BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(55), 2007, Pages 35–48
ISSN 1024–7696

A characterization of the solutions of the Darboux Problem for third order hyperbolic inclusions

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Abstract. In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form $u_{xyz} \in F(x,y,z,u)$ and we prove a characterization of the solutions of the considered problem using the Aumann integral defined for multifunctions.

Mathematics subject classification: 35L30, 35R70, 47H10.
Keywords and phrases: Multifunction, hyperbolic inclusion, upper semi-continuity, initial values, absolutely continuous in Carathéodory's sense function, Aumann integral.

1 Introduction

In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form

$$\frac{\partial^3 u(x,y,z)}{\partial x\, \partial y\, \partial z} \in F(x,y,z,u)\,,\ (x,y,z) \in D = [0,a] \times [0,b] \times [0,c]\,,\ u \in \Omega \subset \mathbb{R}^n \tag{1.1}$$

with initial values

$$\begin{cases} u(x,y,0) = \varphi(x,y)\,, & (x,y) \in D_1 = [0,a] \times [0,b]\,, \\ u(0,y,z) = \psi(y,z)\,, & (y,z) \in D_2 = [0,b] \times [0,c]\,, \\ u(x,0,z) = \chi(x,z)\,, & (x,z) \in D_3 = [0,a] \times [0,c] \end{cases} \tag{1.2}$$

where φ, ψ, χ are absolutely continuous in Carathéodory's sense functions $[2, \S 565-570]$, $\varphi \in C^*(D_1;\mathbb{R}^n)$, $\psi \in C^*(D_2;\mathbb{R}^n)$, $\chi \in C^*(D_3;\mathbb{R}^n)$ and they satisfy the conditions

$$\begin{cases} u(x,0,0) = \varphi(x,0) = \chi(x,0) = v^1(x)\,, & x \in [0,a]\,, \\ u(0,y,0) = \varphi(0,y) = \psi(y,0) = v^2(y)\,, & y \in [0,b]\,, \\ u(0,0,z) = \psi(0,z) = \chi(0,z) = v^3(z)\,, & z \in [0,c]\,, \\ u(0,0,0) = v^1(0) = v^2(0) = v^3(0) = v^0, & \end{cases} \tag{1.3}$$

where $F : D \times \Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact, convex and non-empty values and $\Omega \subset \mathbb{R}^n$ is an open subset.

Under suitable assumptions, we proved in [16] an existence theorem for a local solution of the Darboux Problem (1.1) + (1.2) and that the set of its solutions is compact in Banach space $C(D_0;\mathbb{R}^n)$, $D_0=[0,x_0]\times[0,y_0]\times[0,z_0]\subseteq D$; moreover, as a function of the initial values this set defines an upper semi-continuous multifunction.

In [17] we proved a theorem of prolongation for the solutions of the considered problem and also an existence theorem for a saturated solution.

In this paper we prove a characterization of the solutions of Darboux Problem (1.1) + (1.2) using the Aumann integral defined for multifunctions.

This study has been suggested by [15] and it provides an extension of the results in that article.

2 Preliminaries

The definitions and Theorems 2.1–2.5 plus Propositions 2.1–2.4 in this section are taken from [1, 2, 5–14].

Definition 2.1. Let X and Y be two non-empty sets. A *multifunction* $\Phi : X \to 2^Y$ is a function from X into the family of all non-empty subsets of Y.

To each $x \in X$, a subset $\Phi(x)$ of Y is associated by the multifunction Φ. The set $\bigcup\limits_{x\in X} \Phi(x)$ is the *range* of Φ. $\Phi(X) = \{\bigcup \Phi(x) \mid x \in X\}$.

Definition 2.2. Let us consider $\Phi : X \to 2^Y$.

a) If $A \subset X$, the *image* of A by Φ is $\Phi(A) = \bigcup\limits_{x\in A} \Phi(x)$;

b) If $B \subset Y$, the *counterimage* of B by Φ is

$$\Phi^-(B) = \{x \in X \mid \Phi(x) \cap B \neq \emptyset\};$$

c) The *graph* of Φ, denoted graph Φ, is the set

$$\operatorname{graph}\Phi = \{(x,y) \in X \times Y \mid y \in \Phi(x)\}.$$

Definition 2.3. Let us now take $\Phi : X \to 2^Y$. An element $x \in X$ with the property $x \in \Phi(x)$ is called a *fixed point* of the multifunction Φ.

Definition 2.4. A univalued function $\varphi : X \to Y$ is said to be a *selection* of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

Definition 2.5. Let X and Y be two topological spaces. The multifunction $\Phi : X \to 2^Y$ is *upper semi-continuous* if, for any closed $B \subset Y$, $\Phi^-(B)$ is closed in X.

Definition 2.6. If $(X,\mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is *measurable* if $\Phi^-(B) \in \mathcal{F}$ for every closed subset $B \subset Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^-(B)$ is measurable.

Theorem 2.1 [13]. *Let X and Y be two metric spaces, Y compact and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi(x)$ is a closed subset of Y for any $x \in X$. The following assertions are equivalent:*

i) the multifunction Φ *is upper semi-continuous;*

ii) the graph of Φ *is a closed subset of* $X \times Y$*;*

iii) any would be the seguences $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$*, from* $x_n \to x$, $y_n \in \Phi(x_n)$ *and* $y_n \to y$ *it follows that* $y \in \Phi(x)$*.*

Definition 2.7 [2, 7, 8]. The function $u : \triangle \to \mathbb{R}^n, \triangle \subset \mathbb{R}^2$, is *absolutely continuous in Carathéodory's sense* $[2, \S 565 - 570]$ if and only if it is continuous on $\triangle$, absolutely continuous in x (for any y), absolutely continuous in y (for any x), $u_x(x, y)$ is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in y (for any x) and u_{xy} is Lebesgue-integrable on $\triangle$.

