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The intersection and the union of the asynchronous systems

Serban E. Vlad

Abstract. The asynchronous systems f are the models of the asynchronous circuits from digital electrical engineering. They are multi-valued functions that associate to each input $u : \mathbf{R} \rightarrow \{0,1\}^m$ a set of states $x \in f(u)$, where $x : \mathbf{R} \rightarrow \{0,1\}^n$. The intersection of the systems allows adding supplementary conditions in modeling and the union of the systems allows considering the validity of one of two systems in modeling, for example when testing the asynchronous circuits and the circuit is supposed to be 'good' or 'bad'. The purpose of the paper is that of analyzing the intersection and the union against the initial/final states, initial/final time, initial/final state functions, subsystems, dual systems, inverse systems, Cartesian product of systems, parallel connection and serial connection of systems.

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1 Preliminary definitions

Definition 1. *The set* $\mathbf{B} = \{0,1\}$ *endowed with the laws: the complement* '$\overline{}$', *the union* '$\cup$', *the intersection* '$\cdot$', *the modulo 2 sum* '$\oplus$' *etc is called the binary Boole algebra.*

Definition 2. *We denote by* $\mathbf{R}$ *the set of the real numbers. The initial value* $x(-\infty+0) \in \mathbf{B}$ *and the final value* $x(\infty - 0) \in \mathbf{B}$ *of the function* $x : \mathbf{R} \rightarrow \mathbf{B}$ *are defined by*

$$\exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = x(-\infty + 0),$$

$$\exists t_f \in \mathbf{R}, \forall t > t_f, x(t) = x(\infty - 0).$$

The definition and the notations are similar for the $\mathbf{R} \rightarrow \mathbf{B}^n$ *functions,* $n \geq 1$.

Definition 3. *The characteristic function* $\chi_A : \mathbf{R} \rightarrow \mathbf{B}$ *of the set* $A \subset \mathbf{R}$ *is defined by*

$$\forall t \in \mathbf{R}, \chi_A(t) = \begin{cases} 1, if\ t \in A, \\ 0, if\ t \notin A. \end{cases}$$

Definition 4. *The set* $S^{(n)}$ *of the* $n-$*signals consists by definition in the functions* $x : \mathbf{R} \rightarrow \mathbf{B}^n$ *of the form*

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty,t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0,t_1)}(t) \oplus x(t_1) \cdot \chi_{[t_1,t_2)}(t) \oplus \ldots$$

where $x(-\infty+0)\in\mathbf{B}^n$, $t_0<t_1<t_2<...$ *is some strictly increasing unbounded sequence of real numbers and the laws '$\cdot$', '$\oplus$' are induced by those from* $\mathbf{B}$.

Notation 1. For an arbitrary set H, we use the notation

$$P^*(H)=\{H'|H'\subset H,H'\neq\emptyset\}.$$

Definition 5. *The functions* $f:U\to P^*(S^{(n)}),U\in P^*(S^{(m)})$ *are called (asynchronous) systems. Any* $u\in U$ *is called (admissible) input of* f *and the functions* $x\in f(u)$ *are the (possible) states of* f.

Remark 1. In the paper $t\in\mathbf{R}$ represents time. The $n-$signals model the tensions in digital electrical engineering and the asynchronous systems are the models of the asynchronous circuits. They represent multi-valued associations between a cause u and a set $f(u)$ of effects because of the uncertainties that occur in modeling.

Definition 5 represents the definition of the systems given under the explicit form. In previous works (such as [1]) we used equations and inequalities for defining systems under the implicit form.

Definition 6. *We have the systems* $f:U\to P^*(S^{(n)})$, $g:V\to P^*(S^{(n)})$ *with* $U,V\in P^*(S^{(m)})$. *If* $\exists u\in U\cap V,f(u)\cap g(u)\neq\emptyset$, *the system* $f\cap g:W\to P^*(S^{(n)})$ *defined by*

$$W=\{u|u\in U\cap V,f(u)\cap g(u)\neq\emptyset\}, \tag{1.1}$$

$$\forall u\in W,(f\cap g)(u)=f(u)\cap g(u)$$

is called the intersection of f *and* g.

Remark 2. The intersection of the systems represents the gain of information (of precision) in the modeling of a circuit that results by considering the validity of two (compatible!) models at the same time.

We have the special case when $V=S^{(m)}$ and the system g is constant (such systems are called autonomous): $\forall u\in S^{(m)}$, $g(u)=X$ where $X\in P^*(S^{(n)})$. Then $f\cap X:W\to P^*(S^{(n)})$ is the system given by

$$W=\{u|u\in U,f(u)\cap X\neq\emptyset\},$$

$$\forall u\in W,(f\cap X)(u)=f(u)\cap X.$$

We interpret $f\cap X$ in the next manner. When f models a circuit, $f\cap X$ represents a gain of information resulting by the statement of a request that does not depend on u.

