in 1D case [11]

energy to the infinity. This ejection can be interpreted as explosion of the core collapse supernova.

The first 2D simulations of the collapse of the rotating magnetized star were presented in the paper [15], with unrealistically large values of the magnetic field. The differential rotation and amplification of the magnetic field resulted in the formation of the axial jet.

2 Results of 1D and 2D MR supernova simulations

The 1D simulations of MR supernova had been made in papers [4, 11]. In 1D case a star was represented as an infinite cylinder (Fig.1). For the simulations a set of ideal MHD equations with self gravitation in Lagrangian variables was used. Initial magnetic field had only r component. Differential rotation led to appearance and amplification of the toroidal φ component of the magnetic field. Numerical simulations of 1D MR supernova had shown that amplified due to the differential rotation toroidal field produced MHD shock wave which moved outwards. Part of the matter was ejected by the shock wave. The amount of the ejected energy $\approx 10^{51} erg$ is enough for the explanation of the supernova event. 1D simulations show that time of the evolution of MR supernova t_{expl} depends on the relation of the initial magnetic E_{mag} and gravitational E_{grav} energies $\alpha = \dfrac{E_{mag}}{E_{grav}}$ as $t_{expl} \sim \dfrac{1}{\sqrt{\alpha}}$. It means that for real values of the magnetic field ($\alpha \approx 10^{-6-8}$) t_{expl} becomes rather large. Parameter α characterizes a stiffness of the MHD equations describing MR supernova. The smallness of the parameter α is one of the main difficulties for the numerical simulation of MR supernova. From the physical point of view small α

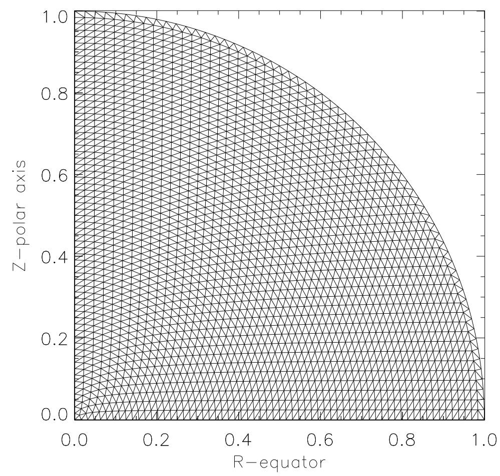

Figure 2. Triangular Lagrangian grid for 2D simulations of the magnetorotational supernova explosion

means existence two significantly different time scales. Very small acoustic time scale and huge time scale proportional to the time of the magnetic field amplification.

More realistic model of magnetorotational supernova was calculated in 2D approximation. The star was represented by a rotating self-gravitating gaseous body. The basic set of equations is a set of ideal MHD equations with self gravity in Lagrangian variables:

$$
\begin{gathered}
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}, \qquad \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \mathbf{v} = 0, \\
\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\mathrm{grad}\left(P + \frac{\mathbf{H}\cdot\mathbf{H}}{8\pi}\right) + \frac{\nabla\cdot(\mathbf{H}\otimes\mathbf{H})}{4\pi} - \rho\nabla\Phi, \\
\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{H}}{\rho}\right) = \mathbf{H}\cdot\nabla\mathbf{v},\ \Delta\Phi = 4\pi G\rho, \\
\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} + P\nabla\cdot\mathbf{v} + \rho F(\rho, T) = 0, \\
P = P(\rho, T),\ \varepsilon = \varepsilon(\rho, T),
\end{gathered}
\tag{1}
$$

where $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$ is the total time derivative, $\mathbf{x} = (r, \varphi, z)$, $\mathbf{v} = (v_r, v_\varphi, v_z)$ is the velocity vector, ρ is the density, P is the pressure, $\mathbf{H} = (H_r,\ H_\varphi,\ H_z)$ is the magnetic field vector, Φ is the gravitational potential, ε is the internal energy, G is gravitational constant, $\mathbf{H}\otimes\mathbf{H}$ is the tensor of rank 2, and $F(\rho, T)$ is the rate of neutrino losses.

Spatial Lagrangian coordinates are r, φ and z, i.e. $r = r(r_0, \varphi_0, z_0, t)$, $\varphi = \varphi(r_0, \varphi_0, z_0, t)$, and $z = z(r_0, \varphi_0, z_0, t)$, where r_0, φ_0, z_0 are the initial coordinates of material points of the matter.