Theorem 2.2 [2, 6, 14]. *The function* $u : \triangle \to \mathbb{R}^n$, $\triangle = [0, a] \times [0, b] \subset \mathbb{R}^2$, *is absolutely continuous in Carathéodory's sense on* $\triangle$ *if and only if there exist* $f \in L^1(\triangle; \mathbb{R}^n)$, $g \in L^1([0, a]; \mathbb{R}^n)$, $h \in L^1([0, b]; \mathbb{R}^n)$ *such that*

$$u(x, y) = \int_0^x \int_0^y f(s, t)\, ds\, dt + \int_0^x g(s)\, ds + \int_0^y h(t)\, dt + u(0, 0).$$

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(\triangle; \mathbb{R}^n)$ [7, 8]. In [6], this space is denoted by $AC(\triangle; \mathbb{R}^n)$.

Theorem 2.3 [6]. *The space* $C^*(\triangle; \mathbb{R}^n)$ *endowed with the norm*

$$\|u(\cdot, \cdot)\| = \int_0^a \int_0^b \|u_{xy}(s, t)\|\, ds\, dt + \int_0^a \|u_x(s, 0)\|\, ds + \int_0^b \|u_y(0, t)\|\, dt + \|u(0, 0)\|,$$

$\triangle = [0, a] \times [0, b] \subset \mathbb{R}^2$*, where* $\|\cdot\|$ *is the Euclidean norm, is a Banach space.*

Definition 2.8 [2, 9]. The function $u : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^3$, is absolutely *continuous in Carathéodory's sense* $[2, \S 565 - 570]$ if and only if $u(x, y, z)$ is continuous on D, absolutely continuous in each variable (for any pair of the other two variables) and similarly for $u_x(x, y, z)$, $u_y(x, y, z)$, $u_z(x, y, z)$, $u_{xy}(x, y, z)$, $u_{yz}(x, y, z)$, $u_{xz}(x, y, z)$, and u_{xyz} is Lebesgue-integrable on D.

Theorem 2.4 [6]. *The function* $u : D \to \mathbb{R}^n$, $D = [0, a] \times [0, b] \times [0, c] \subset \mathbb{R}^3$, *is absolutely continuous in Carathéodory's sense on* D *if and only if there exist* $f \in L^1(D; \mathbb{R}^n)$, $g_1 \in L^1(D_1; \mathbb{R}^n)$, $g_2 \in L^1(D_2; \mathbb{R}^n)$, $g_3 \in L^1(D_3; \mathbb{R}^n)$, $h_1 \in L^1([0, a]; \mathbb{R}^n)$, $h_2 \in L^1([0, b]; \mathbb{R}^n)$, $h_3 \in L^1([0, c]; \mathbb{R}^n)$, *such that*

$$\begin{aligned} u(x, y, z) = {} & \int_0^x \int_0^y \int_0^z f(r, s, t)\, dr\, ds\, dt + \int_0^x \int_0^y g_1(r, s)\, dr\, ds + \\ & + \int_0^y \int_0^z g_2(s, t)\, ds\, dt + \int_0^x \int_0^z g_3(r, t)\, dr\, dt + \\ & + \int_0^x h_1(r)\, dr + \int_0^y h_2(s)\, ds + \int_0^z h_3(t)\, dt + u(0, 0, 0). \end{aligned}$$

We denote the class of absolutely continuous functions in Carathéodory's sense on D by $C^*(D;\mathbb{R}^n)$ [9].

Theorem 2.5 [6]. *The space* $C^*(D;\mathbb{R}^n)$ *endowed with the norm*

$$\begin{aligned}\|u(\cdot,\cdot,\cdot)\| = &\int_0^a\int_0^b\int_0^c \|u_{xyz}(r,s,t)\|\,dr\,ds\,dt + \int_0^a\int_0^b \|u_{xy}(r,s,0)\|\,dr\,ds+\\ &+\int_0^b\int_0^c \|u_{yz}(0,s,t)\|\,ds\,dt + \int_0^a\int_0^c \|u_{xz}(r,0,t)\|\,dr\,dt+\\ &+\int_0^a \|u_x(r,0,0)\|\,dr + \int_0^b \|u_y(0,s,0)\|\,ds+\\ &+\int_0^c \|u_z(0,0,t)\|\,dt + \|u(0,0,0)\|,\end{aligned}$$

where $\|\cdot\|$ *is the Euclidean norm, is a Banach space.*

We denote by $d(x,y)$ the Euclidean distance from x to y, $x,y\in\mathbb{R}^n$, $\mathbb{R}^n$ is the Euclidean space. If $A\subset\mathbb{R}^n$, $d(x,A)=\inf\{d(x,y)\mid y\in A\}$.

$B[x,r]$ is the open ball of radius $r>0$ centered at $x\in\mathbb{R}^n$, $\operatorname{Conv}A$ is the convex covering of $A\subset\mathbb{R}^n$ and

$$|A| = \sup\{\|\zeta\| \mid \zeta\in A\}.$$

$\mathcal{C}(\mathbb{R}^n)$ is the set of compact and non-empty subsets of $\mathbb{R}^n$. Similarly with [1, 5, 15], we define the Aumann integral for multifunctions of three variables.