Example 1. We give some possibilities of choosing in the intersection $f\cap g$ the constant system $g=X$:

i) the initial value of the states is null;

ii) the coordinates $x_1,...,x_n$ of the states are monotonous relative to the order $0<1$ (this allows defining the so called hazard-freedom of the systems);

iii) at each time instant, at least one coordinate of the state should be 1:

$$X = \{x | x \in S^{(n)}, \forall t \in \mathbf{R}, x_1(t) \cup ... \cup x_n(t) = 1\};$$

iv) the state can switch[1] with at most one coordinate at a time (a special case when the so called technical condition of good running of the systems is satisfied):

$$X = \{x | x \in S^{(n)}, \forall t \in \mathbf{R}, x(t-0) \neq x(t) \Longrightarrow \exists! i \in \{1, ..., n\}, x_i(t-0) \neq x_i(t)\};$$

v) X represents a 'stuck at 1 fault':

$$\exists i \in \{1, ..., n\}, X = \{x | x \in S^{(n)}, \forall t \in \mathbf{R}, x_i(t) = 1\},$$

this last choice of X is interesting in designing systems for testability, respectively in designing redundant systems;

vi) X consists in all $x \in S^{(n)}$ satisfying the next 'absolute inertia' property: $\delta_r > 0, \delta_f > 0$ are given so that $\forall i \in \{1, ..., n\}, \forall t \in \mathbf{R}$,

$$\overline{x_i(t-0)} \cdot x_i(t) \leq \bigcap_{\xi \in [t, t+\delta_r]} x_i(\xi);$$

$$x_i(t-0) \cdot \overline{x_i(t)} \leq \bigcap_{\xi \in [t, t+\delta_f]} \overline{x_i(\xi)}.$$

The interpretation of these inequalities is the following: if x_i switches from 0 to 1, then it remains 1 for more than δ_r time units and if x_i switches from 1 to 0 then it remains 0 for more than δ_f time units.

Example 2. We show a possibility of choosing in the intersection $f \cap g$, g non-constant. The Boolean function $F : \mathbf{B}^m \to \mathbf{B}^n$ is given and f is the arbitrary model of a circuit that computes F. $V = S^{(m)}$ and the parameters $\delta_r > 0, \delta_f > 0$ exist so that

$$\forall u \in S^{(m)}, g(u) = \{x | x \in S^{(n)}, \forall i \in \{1, ..., n\}, \forall t \in \mathbf{R},$$
$$\overline{x_i(t-0)} \cdot x_i(t) \leq \bigcap_{\xi \in [t-\delta_r, t)} F_i(u(\xi)),$$
$$x_i(t-0) \cdot \overline{x_i(t)} \leq \bigcap_{\xi \in [t-\delta_f, t)} \overline{F_i(u(\xi))}\}$$

meaning that $g(u)$ contains all x with the property that, on all the coordinates i and at all the time instants t:

– x_i switches from 0 to 1 only if $F_i(u(\cdot))$ was 1 for at least δ_r time units;
– x_i switches from 1 to 0 only if $F_i(u(\cdot))$ was 0 for at least δ_f time units.

[1] The left limit $x(t-0)$ of $x(t)$ that occurs in some examples is defined like this:

$$\forall t \in \mathbf{R}, \exists \varepsilon > 0, \forall \xi \in (t-\varepsilon, t), x(\xi) = x(t-0);$$

x switches if $x(t-0) \neq x(t)$, i.e. if it has a (left) discontinuity.

Definition 7. *The union of the systems* $f : U \to P^*(S^{(n)})$ *and* $g : V \to P^*(S^{(n)}), U, V \in P^*(S^{(m)})$ *is the system* $f \cup g : U \cup V \to P^*(S^{(n)})$ *that is defined by*

$$\forall u \in U \cup V, (f \cup g)(u) = \begin{cases} f(u), if\ u \in U \setminus V, \\ g(u), if\ u \in V \setminus U, \\ f(u) \cup g(u), if\ u \in U \cap V. \end{cases}$$

If $U \cap V = \emptyset$, *then* $f \cup g$ *is called the disjoint union of* f *and* g.

Remark 3. The union of the systems is the dual concept to that of intersection representing the loss of information (of precision) in modeling that results in general by considering the validity of one of two models of the same circuit. The disjoint union means no loss of information however.

Another possibility is that in Definition 7 f, g model two different circuits, see Example 3.

We have the special case when in the union $f \cup g$ the system g is constant under the form $V = S^{(m)}$, $g : S^{(m)} \to P^*(S^{(n)})$, $\forall u \in S^{(m)}$, $g(u) = X$, with $X \subset S^{(n)}$. Then $f \cup X : S^{(m)} \to P^*(S^{(n)})$ is defined by:

$$\forall u \in S^{(m)}, (f \cup X)(u) = \begin{cases} X, if\ u \in S^{(m)} \setminus U, \\ f(u) \cup X, if\ u \in U. \end{cases}$$

The interpretation of $f \cup X$ is the next one: when f is the model of an asynchronous circuit, X represents perturbations that are independent on u.

Example 3. In the union $f \cup g$ we presume that $U \cap V \neq \emptyset$ and f, g model two different circuits, the first considered 'good, without errors' and the second 'bad, with a certain error'. The testing problem consists in finding an input $u \in U \cap V$ so that $f(u) \cap g(u) = \emptyset$; after its application to $f \cup g$ and the measurement of a state $x \in (f \cup g)(u)$, we can say if $x \in f(u)$ and the tested circuit is 'good' or perhaps $x \in g(u)$ and the tested circuit is 'bad'.

