BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(55), 2007, Pages 49–54
ISSN 1024–7696

Maximization methods of turbo-machines performances

Mircea Dimitrie Cazacu

Abstract. Due to the important role of the turbo-machines efficiency, concerning the energy economy and environment pollution diminishing [1, 2], we shall present three original methods to maximize their performances, establishing the optimum blade profile and its setting angles at different blade radii.

Mathematics subject classification: ?.
Keywords and phrases: ?.

1 Maximum extracted power by a run-of-river hydraulic turbine or wind turbine

Projecting the two components of hydrodynamic resultant on the rotational peripheral direction, we shall obtain the mechanical power expression

$$P = UF_u = U(F_y \sin\beta - F_x \cos\beta) = \frac{\rho}{2} V^3 bl \left[c_y(i) \frac{\cos\beta}{\sin^2\beta} - c_x(i) \frac{\cos^2\beta}{\sin^2\beta} \right], \qquad (1)$$

and cancelling the partial derivative

$$\frac{\partial P}{\partial \beta} = -c_y(i) \frac{1+\cos^2\beta}{\sin^3\beta} + c_x(i) \frac{\cos\beta(2+\cos^2\beta)}{\sin^4\beta} = 0, \qquad (2)$$

introducing the notation $\sin^2\beta = x$, we must solve the algebraic equation

$$P(x) = \left[f^2(i)+1\right] x^3 - \left[4f^2(i)+7\right] x^2 + \left[4f^2(i)+15\right] x - 9 = 0, \qquad (3)$$

from which the sub-unit solution maximizes really the power, for any chosen profile-shape. Once more, introducing these values i and β in the power expression (1), the maximal power value will indicate the best profile to use [3]. Applying the relation $V = U\mathrm{tg}\beta$ at the outskirts, we obtain the optimal angular velocity, which being the same for all the blade, determines the rotation velocity at any other radius R_j and because the relative angle is thus known, the power maximization will be obtained only by the variation of the incident angle in case of considered profile.

1.1. The best incidence angle of blade profile for other radii

For other flow channel, placed at radius $R_j \neq R_p$, the peripheral radius, we obtain the maximization of the extracted power

$$P_j = \frac{V}{\mathrm{tg}\beta_j} \frac{\rho}{2} V^2 bl_j(R_j) \left[c_y(i) \frac{1}{\sin\beta_j} - c_x(i) \frac{\cos\beta_j}{\sin^2\beta_j} \right] =$$

$$= \frac{\rho}{2} V^3 bl[A(R_j)c_y(i) - B(R_j)c_x(i)]. \tag{4}$$

From the fluid by annulling its partial derivative with respect to the incidence angle i

$$\frac{\partial \lfloor A(R)c_y(i) - B(R)c_x(i) \rfloor}{\partial i} = 0 = \left[A(c_{y1} - 4i^3 c_{y4}) - B(c_{x1} + 2ic_{x2}\right], \tag{5}$$

in which we considered the usual expressions of variation with the incidence angle of the lift and drag coefficients, supposed of the form: $c_y(i) = c_{y0} + ic_{y1} - -i^4 c_{y4}$ and $c_x(i) = c_{x0} + ic_{x1} + i^2 c_{x2}$, finally obtaining the calculation formula of the best incidence angle for any radius

$$i^3 + i\frac{\omega_{\text{opt}} c_{x2}}{2V c_{y4}} R + \frac{\omega_{\text{opt}} c_{x1}}{4V c_{y4}} R - \frac{c_{y1}}{4c_{y4}} = 0, \tag{6}$$

with the interesting remark that the optimum incidence angle rises at the same time with the radius decreasing, to obtain a greater velocity around the profile [3].

The good performance of power coefficient $C_p = 0,42$ obtained for a three blade rotor [4] and $C_p = 0,56$ for a four blade rotor have put into the evidence the validity of this maximization method.

2 Maximization of the propulsion force for an aircraft or ship propeller

The problem of propulsion force increasing for the same consumed mechanical power at the shaft, is very important not only concerning the operation radius enlargement of an aircraft or ship, but also by the fossil fuel savings and environmental protection, being of a greater importance for the ecological boats, which use the solar energy by means of photovoltaic cells [5, 6].

2.1. The determination of the best peripheral relative angle

Taking into account the expressions of the lift and drag forces, exerted on the profiled blade, laid at the incidence angle i with respect to the relative angle β, corresponding to the relative velocity W from the velocity triangle, we can calculate the axial component of these forces, representing the propulsion force

$$F_a = F_y \cos\beta - F_x \sin\beta = \frac{\rho}{2} V^2 bl(R) \left[c_y(i)\frac{\cos\beta}{\sin^2\beta} - c_x(i)\frac{1}{\sin\beta}\right] \tag{7}$$

and also the expression of the shaft driving mechanical power

$$P_m = U(F_y \sin\beta + F_x \cos\beta) \quad \text{or} \quad p_m = \frac{2P_m}{\rho V^3 bl} = c_y(i)\frac{\cos\beta}{\sin^2\beta} + c_x(i)\frac{\cos^2\beta}{\sin^3\beta}. \tag{8}$$