Definition 2.9. Let $D=[0,a]\times[0,b]\times[0,c]\subset\mathbb{R}^3$. For each $(x,y,z)\in D$, let $H(x,y,z)$ be a non-empty subset of $\mathbb{R}^n$. Let $\mathcal{H}$ be the set of functions $h:D\to\mathbb{R}^n$ integrable on D and $h(x,y,z)\in H(x,y,z)$ for each $(x,y,z)\in D$. Then, by the integral of the multifunction $H:D\to 2^{\mathbb{R}^n}$ we mean the set

$$\iiint_D H(x,y,z)\,dx\,dy\,dz = \left\{\iiint_D h(x,y,z)\,dx\,dy\,dz \mid h\in\mathcal{H}\right\}.$$

In what follows we list some properties of the integral defined above.

Proposition 2.1. *If* $H:D\to 2^{\mathbb{R}^n}$ *is an upper semi-continuous multifunction and there exists a positive real number* C *such that*

$$|H(x,y,z)| = \sup\{\|\zeta\| \mid \zeta\in H(x,y,z)\}\le C$$

for each $(x,y,z)\in D$, *then*

$$\iiint_D H(x,y,z)\,dx\,dy\,dz = \iiint_D \operatorname{conv} H(x,y,z)\,dx\,dy\,dz.$$

Proposition 2.2. *If* $H_k:D\to 2^{\mathbb{R}^n}$, $k\in\mathbb{N}$, *are upper semi-continuous multifunctions and there exists a positive real number* C *such that* $|H_k(x,y,z)|\le C$ *for each*

$(x, y, z) \in D$ *and* $k \in \mathbb{N}$, *then*

$$\iiint\limits_D \underline{\lim} H_k(x, y, z)\, dx\, dy\, dz \subset \underline{\lim} \iiint\limits_D H_k(x, y, z)\, dx\, dy\, dz.$$

Taking into account Definition 2 in [5], we have $(x, y, z) \in \underline{\lim} H_k(x, y, z)$ iff each neighbourhood of (x, y, z) intersects all the sets $H_k(x, y, z)$ with k large enough.

Proposition 2.3. *If* A *is a compact subset of* $\mathbb{R}^n$, *independent of* (x, y, z), *then*

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} A\, dx\, dy\, dz = (x_2 - x_1)(y_2 - y_1)(z_2 - z_1) \operatorname{conv} A,$$

where $(x_1, y_1, z_1), (x_2, y_2, z_2) \in D$.

Moreover, we need the following proposition.

Proposition 2.4. *If* K *is a convex set in a Banach space* X, *then the set* $K_\varepsilon = \bigcup\limits_{x \in K} B[x, \varepsilon]$ *is convex.*

3 Results

In [16] the notion of a *local solution* for the Darboux Problem (1.1) + (1.2) is defined and is proved an existence theorem for a local solution of this problem, together with some properties of the set of its solutions, namely that this set is a compact subset in Banach space $C(D_0; \mathbb{R}^n)$ and, as a function of initial values, it defines an upper semi-continuous multifunction on $D_0 = [0, x_0] \times [0, y_0] \times [0, z_0] \subseteq D$.

Let the following hypotheses be satisfied:

(H_1) $F : D \times \Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact convex non-empty values in $\mathbb{R}^n$, $D = [0, a] \times [0, b] \times [0, c] \subset \mathbb{R}^3$, and $\Omega \subset \mathbb{R}^n$ is an open subset.

(H_2) For any $(x, y, z) \in D$, the mapping $u \to F(x, y, z, u)$ is upper semi-continuous on Ω.

(H_3) For any $u \in \Omega$, the mapping $(x, y, z) \to F(x, y, z, u)$ is Lebesgue-measurable on D.

(H_4) There exists a function $k : D \to \mathbb{R}_+, k \in \mathcal{L}^1(D; \mathbb{R}^n)$ such that

$$\|\zeta\| \le k(x, y, z), (\forall)\, \zeta \in F(x, y, z, u), \quad (\forall)\,(x, y, z) \in D, \quad (\forall)\, u \in \Omega.$$

(H_5) The functions $\varphi \in C^*(D_1; \mathbb{R}^n)$, $\psi \in C^*(D_2; \mathbb{R}^n)$, $\chi \in C^*(D_3, \mathbb{R}^n)$ are absolutely continuous in Carathéodory's sense functions and satisfy condition (1.3).

Remark 1. The function $\alpha : D \to \mathbb{R}^n$ defined by

$$\begin{aligned}\alpha(x,y,z) &= \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \\ &\quad - \varphi(0,y) - \psi(0,z) + \psi(0,0) = \\ &= \varphi(x,y) + \psi(y,z) + \chi(x,z) - v^1(x) - v^2(y) - v^3(z) + v^0, \qquad (3.1)\end{aligned}$$

is an absolutely continuous in Carathéodory's sense function on D, $\alpha \in C^*(D;\mathbb{R}^n)$ $[2, \S 565 - 570]$.

Remark 2. Denote by $M \subset \Omega$ the convex compact set in which the function $\alpha : D \to \mathbb{R}^n$, defined by (3.1), takes its values for all $(x,y,z) \in D_0$.