2 Initial states and final states

Remark 4. In the next properties of the system f:

$$\forall u \in U, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu, \tag{2.1}$$

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu, \tag{2.2}$$

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu, \tag{2.3}$$

$$\forall u \in U, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu, \tag{2.4}$$

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu, \tag{2.5}$$

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu \tag{2.6}$$

we have replaced $t > t_f$ from Definition 2 with $t \geq t_f$ and on the other hand (2.1) is always true due to the way that the $n-$signals were defined. We remark the truth of the implications

$$(2.3) \Longrightarrow (2.2) \Longrightarrow (2.1),$$

$$(2.6) \Longrightarrow (2.5) \Longrightarrow (2.4).$$

Definition 8. *Because f satisfies (2.1), we use to say that it has initial states. The vectors μ are called (the) initial states (of f), or (the) initial values of the states (of f).*

Definition 9. *We presume that f satisfies (2.2). We say in this situation that it has race-free initial states and the initial states μ are called race-free themselves.*

Definition 10. *When f satisfies (2.3), we use to say that it has a (constant) initial state μ. We say in this case that f is initialized and that μ is its (constant) initial state.*

Definition 11. *If f satisfies (2.4), it is called absolutely stable and we also say that it has final states. The vectors μ have in this case the name of final states (of f), or of final values of the states (of f).*

Definition 12. *If f fulfills the property (2.5), it is called absolutely race-free stable and we also say that it has race-free final states. The final states μ are called in this case race-free.*

Definition 13. *We presume that the system f satisfies (2.6). Then it is called absolutely constantly stable or equivalently we say that it has a (constant) final state. The vector μ is called in this situation (constant) final state.*

Theorem 1. *Let $f : U \to P^*(S^{(n)})$ and $g : V \to P^*(S^{(n)})$ be some systems, $U, V \in P^*(S^{(m)})$. If $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$ and f has race-free initial states (constant initial state), then $f \cap g$ has race-free initial states (constant initial state).*

Proof. If one of the previous properties is true for the states in $f(u)$, then it is true for the states in the subset $f(u) \cap g(u) \subset f(u)$ also, $u \in U$. □

Theorem 2. *If f has final states (race-free final states, constant final state) and $f \cap g$ exists, then $f \cap g$ has final states (race-free final states, constant final state).*

Theorem 3. *a) If f, g have race-free initial states and $\forall u \in U \cap V, f(u) \cap g(u) \neq \emptyset$ then $f \cup g$ has race-free initial states.*

b) If f, g have constant initial states and $\bigcup_{u \in U} f(u) \cap \bigcup_{u \in V} g(u) \neq \emptyset$ then $f \cup g$ has constant initial states.

Proof. a) The hypothesis states the truth of the next properties

$$\forall u \in U, \exists \mu \in \mathbf{B}^n, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

$$\forall u \in V, \exists \mu \in \mathbf{B}^n, \forall x \in g(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

$$\forall u \in U \cap V, f(u) \cap g(u) \neq \emptyset. \tag{2.7}$$

If $(U \setminus V) \cup (V \setminus U) \neq \emptyset$, then $\forall u \in (U \setminus V) \cup (V \setminus U)$ the statement is true because it states separately for f and g that they have race-free initial states. And if $U \cap V \neq \emptyset$, then $\forall u \in U \cap V, \forall x \in f(u) \cup g(u)$, the initial value $\mu = x(-\infty + 0)$ depends on u only, not also on the fact that $x \in f(u)$ or $x \in g(u)$ due to (2.7). We have that

$$\forall u \in U \cup V, \exists \mu \in \mathbf{B}^n, \forall x \in (f \cup g)(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu$$

is true.

b) Because $\bigcup\limits_{u \in U} f(u) \cap \bigcup\limits_{u \in V} g(u) \neq \emptyset$, in the statements

$$\exists \mu \in \mathbf{B}^n, \forall u \in U, \forall x \in f(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu,$$

$$\exists \mu' \in \mathbf{B}^n, \forall u \in V, \forall x \in g(u), \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu'$$

the two constants μ and μ', whose existence is unique, coincide. □

Theorem 4. *a) If f, g have final states, then $f \cup g$ has final states.*

b) If f, g have race-free final states and $\forall u \in U \cap V, f(u) \cap g(u) \neq \emptyset$ then $f \cup g$ has race-free final states.

c) If f, g have constant final states and $\bigcup\limits_{u \in U} f(u) \cap \bigcup\limits_{u \in V} g(u) \neq \emptyset$ then $f \cup g$ has constant final states.