By annulling the partial differential of the axial force (7) with respect to the relative angle β

$$\partial F_a/\partial\beta = 0 = -c_y(1 + \cos^2\beta) + c_x \sin\beta \cos\beta, \tag{9}$$

one obtains the condition to maximize the propulsion axial force (denoting by $x = \sin^2\beta$ and the profile fineness $f(i) = c_y(i)/c_x(i)$ as function of the incidence angle i) given by the following algebraic relation

$$(f^2 + 1)x^2 - (4f^2 + 1)x + 4f^2 = 0, \tag{10}$$

having two real solutions and putting into the evidence the relative best and respectively worst angle β as function of the fineness of the aerodynamic or hydrodynamic profiles, for the positive value under the root expression, necessary to assure the non-imaginary solutions

$$x = \frac{4f^2 + 1 \pm \sqrt{1 - 8f^2}}{2f^2 + 2}, \quad \text{for} \quad 1 - 8f^2 \geq 0 \to f(i) = \frac{c_y}{c_x} \leq 0.3536\ldots \tag{11}$$

which condition eliminates a lot of profiles too curved and prefers these that have the lift force near by zero for a certain incidence angle i [7].

2.2. The determination of the optimum profile setting angle for other radii

For the other radii, because the peripheral relative angle β_j is already determined by the relation $V = U_j \mathrm{tg}\beta_j$, the power maximization will be obtained only by the election of the optimum incidence angle in case of considered profile, as we shall see below. We have determined the blade profile angle $\beta_b = \beta_j - i$ annulling the expression of the axial force with respect to the incidence angle i of the profile [3], obtaining the relation

$$F_j = \frac{\rho}{2}V^2 bl(R_j)\left[(c_{y0} + ic_{y1}\frac{\cos\beta_j}{\sin^2\beta_j} - (c_{x0} + ic_{x1} + i^2 c_{x2})\frac{1}{\sin\beta_j}\right] \tag{12}$$

the blade spread being $b = \delta R =$ constant and the blade depth l as function of radius R_j having no importance, we can annul the axial propulsion force with respect to the incidence angle to obtain the optimal incidence for each relative radius

$$\frac{\partial F_a}{\partial i} = 0 = \frac{c_{y1}}{\mathrm{tg}\beta_j} - c_{x1} - 2ic_{x2} \to i_{\mathrm{opt}} = \frac{1}{2c_{x2}}(\frac{c_{y1}}{\mathrm{tg}\beta_p}\frac{R_j}{R_p} - c_{x1}), \tag{13}$$

considering the variation approximately linear of the lift coefficient of the profile (for example of the symmetric profile Gö 445 [3, 4]) as function of the incidence angle $C_y(i) \simeq C_{y0} + C_{y1}i = 0.002i$ and the parabolic approximately variation of the drag coefficient of the profile

$$C_x(i) \simeq C_{x0} + C_{x1}i - C_{x2}i^2 = 0.005 + 0.004,5i - 0.000,5i^2. \tag{14}$$

In this manner we can establish the airfoil profile, which realises the best propulsion axial force, as also the value of the relative mechanical driving power.

For the smaller relative radius $r = R_j/R_p < 1$, where we have already the relative angle β_j imposed, to maximize the axial force F_a one calculates the values of the optimal incidence angle i_{opt} given by the relation (13).

2.3. Maximization of the ratio between the axial force and consumed power

In this case [8], by annulling the partial differential with respect to the relative angle $0 \leq \beta \leq \pi/2$

$$\frac{\partial \left(f_a/p_m\right)}{\partial \beta} = \frac{f\mathrm{ctg}^2\beta - 2\mathrm{ctg}\beta - f}{\cos^2\beta(f^2 + 2f\mathrm{ctg}\beta + \mathrm{ctg}^2\beta)} = 0 \rightarrow f\mathrm{ctg}^2\beta - 2\mathrm{ctg}\beta - f = 0, \quad (15)$$

one obtains the maximization condition, that by introducing the notation $x = \mathrm{ctg}\beta$, leads us to the solving of the algebraic equation of 2$^{\mathrm{nd}}$ degree

$$f(i)x^2 - 2x - f(i) = 0, \quad (16)$$

having always two real solutions, one positive and other negative

$$x_{1,2} = \frac{1 \pm \sqrt{1+f^2}}{f(i)}, \quad (17)$$

as one can see for the case of Göttingen 450 profile [3], which are vindicated again as the best performing, and where we put also the value of the ratio f_a/p_m for the confirmation of the maximal value of the axial force, obtained at the approximate incidence angle $i \approx 1°$.

3 The obtaining of the fluid current maximal velocity

This optimization method presents a special importance in the problem of optimal profiling of the axial rotor blades for a mixer, ventilator or pump. To maximize the fluid current velocity V, we shall present two possibilities to solve this problem: using the velocity relation deduced by the axial force expression (7) or from that of the rotor driving power (8).