Remark 3. Let $(x_0, y_0, z_0) \in]0,a] \times]0,b] \times]0,c]$ be a point such that

$$\int_0^{x_0} \int_0^{y_0} \int_0^{z_0} k(r,s,t)\, dr\, ds\, dt < d(M, C_\Omega),$$

where $d(M, C_\Omega)$ is the distance from M to $C_\Omega = \mathbb{R}^n - \Omega$, an inequality immediately resulting from the integrability of function k.

Definition 3.1 [16]. The *Darboux Problem* for the hyperbolic inclusion (1.1) means to determine a *solution* of this inclusion which satisfies the initial conditions (1.2).

Definition 3.2 [16]. A *local solution* of Darboux Problem $(1.1) + (1.2)$ is defined as a function $U : D_0 \to \Omega, U \in C^*(D_0;\mathbb{R}^n)$, absolutely continuous in Carathéodory's sense $[2, \S 565 - 570]$, which satisfies (1.1) for a.e $(x,y,z) \in D_0$, and also initial conditions (1.2) for all $(x,y) \in [0,x_0] \times [0,y_0]$, all $(y,z) \in [0,y_0] \times [0,z_0]$, all $(x,z) \in [0,x_0] \times [0,z_0]$.

In [16] we proved the following

Theorem 3.1 [16]. *Let the hypotheses* $(H_1) - (H_5)$ *be satisfied. Then:*

- (i) *there exists at least a local solution* U *of Darboux Problem* $(1.1) + (1.2)$*;*
- (ii) *the set* S_α *of the local solutions* U *is compact in Banach space* $C(D_0;\mathbb{R}^n)$*;*
- (iii) *the multifunction* $\alpha \to S_\alpha$ *is upper semi-continuous on* $C^*(D_0;\mathbb{R}^n)$*, taking values in* $C(D_0;\mathbb{R}^n)$*.*

The solution U is a fixed point of a suitable multifunction which satisfies the Kakutani-Ky Fan fixed point theorem and it is of the form

$$U(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z \beta(r,s,t)\, dr\, ds\, dt, \quad (x,y,z) \in D_0, \qquad (3.2)$$

where

$$\beta(x,y,z) \in \Gamma(x,y,z) \subset F(x,y,z,U(x,y,z)) \text{ for a.e. } (x,y,z) \in D_0, \qquad (3.3)$$

β is a measurable selection of the multifunction $\Gamma : D_0 \to \mathcal{C}(\mathbb{R}^n)$ [3, 4, 16].

Definition 3.3 [17]. A local solution for the Darboux Problem (1.1) + (1.2) $U : D_0 \to \Omega$ is *prolongable* (or *non-saturated*) if there exists a solution $\widetilde{U} : \widetilde{D} \to \mathbb{R}^n$ for the Darboux Problem (1.1) + (1.2) such that

$$\begin{cases} D_0 \subseteq \widetilde{D}, \quad D_0 \neq \widetilde{D}, \\ \widetilde{U}(x,y,z) = U(x,y,z), \quad (x,y,z) \in D_0, \end{cases}$$

where $\widetilde{D} \subseteq D$ is a union of D_0 with a finite number of adjacent parallelepipeds.

In [17] we proved the following theorems:

Theorem 3.2 [17]. *Let the hypotheses* $(H_1) - (H_5)$ *be satisfied together with the hypotheses:*

(H_6) *The set* Ω *is bounded, that is there exists a constant* $C \in \mathbb{R}_+$ *such that* $\|u\| \leq C,$, $(\forall)\, u \in \Omega$.

(H_7) *The multifunction* F *maps bounded sets onto bounded sets, hence a constant* $K \in \mathbb{R}_+$ *exists such that*

$$\sup \{\|\zeta\| \mid \zeta \in F(x,y,z,u)\} \leq K,$$

for any $(x,y,z,u) \in D \times \Omega$.

Then the local solution U *is prolongable.*

Theorem 3.3 [17]. *We assume the hypotheses* $(H_1) - (H_7)$ *to be satisfied. If* $U : D_0 \to \Omega$ *is a local solution of Darboux Problem* (1.1) + (1.2) *that is non-saturated, hence prolongable, then there exists a saturated solution* $U^* : D^* \to \Omega$ *of the Darboux Problem* (1.1) + (1.2) *such that*

$$\begin{cases} D_0 \subseteq D^*, \quad D_0 \neq D^*, \quad D^* \subseteq D, \\ U^*(x,y,z) = U(x,y,z), \quad (x,y,z) \in D_0, \end{cases}$$

hence U^* *is a prolongation of* U *onto* D^* *that has been built by joining* D_0 *with a union of parallelepipeds adjacent to* D_0.

Theorem 3.4 [17]. *Let the hypotheses* $(H_1) - (H_7)$ *be satisfied. If the saturated solution* U^* *is bounded on* D^*, *then* $D^* = D$.

Theorem 3.5 [17]. *Let the hypotheses* $(H_1) - (H_7)$ *be satisfied together with the hypothesis:*

(H_8) *The multifunction* $F : D \times \Omega \to 2^{\mathbb{R}^n}$ *is sub-linear, hence two constants* $k_1 > 0$ *and* $k_2 \in \mathbb{R}$ *exist with the property*

$$\sup \{\|\zeta\| \mid \zeta \in F(x,y,z,u)\} \leq k_1 \|u\| + k_2, \quad \text{for a.e. } (x,y,z) \in D, \quad u \in \Omega. \tag{3.4}$$

Then the saturated solution $U^*: D \to \Omega$ *is bounded on* D.