3 Initial time and final time

Notation 2. The set of the $n-$signals with final values is denoted by $S_c^{(n)}$. It consists in the functions $x : \mathbf{R} \to \mathbf{B}^n$ of the form

$$x(t) = x(-\infty + 0) \cdot \chi_{(-\infty, t_0)}(t) \oplus x(t_0) \cdot \chi_{[t_0, t_1)}(t) \oplus x(t_1) \cdot \chi_{[t_1, t_2)}(t) \oplus \ldots$$

$$\cdots \oplus x(t_k) \cdot \chi_{[t_k, t_{k+1})}(t) \oplus x(\infty - 0) \cdot \chi_{[t_{k+1}, \infty)}(t)$$

where $x(-\infty + 0), x(\infty - 0) \in \mathbf{B}^n$ and $t_0 < t_1 < ... < t_k < t_{k+1}$ is a finite family of real numbers, $k \geq 0$.

Remark 5. We state the next properties on the asynchronous system $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$:

$$\forall u \in U, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \exists t_0 \in \mathbf{R}, \forall t < t_0, x(t) = \mu, \tag{3.1}$$

$$\forall u \in U, \exists t_0 \in \mathbf{R}, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu, \tag{3.2}$$

$$\exists t_0 \in \mathbf{R}, \forall u \in U, \forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu, \tag{3.3}$$

$$\forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \exists \mu \in \mathbf{B}^n, \exists t_f \in \mathbf{R}, \forall t \geq t_f, x(t) = \mu, \tag{3.4}$$

$$\forall u \in U, \exists t_f \in \mathbf{R}, \forall x \in f(u) \cap S_c^{(n)}, \exists \mu \in \mathbf{B}^n, \forall t \geq t_f, x(t) = \mu, \tag{3.5}$$

$$\exists t_f \in \mathbf{R}, \forall u \in U, \forall x \in f(u) \cap S_c^{(n)}, \exists \mu \in \mathbf{B}^n, \forall t \geq t_f, x(t) = \mu. \tag{3.6}$$

The properties (3.1) and (3.4) are fulfilled by all the systems and the next implications hold:

$$(3.3) \Longrightarrow (3.2) \Longrightarrow (3.1),$$

$$(3.6) \Longrightarrow (3.5) \Longrightarrow (3.4).$$

Definition 14. *The fact that f satisfies (3.1) is expressed sometimes by saying that it has unbounded initial time and any t_0 satisfying this property is called unbounded initial time (instant).*

Definition 15. *Let f be a system that fulfills the property (3.2). We say that it has bounded initial time and any t_0 making this property true is called bounded initial time (instant).*

Definition 16. *When f satisfies (3.3), we use to say that it has fixed initial time and any t_0 fulfilling (3.3) is called fixed initial time (instant).*

Definition 17. *The fact that f satisfies (3.4) is expressed by saying that it has unbounded final time and any t_f satisfying this property is called unbounded final time (instant).*

Definition 18. *If f fulfills the property (3.5), we say that it has bounded final time. Any number t_f satisfying (3.5) is called bounded final time (instant).*

Definition 19. *We presume that the system f satisfies the property (3.6). Then we say that it has fixed final time and any number t_f satisfying (3.6) is called fixed final time (instant).*

Theorem 5. *If f has bounded initial time (fixed initial time) and $f \cap g$ exists, then $f \cap g$ has bounded initial time (fixed initial time).*

Proof. Like previously, if one of the above properties is true for the states in $f(u)$, then it is true for the states in $f(u) \cap g(u) \subset f(u), u \in U$. □

Theorem 6. *If f has bounded final time (fixed final time) and $f \cap g$ exists, then $f \cap g$ has bounded final time (fixed final time).*

Theorem 7. *If f, g have bounded initial time (fixed initial time), then $f \cup g$ has bounded initial time (fixed initial time).*

Proof. We presume that f, g have bounded initial time. If $(U \setminus V) \cup (V \setminus U) \neq \emptyset$, then $\forall u \in (U \setminus V) \cup (V \setminus U)$, $(f \cup g)(u)$ has the desired property, that refers to exactly one of f, g. We presume that $U \cap V \neq \emptyset$ and let $u \in U \cap V$ be arbitrary. $t_0', t_0" \in \mathbf{R}$ exist, depending on u, so that

$$\forall x \in f(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0', x(t) = \mu,$$

$$\forall x \in g(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0^{"}, x(t) = \mu,$$

$t_0 = \min\{t_0', t_0^{"}\}$ satisfies

$$\forall x \in f(u) \cup g(u), \exists \mu \in \mathbf{B}^n, \forall t < t_0, x(t) = \mu.$$

□

Theorem 8. *If f, g have bounded final time (fixed final time), then $f \cup g$ has bounded final time (fixed final time).*

4 Initial state function and set of initial states. Final state function and set of final states

Definition 20. *Let $f : U \to P^*(S^{(n)}), U \in P^*(S^{(m)})$ be a system. The initial state function $\phi_0 : U \to P^*(\mathbf{B}^n)$ and the set of the initial states $\Theta_0 \in P^*(\mathbf{B}^n)$ of f are defined by*

$$\forall u \in U, \phi_0(u) = \{x(-\infty + 0) | x \in f(u)\},$$

$$\Theta_0 = \bigcup_{u \in U} \phi_0(u).$$

Definition 21. *If f has final states, i.e. if (2.4) is satisfied, the final state function $\phi_f : U \to P^*(\mathbf{B}^n)$ and the set of the final states $\Theta_f \in P^*(\mathbf{B}^n)$ of f are*