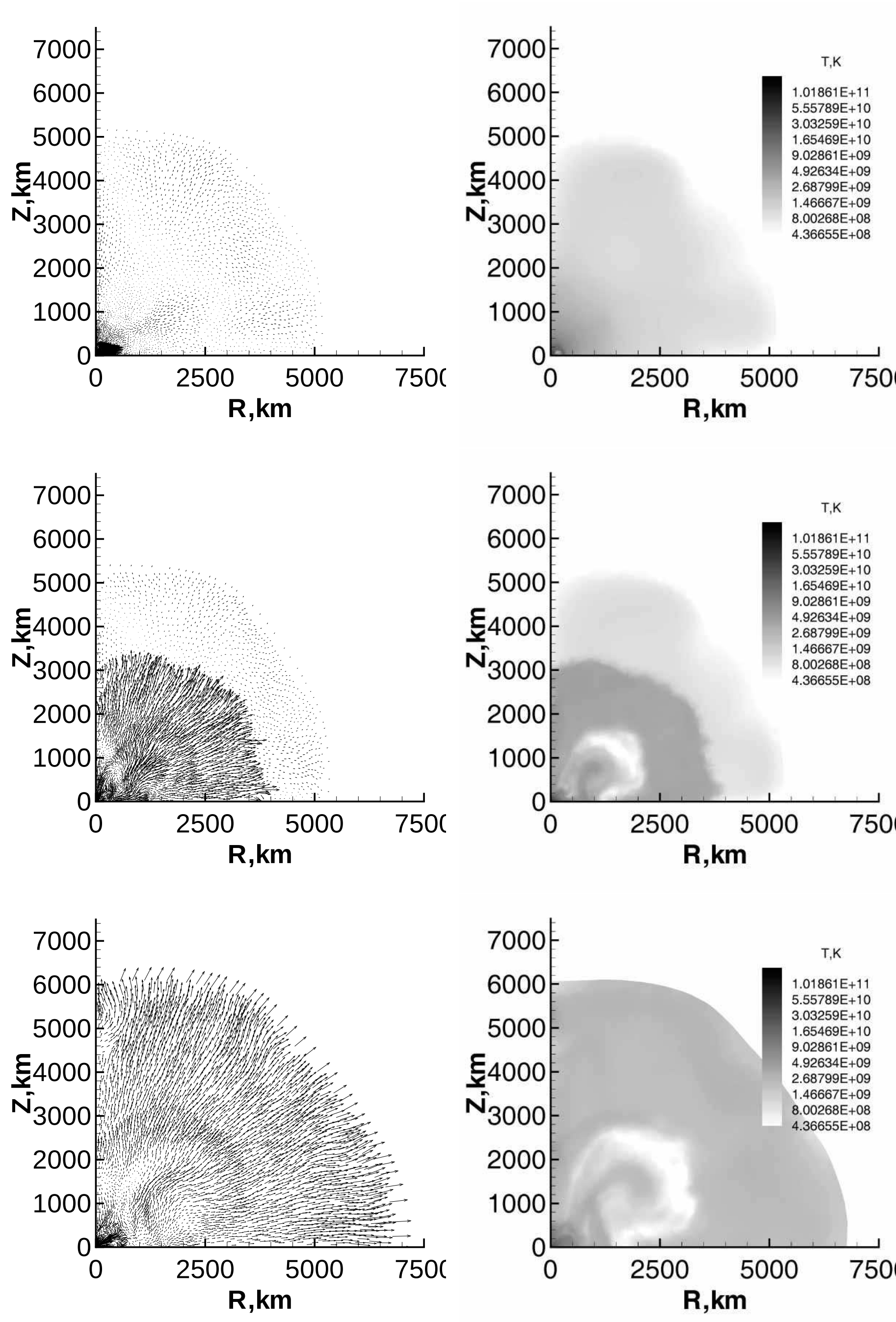

Figure 3. The distribution of the velocity field (left column) and the temperature (right column) for the time moments $t = 0.07,\ 0.2,\ 0.3s$ for the initial *quadrupole*-like magnetic field