The saturated solution U^* has the form, by Theorem 3.1 [16],

$$U^*(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z \beta^*(r,s,t)\, dr\, ds\, dt, \quad (x,y,z) \in D, \tag{3.5}$$

where $\alpha(x,y,z)$ is given by (3.1) and β^* is a measurable selection of the multivalued mapping Γ^* [3, 4, 16], defined on D with compact non-empty values in $\mathbb{R}^n$, i.e. $\Gamma^*: D \to \mathcal{C}(\mathbb{R}^n)$, such that

$$\beta^*(x,y,z) \in \Gamma^*(x,y,z) \subseteq F(x,y,z,U^*(x,y,z)) \text{ for a.e. } (x,y,z) \in D. \tag{3.6}$$

Definition 3.4. A function $U: D \to \mathbb{R}^n$ is called a *solution* of the Darboux Problem (1.1)+(1.2) if it is absolutely continuous in Carathéodory's sense on $D, U \in C^*(D; \mathbb{R}^n)$ [2, §565 − 570] and satisfies (1.1) for a.e. $(x,y,z) \in D$, and also initial conditions (1.2) for all $(x,y) \in D_1$, all $(y,z) \in D_2$, all $(x,z) \in D_3$.

Similarly with [5, 15] in this paper we prove a theorem of characterization of the solutions for Darboux Problem (1.1) + (1.2).

Theorem 3.6. *Let the hypotheses* (H_1'), (H_3), (H_4), (H_5) *of Theorem 3.1 be satisfied:*

(H_1') $F: D \times \Omega \to 2^{\mathbb{R}^n}$ *is an upper semi-continuous multifunction with compact convex non-empty values in* $\mathbb{R}^n$, $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$ *and* $\Omega \subset \mathbb{R}^n$ *is an open bounded set.*

The hypothesis (H_1') *includes the hypothesis* (H_6).

Then, the continuous function $U: D \to \mathbb{R}^n$ *is a solution of Darboux Problem* (1.1) + (1.2) *if and only if for each* $(x_1,y_1,z_1), (x_2,y_2,z_2) \in D$ *the membership relation*

$$\begin{aligned} &[U(x_2,y_2,z_2) - U(x_1,y_2,z_2) - U(x_2,y_1,z_2) + U(x_1,y_1,z_2)] - \\ &- [U(x_2,y_2,z_1) - U(x_1,y_2,z_1) - U(x_2,y_1,z_1) + U(x_1,y_1,z_1)] \in \\ &\in \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F(x,y,z,U(x,y,z))\, dx\, dy\, dz, \end{aligned} \tag{3.7}$$

holds, and U *satisfies the conditions* (1.2).

The difference in (3.7) is an extension of hyperbolic difference for functions in two variables.

Proof. The *necessity* of (3.7) is a consequence of the following arguments. Let U be a solution of (1.1) + (1.2) on D. It exists from Theorem 3.4 and has the form (3.5). We denote $U^* = U$.

$$U(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z \beta(r,s,t)\, dr\, ds\, dt, \quad (x,y,z) \in D, \tag{3.8}$$

$\beta^* = \beta$ is a measurable selection of multivalued mapping $\Gamma^* = \Gamma$ [3, 4, 16] defined on D with compact non-empty values in $\mathbb{R}^n, \Gamma : D \to \mathcal{C}(\mathbb{R}^n)$,

$$\beta(x,y,z) \in \Gamma(x,y,z) \subseteq F(x,y,z,U(x,y,z)) \quad \text{for a.e} \quad (x,y,z) \in D. \tag{3.9}$$

We denote $\delta = [x_1, x_2] \times [y_1, y_2] \times [z_1, z_2] \subseteq D$. By (3.8) it follows that

$$\frac{\partial^3 U(x,y,z)}{\partial x \partial y \partial z} = \beta(x,y,z) \in \Gamma(x,y,z) \subseteq F(x,y,z,U(x,y,z)) \tag{3.10}$$
$$\text{for a.e. } (x,y,z) \in D$$

and U satisfies the conditions (1.2).

Choosing two points $(x_1, y_1, z_1), (x_2, y_2, z_2) \in D$ and integrating the equation (3.10) on δ we get