$$\forall u \in U, \phi_f(u) = \{x(\infty - 0) | x \in f(u)\},$$

$$\Theta_f = \bigcup_{u \in U} \phi_f(u).$$

Theorem 9. *For the systems f, g we have $(\phi \cap \gamma)_0 : W \to P^*(\mathbf{B}^n)$,*

$$\forall u \in W, (\phi \cap \gamma)_0(u) = \phi_0(u) \cap \gamma_0(u),$$

$$(\Theta \cap \Gamma)_0 = \bigcup_{u \in W} (\phi \cap \gamma)_0(u).$$

We have presumed that the domain W of $f \cap g$ is non-empty and we have denoted by $\phi_0, \gamma_0, (\phi \cap \gamma)_0$ the initial state functions of $f, g, f \cap g$ and respectively by $(\Theta \cap \Gamma)_0$ the set of initial states of $f \cap g$.

Proof. We can write that $\forall u \in W$,

$$(\phi \cap \gamma)_0(u) = \{x(-\infty + 0) | x \in (f \cap g)(u)\} = \{x(-\infty + 0) | x \in f(u) \cap g(u)\} =$$

$$= \{x(-\infty + 0) | x \in f(u)\} \cap \{x(-\infty + 0) | x \in g(u)\} = \phi_0(u) \cap \gamma_0(u).$$

□

Theorem 10. *If f, g have final states, then we have $(\phi \cap \gamma)_f : W \to P^*(\mathbf{B}^n)$,*

$$\forall u \in W, (\phi \cap \gamma)_f(u) = \phi_f(u) \cap \gamma_f(u),$$

$$(\Theta \cap \Gamma)_f = \bigcup_{u \in W} (\phi \cap \gamma)_f(u).$$

We have presumed that $W \neq \emptyset$ and the notations are obvious and similar with those from the previous theorem.

Theorem 11. *For the systems f, g we have $(\phi \cup \gamma)_0 : U \cup V \to P^*(\mathbf{B}^n)$,*

$$\forall u \in U \cup V, (\phi \cup \gamma)_0(u) = \begin{cases} \phi_0(u),\ u \in U \setminus V, \\ \gamma_0(u),\ u \in V \setminus U, \\ \phi_0(u) \cup \gamma_0(u),\ u \in U \cap V, \end{cases}$$

$$(\Theta \cup \Gamma)_0 = \bigcup_{u \in U \cup V} (\phi \cup \gamma)_0(u).$$

We have denoted by $(\phi \cup \gamma)_0$ the initial state function of $f \cup g$ and respectively by $(\Theta \cup \Gamma)_0$ the set of initial states of $f \cup g$.

Proof. Three possibilities exist for an arbitrary $u \in U \cup V : u \in U \setminus V, u \in V \setminus U$ and $u \in U \cap V$. If for example $u \in U \setminus V$, then:

$$(\phi \cup \gamma)_0(u) = \{x(-\infty + 0) | x \in (f \cup g)(u)\} = \{x(-\infty + 0) | x \in f(u)\} = \phi_0(u).$$

□

Theorem 12. *We presume that f, g have final states. We have $(\phi \cup \gamma)_f : U \cup V \to P^*(\mathbf{B}^n)$,*

$$\forall u \in U \cup V, (\phi \cup \gamma)_f(u) = \begin{cases} \phi_f(u),\ u \in U \setminus V, \\ \gamma_f(u),\ u \in V \setminus U, \\ \phi_f(u) \cup \gamma_f(u),\ u \in U \cap V, \end{cases}$$

$$(\Theta \cup \Gamma)_f = \bigcup_{u \in U \cup V} (\phi \cup \gamma)_f(u)$$

where the notations are obvious and similar with those from the previous theorem.

5 Subsystem

Definition 22. *Let $f : U \to P^*(S^{(n)})$ and $g : V \to P^*(S^{(n)}), U, V \in P^*(S^{(m)})$ be two systems. f is called a subsystem of g if*

$$U \subset V \text{ and } \forall u \in U, f(u) \subset g(u).$$

Remark 6. The subsystem of a system represents a more precise model of the same circuit, obtained perhaps after restricting the inputs set.

A special case in the inclusion $f \subset g$ is the one when f is uni-valued (it is called deterministic in this situation). This is considered to be non-realistic in modeling.

Example 4. Let f be a system and we take some arbitrary $\mu \in \Theta_0$. The subsystem $f_\mu : U_\mu \to P^*(S^{(n)})$ defined by

$$U_\mu = \{u|u \in U, \mu \in \phi_0(u)\},$$

$$\forall u \in U_\mu, f_\mu(u) = \{x|x \in f(u), x(-\infty + 0) = \mu\}$$

is called the restriction of f at μ. The next property is satisfied: for $\Theta_0 = \{\mu^1, ..., \mu^k\}$, we have $f = f_{\mu^1} \cup ... \cup f_{\mu^k}$ (the union is not disjoint).