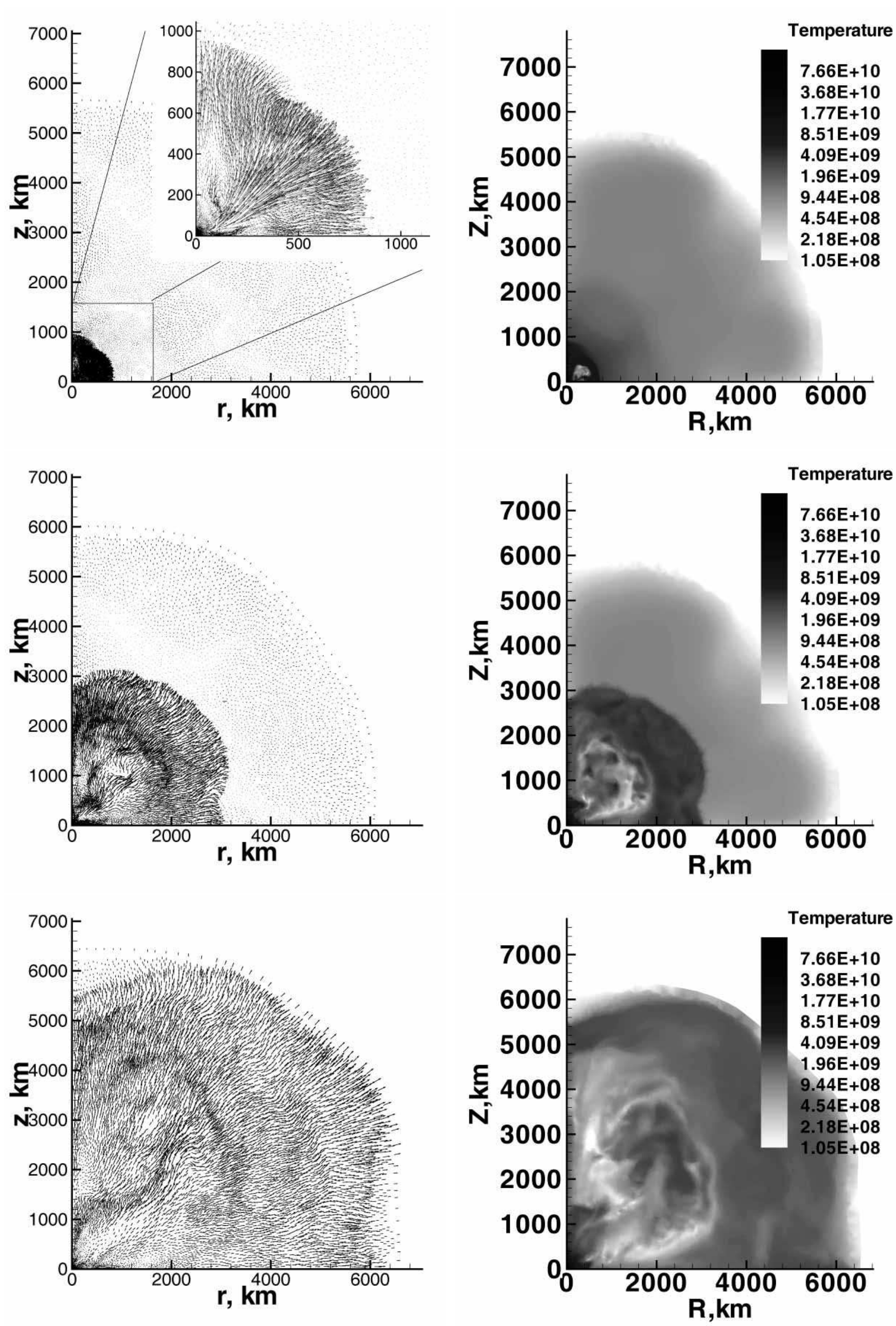

Figure 4. The distribution of the velocity field (left column) and the temperature (right column) for the time moments $t = 0.075$, 0.1, $0.25s$ for the initial *dipole*-like magnetic field

Taking into account symmetry assumptions $(\frac{\partial}{\partial\varphi} = 0)$, the divergency of the tensor $\mathbf{H}\otimes\mathbf{H}$ can be presented in the following form:

$$\nabla\cdot(\mathbf{H}\otimes\mathbf{H}) = \begin{pmatrix} \frac{1}{r}\frac{\partial(rH_rH_r)}{\partial r} + \frac{\partial(H_zH_r)}{\partial z} - \frac{1}{r}H_\varphi H_\varphi \\ \frac{1}{r}\frac{\partial(rH_rH_\varphi)}{\partial r} + \frac{\partial(H_zH_\varphi)}{\partial z} + \frac{1}{r}H_\varphi H_r \\ \frac{1}{r}\frac{\partial(rH_rH_z)}{\partial r} + \frac{\partial(H_zH_z)}{\partial z} \end{pmatrix}.$$