$$\begin{aligned}
&\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \frac{\partial^3 U(x,y,z)}{\partial x \partial y \partial z} dx\,dy\,dz = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{\partial^2 U(x,y,z)}{\partial x \partial y}\Big|_{z=z_1}^{z=z_2} dx\,dy = \\
&= \int_{x_1}^{x_2}\int_{y_1}^{y_2} \left[\frac{\partial^2 U(x,y,z_2)}{\partial x \partial y} - \frac{\partial^2 U(x,y,z_1)}{\partial x \partial y}\right] dx\,dy = \\
&= \int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{\partial^2 U(x,y,z_2)}{\partial x \partial y} d\dot{x}\,dy - \int_{x_1}^{x_2}\int_{y_1}^{y_2} \frac{\partial^2 U(x,y,z_1)}{\partial x \partial y} dx\,dy = \\
&= \int_{x_1}^{x_2} \frac{\partial U(x,y,z_2)}{\partial x}\Big|_{y=y_1}^{y=y_2} dx - \int_{x_1}^{x_2} \frac{\partial U(x,y,z_1)}{\partial x}\Big|_{y=y_1}^{y=y_2} dx = \\
&= \int_{x_1}^{x_2} \left[\frac{\partial U(x,y_{2,}z_2)}{\partial x} - \frac{\partial U(x,y_1,z_2)}{\partial x}\right] dx - \\
&- \int_{x_1}^{x_2} \left[\frac{\partial U(x,y_{2,}z_1)}{\partial x} - \frac{\partial U(x,y_1,z_1)}{\partial x}\right] dx = \\
&= \left(U(x,y_2,z_2)\Big|_{x=x_1}^{x=x_2} - U(x,y_1,z_2)\Big|_{x=x_1}^{x=x_2}\right) - \\
&- \left(U(x,y,z_1)\Big|_{x=x_1}^{x=x_2} - U(x,y_1,z_1)\Big|_{x=x_1}^{x=x_2}\right) = \\
&= [(U(x_2,y_2,z_2) - U(x_1,y_2,z_2)) - (U(x_2,y_1,z_2) - U(x_1,y_1,z_2))] - \\
&- [(U(x_2,y_2,z_1) - U(x_1,y_2,z_1)) - (U(x_2,y_1,z_1) - U(x_1,y_1,z_1))] = \\
&= [U(x_2,y_2,z_2) - U(x_1,y_2,z_2) - U(x_2,y_1,z_2) + U(x_1,y_1,z_2)] - \\
&- [U(x_2,y_2,z_1) - U(x_1,y_2,z_1) - U(x_2,y_1,z_1) + U(x_1,y_1,z_1)] = \\
&= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \beta(x,y,z)\,dx\,dy\,dz \in \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} \Gamma(x,y,z)\,dx\,dy\,dz \subseteq \\
&\subseteq \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} F(x,y,z,U(x,y,z))\,dx\,dy\,dz.
\end{aligned} \tag{3.11}$$

According to (3.11), we have (3.7) satisfied it was stated.

In order to prove the *sufficiency* of (3.7), we firstly prove that the continuous function U, satisfying (3.7) and (1.2), has the derivative $\dfrac{\partial^3 U(x,y,z)}{\partial x \partial y \partial z}$ for a.e $(x,y,z) \in D$. For this, we prove that U is absolutely continuous in Carathéodory's sense on D. We associate to the continuous function U, the interval function $[2, \S 453, 565]$,

$$\begin{aligned} \Phi(\delta) = &[U(x_2,y_2,z_2) - U(x_1,y_2,z_2) - U(x_2,y_1,z_2) + U(x_1,y_1,z_2)] - \\ &-[U(x_2,y_2,z_1) - U(x_1,y_2,z_1) - U(x_2,y_1,z_1) + U(x_1,y_1,z_1)]. \end{aligned} \tag{3.12}$$

We prove that $\Phi(\delta)$ is absolutely continuous, using the Theorem 1 in [2, §453]. From (3.7) and (3.12) we get

$$\Phi(\delta) \in \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F(x,y,z,U(x,y,z))\, dx\, dy\, dz. \tag{3.13}$$

In view of Definition 2.9 and (3.11), the relation (3.7) holds for $(x,y,z) \in \delta$. Then (3.7), (3.11), (3.13) yield

$$\begin{aligned} \Phi(\delta) = &\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \beta(x,y,z)\, dx\, dy\, dz \in \\ &\in \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F(x,y,z,U(x,y,z_1))\, dx\, dy\, dz. \end{aligned} \tag{3.14}$$

According to the hypothesis (H_4), we obtain

$$\begin{aligned} \|\Phi(\delta)\| \le &\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \|\beta(x,y,z)\|\, dx\, dy\, dz \le \\ &\le \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} k(x,y,z)\, dx\, dy\, dz. \end{aligned} \tag{3.15}$$

We set

$$\eta(\lambda) = \sup_{\mu(\delta) \le \lambda} \|\Phi(\delta)\|, \text{ for any } \lambda \in \mathbb{R}_+. \tag{3.16}$$

In view of the absolute continuity of the integral, for each $\varepsilon > 0$ there exists a $\delta_1(\varepsilon) > 0$ such that

$$\iiint_{\delta} k(x,y,z)\, dx\, dy\, dz = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} k(x,y,z)\, dx\, dy\, dz < \varepsilon, \tag{3.17}$$

whenever $\mu(\delta) < \delta_1(\varepsilon)$.

Let $\lambda < \delta_1(\varepsilon)$. According to (3.15), (3.16), (3.17) we obtain

$$\eta(\lambda) \le \sup \iiint_{\delta} k(x,y,z)\, dx\, dy\, dz = \int_{x_1}^{x_2} \int_{y}^{y_2} \int_{z_1}^{z_2} k(x,y,z)\, dx\, dy\, dz < \varepsilon, \tag{3.18}$$

whenever $\mu(\delta) \leq \lambda < \delta_1(\varepsilon)$, or

$$\lim_{\lambda \to 0} \eta(\lambda) = 0. \tag{3.19}$$

According to Theorem in [2, §453] the interval function $\Phi(\delta)$ is absolutely continuous. Since the continuous function U satisfies the conditions (1.3) the hypothesis (H_5) holds too. In view of [2, §567]the function U is absolutely continuous in Carathéodory's sense. From Theorems 5, 6 [2, §569 − 570] the function U has the derivative $\dfrac{\partial^3 U(x,y,z)}{\partial x\,\partial y\,\partial z}$ for a.e. $(x,y,z) \in D$.