3.1. The fluid velocity maximizing using the axial force relation

We shall consider the mathematical problem of linked maximum, corresponding to the obtaining of maximal velocity of the axial fluid current using the axial force expression (7).

3.1.1. The optimal profiling of the blade and the optimal peripheral setting angle

Considering the relation (7), we can write

$$\frac{V^2\rho bl}{2F_a} = \frac{1}{c_y\frac{\cos\beta}{\sin^2\beta} - c_x\frac{1}{\sin\beta}} \rightarrow \frac{V}{\sqrt{2F_a/\rho bl}} = \left(c_y\frac{\cos\beta}{\sin^2\beta} - c_x\frac{1}{\sin\beta}\right)^{-1/2}, \quad (18)$$

from that by annulment of its partial derivation, we shall obtain the value of the relative angle β, introducing the profile fineness $f = c_y/c_x$ and denoting $\sin^2\beta = x$, the problem reduces to the solving of the same algebraic equation (10).

3.1.2. The optimal setting angle of the profile to the other blade radii

Because for the other blade radii the relative angle is already determined, we shall maximize the current velocity by annulment of its derivative with respect to profile incidence angle, obtaining the new expression of the fluid velocity

$$\frac{V}{\sqrt{2F_a/\rho bl}} = \left[(c_{y0} + ic_{y1} - i^2 c_{y2})\frac{\cos\beta_j}{\sin^2\beta_j} - (c_{x0} + ic_{x1} + i^2 c_{x2})\frac{1}{\sin\beta_j}\right]^{-1/2}, \quad (19)$$

which by annulment of its partial derivation with respect to the incidence angle i, gives us the necessary relation

$$i_{\text{opt}} = \frac{c_{y1}\text{ctg}\beta_j - c_{x1}}{c_{y2}\text{ctg}\beta_j + c_{x2}}. \quad (20)$$

3.2. The obtaining of the maximal velocity for the minimum consumed power

We solved this problem reporting the fluid velocity to the rotor driving mechanical power

$$\frac{V}{\sqrt[3]{2P_m/\rho bl}} = \left[c_y(i)\frac{\cos\beta}{\sin^2\beta} + c_x(i)\frac{\cos^2\beta}{sin^3\beta}\right]^{-1/3}, \quad (21)$$

in which we shall annul the partial derivative

$$\frac{\partial V}{\partial\beta} \approx \frac{c_y\sin\beta(2-\sin^2\beta) + c_x(3-\sin^2\beta)\cos\beta}{3[c_y\sin\beta\cos\beta + c_x(1-\sin^2\beta)]^{4/3}} = 0 \quad (22)$$

and because the denominator can never become infinite, the annulment of the numerator leads us to a same algebraic equation as (3).

References

[1] Cazacu M.D. *Tehnologii pentru o dezvoltare durabilă*. Acad. Oamenilor de Ştiinţă din România. Congresul "Dezvoltarea în pragul mileniului al III-lea", Secţia "Dezvoltarea durabilă", 27–29 septembrie 1998, Bucureşti. Editura Europa Nova, 1999, p. 533–539.

[2] Cazacu M.D. *A survey of the technics which might preserve the biosphere reservation Danube Delta*. The 8th Symposium "Technologies, installations and equipments for improvement of environment quality", 9–12 November 1999, Bucharest.

[3] Cazacu M.D., Nicolaie S. *Micro-hydroturbine for run-of-river power station.* A 2-Conferinţă a Hidroenergeticienilor din România, 24–25 mai 2002, Univ. Politehnica, Bucureşti, Vol. II, p. 443–448.

[4] Cazacu M.D. *Microagregat hidroelectric pentru asigurarea autonomiei energetice a balizelor luminoase sau a unor bărci fluviale*. Rev. Ştiinţă, Industrie, Tehnologie, Nr. 2, Bucureşti, 2005, p. 44–45.

[5] Tudor G., Mocanu Z. *Solar energy propelled craft intended for the ecological tourism and transport in "Danube Delta Biosphere" and other protected zones.* The 1st International Symposium "Renewable energies and Sustainable development", 23–25 September 2004, Tulcea.

[6] Aucouturier J.L., Henry H., Stempin E. *Electro solar boats for passengers transport: a reality now.* The 2nd Internat. Symposyum "Renewable energies and Sustainable development", 22–24 September 2005, Tulcea.

[7] Cazacu M.D. *The maximization of the propulsion force for an aircraft or ship propeller.* The 30th "Caius Iacob" Conference on Fluid Mechanics and its Technical Applications, Bucharest, 25–26 November 2005, CD.

[8] Cazacu M.D. *Optimization of the axial propulsion force.* A 31-a Conferinţă Naţională "Caius Iacob" de Mecanica fluidelor şi Aplicaţiile ei tehnice, Univ. Transilvania din Braşov, 19–21 oct. 2006, Bull. of Transilvania University, series B1 – Mathematics, Informatics, Physics, 2006, Vol. 13(48), p. 71–76.

University Politehnica of Bucharest
Romania
E-mail: *cazacumircea@yahoo.com*

Received March 16, 2007