Axial symmetry $(\frac{\partial}{\partial\varphi} = 0, \quad r \geq 0)$ and symmetry to the equatorial plane $(z \geq 0)$ are assumed. The problem is solved in the restricted domain [5]. At $t = 0$ the domain is restricted by the rotational axis $r \geq 0$, equatorial plane $z \geq 0$, and outer boundary of the star where the density of the matter is zero, while poloidal components of the magnetic field H_r and H_z can be non-zero.

We assume axial and equatorial symmetry ($r \geq 0$, $z \geq 0$). On the rotational axis ($r = 0$) the following boundary conditions are defined: $(\nabla\Phi)_r = 0$, $v_r = 0$. On the equatorial plane ($z = 0$) the boundary conditions are: $(\nabla\Phi)_z = 0$, $v_z = 0$. On the outer boundary (boundary with vacuum) the following condition is defined: $P_{\text{outer boundary}} = 0$.

The equation of state, expression for the internal energy and formula for neutrino losses are the same as in [3].

At the initial moment we start with rigidly rotating sphere of $1.2M_\odot$ mass without magnetic field [2]. As first stage we calculate a rotating core collapse and formation of the protoneutron star. The ratios between the initial rotational and gravitational energies and between the internal and gravitational energies of the star are the following:

$$\frac{E_{rot}}{E_{grav}} = 0.0057, \qquad \frac{E_{int}}{E_{grav}} = 0.727.$$

During the collapse the bounce shock appears and moves outwards. The shock leads to the ejection of $\approx 2.9 \times 10^{48}$erg of energy. The amount of the ejected energy is too small for the explanation of the supernova explosion.

For the simulations we used completely conservative operator-difference scheme on triangular Lagrangian grid (Fig.2) of variable structure [6].

Results of the 2D simulations of the magnetorotational supernova are qualitatively different from 1D results. In the 2D case the magnetorotational instability (MRI) appears, leading to the exponential growth of all components of the magnetic field. MRI significantly reduce the time for the magnetorotational explosion. In the paper [3] a toy model for the explanation of MRI development in the magnetorotational supernova was suggested.

Due to MRI the dependence of the explosion time on the strength of the initial magnetic field can be expressed by the approximate formula: $t_{expl} \approx |log(\alpha)|$, where $\alpha = \dfrac{E_{mag}}{E_{grav}}$ is a relation between initial magnetic and gravitational energies.

In the 1D case the development of MRI is not possible due to the restricted number of the freedom degrees.

The results of 2D simulations [3,17] show that the magnetorotational mechanism allows to produce $0.5-0.6 \cdot 10^{51}$ ergs energy of explosion. These values of SN explosion energy correspond to estimations made from core collapse SN observations.

The shape of the magnetorotational explosion qualitatively depends on the configuration of the initial magnetic field. For the initial quadrupole like configuration [3] the explosion develops mainly near equatorial plane (Fig.3). The dipole like initial magnetic field [17] leads to the formation and development of mildly collimated axial jet (Fig.4).

3 Simulation of MR supernovae in 3D case

3D models of the magnetorotational supernova are the more realistic, and have no constraints connected with the symmetry assumptions.

3D models allow us to simulate the magnetorotational supernova explosion in the case when rotational axis and axis of dipole magnetic field (if dipole is taken as initial magnetic field) do not coincide (inclined rotator).

The application of numerical method in Lagrangian variables, similar to the method used for the 2D case, leads in 3D case to serious difficulties.

In 2D case the matter of the star is slipping in φ direction. To produce the magnetorotational explosion the protoneutron star has to make thousands of revolutions. The rotation of the matter in the outer layers of the protoneutron star is highly differential. If 3D Lagrangian grid consisting of tetrahedrons would be applied for the simulations, then in the region of strong differential rotation the grid would require reconstruction almost at every time step. The reconstruction of the grid leads to the interpolation of the grid functions to a new grid structure. Frequent application of the grid reconstruction procedure and interpolation of grid functions for the same parts of the Lagrangian grid can lead to the significant perturbation of the solution of initial set of MHD equations with self gravitation.