It remains to prove that the function U satisfies the inclusion (1.1).

Taking into account the hypothesis (H_1) and the continuity of the function U, it follows that the multifunction$\widetilde{F} : D \to 2^{\mathbb{R}^n}$, given by

$$\widetilde{F}(x,y,z) = F(x,y,z,U(x,y,z)), \quad (x,y,z) \in D, \tag{3.20}$$

is upper semi-continuous on D. Then by Theorem 9.3.1 [13] and [5], Definition 1, we deduce

$$\widetilde{F}([B(x,y,z),\delta_2]) \subset B\left[\widetilde{F}(x,y,z),\varepsilon\right], \quad (x,y,z) \in D, \tag{3.21}$$

where $B[(x,y,z),\delta_2]$ is the open ball centered at $(x,y,z) \in D$ of radius $\delta_2 = \delta_2(\varepsilon) > 0$ and

$$B\left[\widetilde{F}(x,y,z),\varepsilon\right] = \left\{\omega \in \mathbb{R}^n \mid d\left(\omega, \widetilde{F}(x,y,z)\right) < \varepsilon\right\}. \tag{3.22}$$

Fix $(x,y,z) \in D$. If $(x',y',z') \in B[(x,y,z),\delta_2]$, then

$$\widetilde{F}\left(x',y',z'\right) \subset B\left[\widetilde{F}(x,y,z),\varepsilon\right] \tag{3.23}$$

because by Definition 2.1, and by Definition 9.1.2 [13, p.67) and also [5, 2] we have

$$\widetilde{F}(B[(x,y,z),\delta_2]) = \left\{\bigcup \widetilde{F}\left(x',y',z'\right) \mid \left(x',y',z'\right) \in B[(x,y,z),\delta_2]\right\}. \tag{3.24}$$

The condition (3.7) may be rewritten as

$$\begin{aligned}
&\left[U\left(x',y',z'\right) - U\left(x,y',z'\right) - U\left(x',y,z'\right) + U\left(x,y,z'\right)\right] - \\
&- \left[U\left(x',y',z\right) - U\left(x,y',z\right) - U\left(x',y,z\right) + U(x,y,z)\right] \in \\
&\in \int_x^{x'} \int_y^{y'} \int_z^{z'} F(r,s,t,U(r,s,t))\,dr\,ds\,dt,
\end{aligned} \tag{3.25}$$

for the domain $[x,x'] \times [y,y'] \times [z,z'] \subseteq D$.

According to (3.20), we deduce from (3.25) that

$$\begin{aligned}
&\left[U\left(x',y',z'\right) - U\left(x,y',z'\right) - U\left(x',y,z'\right) + U\left(x,y,z'\right)\right] - \\
&- \left[U\left(x',y',z\right) - U\left(x,y',z\right) - U\left(x',y,z\right) + U(x,y,z)\right] \in
\end{aligned}$$

$$\in \int_{x}^{x'} \int_{y}^{y'} \int_{z}^{z'} \widetilde{F}(r,s,t,)\, dr\, ds\, dt. \tag{3.26}$$

By $(x',y',z') \in B[(x,y,z),\delta_2]$, we obtain $|x-x'| < \delta_2$, $|y-y'| < \delta_2$, $|z-z'| < \delta_2$. Moreover $|r-x| < \delta_2$, $|s-y| < \delta_2$, $|t-z| < \delta_2$ for $x \le r \le x'$, $y \le s \le y'$, $z \le t \le z'$.

By (3.23) we have

$$\widetilde{F}(r,s,t) \subset B\left[\widetilde{F}(r,s,t),\varepsilon\right]. \tag{3.27}$$

Then, by (3.27), the relation (3.26) yields

$$\begin{gathered}\left[U\left(x',y',z'\right) - U\left(x,y',z'\right) - U\left(x',y,z\right) + U\left(x,y,z'\right)\right] - \\ - \left[U\left(x',y',z\right) - U\left(x,y',z\right) - U\left(x',y,z\right) + U\left(x,y,z\right)\right] \in \\ \in \int_{x}^{x'} \int_{y}^{y'} \int_{z}^{z'} B\left[\widetilde{F}(r,s,t,),\varepsilon\right] dr\, ds\, dt.\end{gathered} \tag{3.28}$$

As the multifunction $\widetilde{F}$, given by (3.20), is upper semi-continuous on D, the set $B\left[\widetilde{F}(x,y,z),\varepsilon\right]$ is closed in $\mathbb{R}^n$.

In view of (3.22) it follows that $B\left[\widetilde{F}(x,y,z),\varepsilon\right]$ is also bounded in $\mathbb{R}^n$ and therefore it is a compact set. Then we can use Proposition 2.3, setting $A = B\left[\widetilde{F}(x,y,z),\varepsilon\right]$ and $[x,x'] \times [y,y'] \times [z,z']$ instead of $\delta = [x_1,x_2] \times [y_1,y_2] \times [z_1,z_2]$, we obtain

$$\begin{gathered}\int_{x}^{x'} \int_{y}^{y'} \int_{z}^{z'} B\left[\widetilde{F}(x,y,z),\varepsilon\right] dr\, ds\, dt = \\ = \left(x'-x\right)\left(y'-y\right)\left(z'-z\right) \operatorname{conv} B\left[\widetilde{F}(x,y,z),\varepsilon\right].\end{gathered} \tag{3.29}$$