Theorem 13. *Let $f : U \to P^*(S^{(n)})$, $f_1 : U_1 \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $g_1 : V_1 \to P^*(S^{(n)})$ be some systems with $U, U_1, V, V_1 \in P^*(S^{(m)})$. If $f \subset f_1, g \subset g_1$ and if $f \cap g$ exists, then $f_1 \cap g_1$ exists and the inclusion $f \cap g \subset f_1 \cap g_1$ is true.*

Proof. We denote by W the set from (1.1) and with W_1 the set

$$W_1 = \{u|u \in U_1 \cap V_1, f_1(u) \cap g_1(u) \neq \emptyset\}$$

From the fact that $U \subset U_1$, $\forall u \in U, f(u) \subset f_1(u)$, $V \subset V_1$, $\forall v \in V, g(v) \subset g_1(v)$ and $W \neq \emptyset$ we infer $W \subset W_1$, $W_1 \neq \emptyset$ and furthermore we have $\forall u \in W, (f \cap g)(u) = f(u) \cap g(u) \subset f_1(u) \cap g_1(u) = (f_1 \cap g_1)(u)$. □

Theorem 14. *We consider the systems $f : U \to P^*(S^{(n)})$, $f_1 : U_1 \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $g_1 : V_1 \to P^*(S^{(n)})$ with $U, U_1, V, V_1 \in P^*(S^{(m)})$. If $f \subset f_1$, $g \subset g_1$ then $f \cup g \subset f_1 \cup g_1$.*

Proof. From $U \subset U_1, V \subset V_1$ we infer that $U \cup V \subset U_1 \cup V_1$. It is shown that $\forall u \in U \cup V, (f \cup g)(u) \subset (f_1 \cup g_1)(u)$ is true in all the three situations $u \in U \setminus V, u \in V \setminus U$ and $u \in U \cap V$. For example if $u \in U \setminus V$, then two possibilities exist:

– $u \in U_1 \setminus V_1$, thus

$$(f \cup g)(u) = f(u) \subset f_1(u) = (f_1 \cup g_1)(u),$$

– $u \in U_1 \cap V_1$, when

$$(f \cup g)(u) = f(u) \subset f_1(u) \subset f_1(u) \cup g_1(u) = (f_1 \cup g_1)(u)$$

is true. We observe that $u \in V_1 \setminus U_1$ is impossible, since $u \notin U_1$ implies $u \notin U$, contradiction. □

6 Dual system

Notation 3. For $u \in S^{(m)}$, we denote by $\overline{u} \in S^{(m)}$ the complement of u satisfying

$$\forall t \in \mathbf{R}, \overline{u}(t) = (\overline{u_1(t)}, ..., \overline{u_m(t)})$$

Definition 23. *The dual system of f is the system $f^* : U^* \to P^*(S^{(n)})$ defined in the next way*

$$U^* = \{\overline{u}|u \in U\},$$

$$\forall u \in U^*, f^*(u) = \{\overline{x}|x \in f(\overline{u})\}.$$

Remark 7. For any $u \in U^*$, $\overline{u} \in U$ and Definition 23 is correct.

If f models a circuit, then f^* models the circuit that is obtained from the previous one after the replacement of the OR gates with AND gates and viceversa and respectively of the input and state tensions with their complements (the complement of the 'HIGH' tension is by definition the 'LOW' tension and viceversa).

Theorem 15. *If $f \cap g$ exists, then $(f \cap g)^*$, $f^* \cap g^*$ exist and*

$$(f \cap g)^* = f^* \cap g^*$$

Proof. We denote by W the domain (1.1) of $f \cap g$. The domain of $(f \cap g)^*$ is W^* and the domain W_1 of $f^* \cap g^*$ is:

$$W_1 = \{u | u \in U^* \cap V^*, f^*(u) \cap g^*(u) \neq \emptyset\} =$$

$$= \{u | \overline{u} \in U \cap V, \{\overline{x} | x \in f(\overline{u})\} \cap \{\overline{x} | x \in g(\overline{u})\} \neq \emptyset\} =$$

$$= \{\overline{u} | u \in U \cap V, \{\overline{x} | x \in f(u)\} \cap \{\overline{x} | x \in g(u)\} \neq \emptyset\} =$$

$$= \{\overline{u} | u \in U \cap V, \{x | x \in f(u)\} \cap \{x | x \in g(u)\} \neq \emptyset\} = W^*.$$

Moreover, for any $u \in W^*$ we infer

$$(f \cap g)^*(u) = \{\overline{x} | x \in (f \cap g)(\overline{u})\} = \{\overline{x} | x \in f(\overline{u}) \cap g(\overline{u})\} =$$

$$= \{\overline{x} | x \in f(\overline{u})\} \cap \{\overline{x} | x \in g(\overline{u})\} = f^*(u) \cap g^*(u) = (f^* \cap g^*)(u)$$

□

Theorem 16. *We have*

$$(f \cup g)^* = f^* \cup g^*.$$

Proof. We remark that the equal domains of the two systems are $(U \cup V)^* = U^* \cup V^*$. Let $u \in U^* \cup V^*$ be an arbitrary input. If $u \in U^* \setminus V^*$, then $f^*(u) = (f^* \cup g^*)(u)$ and the fact that $\overline{u} \in U \setminus V$ implies $(f \cup g)(\overline{u}) = f(\overline{u})$, thus

$$(f \cup g)^*(u) = \{\overline{x} | x \in (f \cup g)(\overline{u})\} = \{\overline{x} | x \in f(\overline{u})\} = f^*(u) = (f^* \cup g^*)(u).$$