One of the possible ways to simulate magnetorotational supernova in 3D case is to apply methods based on the unstructured grids of Dirichlet cells (see for example [18]). This type of methods can be effectively applied for the simulations of the different types of gas dynamical flows, but the procedure of the construction of the grid of Dirichlet cells is rather expensive, especially in 3D case.

Another method widely applied for the simulations of astrophysical problems is Smooth Particle Hydrodynamics (SPH) [13, 16] method. Codes based on the SPH approach can be easily developed, but to achieve a high accuracy in simulations SPH requires huge number of particles. The simulation of the problems of gravitational gas dynamics using SPH leads to the concentration of the particles near the gravitational center, on the periphery of the computational domain the number of particles is rather small and it leads to the significant loss of the accuracy of the results of simulations.

One of the most suitable approaches for the simulations of the explosive magnetorotational phenomena in 3D case is an application of the numerical methods in Eulerian variables based on the solution of the decomposition of discontinuity (Riemann solver) problem. This type of methods was successfully applied for the solution of the different astrophysical magnetorotational problems. Application of the Eulerian grid does not require grid reconstruction and interpolation of grid functions. The methods of this type are described in [14]. The methods based on the approximate MHD Riemann solver in Eulerian variables are the most suitable for the simulations of the explosive magnetorotational phenomena

For the simulations of astrophysical magnetorotational explosive phenomena it is important to calculate gravitational potential with sufficient accuracy. The procedure of the calculation of the gravitational potential is rather expensive (up to 40% of the computer time for the time step).

For our simulations we plan to apply Adaptive Mesh Refinement (AMR) approach. The adaptive refinement and rarefaction of the grid can increase the accuracy of the calculations significantly with the reasonable number of the grid points. We expect to apply AMR using two approaches. First one is a construction of the hierarchical tree which root is our initial 3D grid [1,7]. The second approach consists in construction of the rectangular (for the 2D case) [8] or parallelepiped (for the 3D case) patches consisting of specially chosen association of the cells of one level.

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References

[1] Aftersmoth M.J., Berger M.J., Melton J.E. J. AIAA, 1998, N 36(6), p. 952.

[2] Ardeljan N.V., Bisnovatyi-Kogan G.S., Kosmachevskii K.V., Moiseenko S.G. Astrophysics, 2004, **47**, 1.

[3] Ardeljan N.V., Bisnovatyi-Kogan G.S., Moiseenko S.G. MNRAS, 2005, **359**, p. 333.

[4] Ardelyan N.V., Bisnovatyi-Kogan G.S., Popov Yu.P. Astron. Zh., 1979, **56**, p. 1244.

[5] Ardelyan N.V., Bisnovatyi-Kogan G.S., Popov Yu.P., Chernigovsky S.V. Astron. Zh., 1987, **64**, p. 761.

[6] Ardeljan N.V., Kosmachevskii K.V. Comput. Math. Modelling, 1995, **6**, p. 209.

[7] Balsara D.S., Norton, G.D. Parallel Computing, 2001, **27**, p. 37.

[8] Berger M., Rigoutsos I. IEEE Trans. System Man Cybernetics, 1991, **21**, p. 61.

[9] Bisnovatyi-Kogan G.S. Astron. Zh., 1970, **47**, p. 813.

[10] Bisnovatyi-Kogan G.S. Stellar Physics, 2 volumes, Springer, 2001.

[11] Bisnovatyi-Kogan G.S., Popov Yu.P., Samochin, A.A. Astrophys. and Space Sci., 1976, **41**, p. 321.

[12] Buras R., Rampp M., Janka H.Th., Kifonidis K. Phys. Rev. Lett., 2003, **90**, p. 241101.

[13] Gingold R.A., Monaghan J.J. MNRAS, 1977, **181**, p. 375.

[14] Kulikovsky A.G., Pogorelov N.V., Semenov A.Yu. *Mathematical problems of the numerical solution of sets of hyperbolic equations*. Moskva, FIZMATLIT, 2001.

[15] Leblanc L.M., Wilson J.R. ApJ, 1970, **161**, p. 541.

[16] Lucy L.B. ApJ, 1977, **82**, p. 1013.

[17] Moiseenko S.G., Bisnovatyi-Kogan G.S., Ardeljan N.V. MNRAS, 2006, **370**, p. 501.

[18] Pushkina I.G., Tishkin V.F. Mathematical modeling, 2000, **12**, p. 4.

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