According to (3.29), the relation (3.28) yields

$$\begin{gathered}\left[U\left(x',y',z'\right) - U\left(x,y',z'\right) - U\left(x',y,z'\right) + U\left(x,y,z'\right)\right] - \\ - \left[U\left(x',y',z\right) - U\left(x,y',z\right) - U\left(x',y,z\right) + U\left(x,y,z\right)\right] \in \\ \in \left(x'-x\right)\left(y'-y\right)\left(z'-z\right) \operatorname{conv} B\left[\widetilde{F}(x,y,z),\varepsilon\right].\end{gathered} \tag{3.30}$$

By Proposition (2.4), the set $B\left[\widetilde{F}(x,y,z),\varepsilon\right]$ is convex and therefore

$$\operatorname{conv} B\left[\widetilde{F}(x,y,z),\varepsilon\right] = B\left[\widetilde{F}(x,y,z),\varepsilon\right]. \tag{3.31}$$

Using (3.31), the relation (3.30) yields

$$\begin{gathered}\left[U\left(x',y',z'\right) - U\left(x,y',z'\right) - U\left(x',y,z'\right) + U\left(x,y,z'\right)\right] - \\ - \left[U\left(x',y',z\right) - U\left(x,y',z\right) - U\left(x',y,z\right) + U\left(x,y,z\right)\right] \in\end{gathered}$$

$$\in \left(x' - x\right)\left(y' - y\right)\left(z' - z\right) B\left[\widetilde{F}\left(x, y, z\right), \varepsilon\right]. \tag{3.32}$$

From (3.32) we get

$$\left[\frac{U\left(x', y', z'\right) - U\left(x, y', z'\right)}{x' - x} - \frac{U\left(x', y, z'\right) - U\left(x, y, z'\right)}{x' - x}\right] - \\ - \left[\frac{U\left(x', y', z\right) - U\left(x, y', z\right)}{x' - x} - \frac{U\left(x', y, z\right) - U\left(x, y, z\right)}{x' - x}\right] \in \\ \in \left(y' - y\right)\left(z' - z\right) B\left[\widetilde{F}\left(x, y, z\right), \varepsilon\right]. \tag{3.33}$$

Taking into account that $B\left[\widetilde{F}\left(x, y, z\right), \varepsilon\right]$ is closed, the relation (3.33) yields

$$\lim_{x' \to x}\left\{\left[\frac{U\left(x', y', z'\right) - U\left(x, y', z'\right)}{x' - x} - \frac{U\left(x', y, z'\right) - U\left(x, y, z'\right)}{x' - x}\right] - \\ - \left[\frac{U\left(x', y', z\right) - U\left(x, y', z\right)}{x' - x} - \frac{U\left(x', y, z\right) - U\left(x, y, z\right)}{x' - x}\right]\right\} = \\ = \left\{\left[\frac{\partial U}{\partial x}\left(x, y', z'\right) - \frac{\partial U}{\partial x}\left(x, y, z'\right)\right] - \left[\frac{\partial U}{\partial x}\left(x, y', z\right) - \frac{\partial U}{\partial x}\left(x, y, z\right)\right]\right\} \in \\ \in \left(y' - y\right)\left(z' - z\right) B\left[\widetilde{F}\left(x, y, z\right), \varepsilon\right] \tag{3.34}$$

and

$$\lim_{y' \to y}\left[\frac{\frac{\partial U}{\partial x}\left(x, y', z'\right) - \frac{\partial U}{\partial x}\left(x, y, z'\right)}{y' - y} - \frac{\frac{\partial U}{\partial x}\left(x, y', z\right) - \frac{\partial U}{\partial x}\left(x, y, z\right)}{y' - y}\right] \in \\ \in \left(z' - z\right) B\left[\widetilde{F}\left(x, y, z\right), \varepsilon\right] \tag{3.35}$$

or

$$\frac{\partial^2 U}{\partial x \partial y}\left(x, y, z'\right) - \frac{\partial^2 U}{\partial x \partial y}\left(x, y, z\right) \in \left(z' - z\right) B\left[\widetilde{F}\left(x, y, z\right), \varepsilon\right]. \tag{3.36}$$

It results

$$\frac{\frac{\partial^2 U}{\partial x \partial y}\left(x, y, z'\right) - \frac{\partial^2 U}{\partial x \partial y}\left(x, y, z\right)}{z' - z} \in B\left[\widetilde{F}\left(x, y, z\right), \varepsilon\right] \tag{3.37}$$

and

$$\lim_{z' \to z}\frac{\frac{\partial^2 U}{\partial x \partial y}\left(x, y, z'\right) - \frac{\partial^2 U}{\partial x \partial y}\left(x, y, z\right)}{z' - z} = \frac{\partial^3 U}{\partial x \partial y \partial z}\left(x, y, z\right) \in B\left[\widetilde{F}\left(x, y, z\right), \varepsilon\right]. \tag{3.38}$$

Since $\widetilde{F}\left(x, y, z\right)$ is closed and F is an upper semi-continuous multifunction, the relation (3.38) yields, for $\varepsilon \to 0$,

$$\frac{\partial^3 U}{\partial x \partial y \partial z}\left(x, y, z\right) \in \widetilde{F}\left(x, y, z\right) = F\left(x, y, z, U\left(x, y, z\right)\right) \text{ for a.e. } \left(x, y, z\right) \in D. \tag{3.39}$$

Therefore, U satisfies the inclusion (1.1) as stated.

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Received October 4, 2006