If $u \in V^* \setminus U^*$, the situation is similar. We presume in this moment that $u \in U^* \cap V^*$, implying $f^*(u) \cup g^*(u) = (f^* \cup g^*)(u)$, $\overline{u} \in U \cap V, (f \cup g)(\overline{u}) = f(\overline{u}) \cup g(\overline{u})$ and we have:

$$(f \cup g)^*(u) = \{\overline{x} | x \in (f \cup g)(\overline{u})\} = \{\overline{x} | x \in f(\overline{u}) \cup g(\overline{u})\} =$$

$$= \{\overline{x} | x \in f(\overline{u})\} \cup \{\overline{x} | x \in g(\overline{u})\} = f^*(u) \cup g^*(u) = (f^* \cup g^*)(u).$$

In all the three cases the statement of the theorem was proved to be true. □

7 Inverse system

Definition 24. *The inverse system of f is defined by $f^{-1}: X \to P^*(S^{m)})$,*

$$X = \bigcup_{u \in U} f(u),$$

$$\forall x \in X, f^{-1}(x) = \{u | u \in U, x \in f(u)\}.$$

Remark 8. The inputs and the states of f become states and inputs of f^{-1}, meaning that f^{-1} inverts the causes and the effects in modeling: its aim is to answer the question "given an effect x, which are the causes u producing it?"

Theorem 17. *Let $f: U \to P^*(S^{(n)}), g: V \to P^*(S^{(n)}), U, V \in P^*(S^{(m)})$ be some systems. If $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$, then the systems $(f \cap g)^{-1}$, $f^{-1} \cap g^{-1}$ exist and they have the same domain:*

$$Y = \bigcup_{u \in W} (f(u) \cap g(u)).$$

Furthermore, we have

$$(f \cap g)^{-1} = f^{-1} \cap g^{-1}.$$

Proof. Y is obviously the domain of $(f \cap g)^{-1}$. We can write

$$Y = \bigcup_{u \in U \cap V} (f(u) \cap g(u)) = \{x | \exists u \in U \cap V, x \in f(u) \cap g(u)\} =$$

$$= \{x | x \in \bigcup_{v \in U} f(v) \cap \bigcup_{v \in V} g(v), \exists u, u \in U, x \in f(u) \text{ and } u \in V, x \in g(u)\} =$$

$$= \{x | x \in \bigcup_{v \in U} f(v) \cap \bigcup_{v \in V} g(v), \exists u, u \in f^{-1}(x) \text{ and } u \in g^{-1}(x)\} =$$

$$= \{x | x \in \bigcup_{v \in U} f(v) \cap \bigcup_{v \in V} g(v), f^{-1}(x) \cap g^{-1}(x) \neq \emptyset\}$$

thus Y is the domain of $f^{-1} \cap g^{-1}$ too. We have $\forall x \in Y$,

$$(f \cap g)^{-1}(x) = \{u | u \in U \cap V, x \in (f \cap g)(u)\} = \{u | u \in U \cap V, x \in f(u) \cap g(u)\} =$$

$$= \{u | u \in U \cap V, x \in f(u)\} \cap \{u | u \in U \cap V, x \in g(u)\} =$$

$$= (\{u | u \in U \setminus V, x \in f(u)\} \cup \{u | u \in U \cap V, x \in f(u)\}) \cap$$

$$\cap (\{u | u \in V \setminus U, x \in g(u)\} \cup \{u | u \in U \cap V, x \in g(u)\}) =$$

$$= \{u | u \in U, x \in f(u)\} \cap \{u | u \in V, x \in g(u)\} = f^{-1}(x) \cap g^{-1}(x) = (f^{-1} \cap g^{-1})(x).$$

□

Theorem 18. *We consider the systems $f : U \to P^*(S^{(n)}), g : V \to P^*(S^{(n)}), U, V \in P^*(S^{(m)})$. The systems $(f \cup g)^{-1}$, $f^{-1} \cup g^{-1}$ have the domain equal with*

$$Y' = \bigcup_{u \in U} f(u) \cup \bigcup_{u \in V} g(u)$$

and the next equality is true

$$(f \cup g)^{-1} = f^{-1} \cup g^{-1}$$

Proof. The domain of $(f \cup g)^{-1}$ is

$$\bigcup_{u \in U \cup V} (f \cup g)(u) = \bigcup_{u \in U \setminus V} (f \cup g)(u) \cup \bigcup_{u \in U \cap V} (f \cup g)(u) \cup \bigcup_{u \in V \setminus U} (f \cup g)(u) =$$

$$= \bigcup_{u \in U \setminus V} f(u) \cup \bigcup_{u \in U \cap V} (f(u) \cup g(u)) \cup \bigcup_{u \in V \setminus U} g(u) =$$

$$= \bigcup_{u \in U \setminus V} f(u) \cup \bigcup_{u \in U \cap V} f(u) \cup \bigcup_{u \in U \cap V} g(u) \cup \bigcup_{u \in V \setminus U} g(u) = \bigcup_{u \in U} f(u) \cup \bigcup_{u \in V} g(u)$$

and it coincides with Y', that is obviously the domain of $f^{-1} \cup g^{-1}$. For any $x \in Y'$ we have:

$$(f \cup g)^{-1}(x) = \{u | u \in U \cup V, x \in (f \cup g)(u)\} = \{u | u \in U \setminus V, x \in f(u)\} \cup$$

$$\cup \{u | u \in V \setminus U, x \in g(u)\} \cup \{u | u \in U \cap V, x \in f(u)\} \cup \{u | u \in U \cap V, x \in g(u)\} =$$

$$= \{u | u \in U, x \in f(u)\} \cup \{u | u \in V, x \in g(u)\} =$$

$$= \begin{cases} f^{-1}(x), x \in \bigcup\limits_{u \in U} f(u) \setminus \bigcup\limits_{u \in V} g(u) \\ g^{-1}(x), x \in \bigcup\limits_{u \in V} g(u) \setminus \bigcup\limits_{u \in U} f(u) \\ f^{-1}(x) \cup g^{-1}(x), x \in \bigcup\limits_{u \in U} f(u) \cap \bigcup\limits_{u \in V} g(u) \end{cases} = (f^{-1} \cup g^{-1})(x)$$

□

8 Cartesian product

Definition 25. *Let $u \in S^{(m)}, u' \in S^{(m')}$ be two signals. We define the Cartesian product $u \times u' \in S^{(m+m')}$ of the functions u and u' by*

$$\forall t \in \mathbf{R}, (u \times u')(t) = (u_1(t), ..., u_m(t), u'_1(t), ..., u'_{m'}(t))$$

Definition 26. *For any sets $U \in P^*(S^{(m)}), U' \in P^*(S^{(m')})$ we define the Cartesian product $U \times U' \in P^*(S^{(m+m')})$,*

$$U \times U' = \{u \times u' | u \in U, u' \in U'\}$$

Definition 27. *The Cartesian product of the systems f and $f' : U' \to P^*(S^{(n')}), U' \in P^*(S^{(m')})$ is $f \times f' : U \times U' \to P^*(S^{(n+n')})$,*

$$\forall u \times u' \in U \times U', (f \times f')(u \times u') = f(u) \times f'(u')$$

Remark 9. The Cartesian product of the systems models two circuits that are not interconnected and run under different inputs.

Theorem 19. *Let $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ and $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$ be three systems. If $\exists u \in U \cap V$, $f(u) \cap g(u) \neq \emptyset$ then the systems $(f \cap g) \times f'$, $(f \times f') \cap (g \times f')$ are defined and $W \times U'$ is their common domain, where we have used again the notation*

$$W = \{u | u \in U \cap V, f(u) \cap g(u) \neq \emptyset\}.$$

The next equality is true

$$(f \cap g) \times f' = (f \times f') \cap (g \times f').$$

Proof. We show that $W \times U'$, that is the domain of $(f \cap g) \times f'$, is also the domain of $(f \times f') \cap (g \times f')$:

$$W \times U' = \{u \times u' | u \in W, u' \in U'\} =$$

$$= \{u \times u' | u \in U \cap V, u' \in U', f(u) \cap g(u) \neq \emptyset \text{ and } f'(u') \neq \emptyset\} =$$

$$= \{u \times u' | u \times u' \in (U \cap V) \times U', (f(u) \times f'(u')) \cap (g(u) \times f'(u')) \neq \emptyset\} =$$

$$= \{u \times u' | u \times u' \in (U \times U') \cap (V \times U'), (f \times f')(u \times u') \cap (g \times f')(u \times u') \neq \emptyset\}.$$

Furthermore for any $u \times u' \in W \times U'$ we have

$$((f \cap g) \times f')(u \times u') = (f \cap g)(u) \times f'(u') = (f(u) \cap g(u)) \times f'(u') =$$

$$= (f(u) \times f'(u')) \cap (g(u) \times f'(u')) = (f \times f')(u \times u') \cap (g \times f')(u \times u') =$$

$$= ((f \times f') \cap (g \times f'))(u \times u').$$

□

Theorem 20. *Let $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ and $f' : U' \to P^*(S^{(n')})$, $U' \in P^*(S^{(m')})$ be some systems. The common domain of $(f \cup g) \times f'$, $(f \times f') \cup (g \times f')$ is $(U \cup V) \times U' = (U \times U') \cup (V \times U')$ and the next equality holds*

$$(f \cup g) \times f' = (f \times f') \cup (g \times f').$$

Proof. $\forall u \times u' \in (U \cup V) \times U'$ we have one of the next possibilities:

Case $u \times u' \in (U \setminus V) \times U' = (U \times U') \setminus (V \times U')$

$$((f \cup g) \times f')(u \times u') = (f \cup g)(u) \times f'(u') = f(u) \times f'(u') = (f \times f')(u \times u') =$$
$$= ((f \times f') \cup (g \times f'))(u \times u');$$

Case $u \times u' \in (V \setminus U) \times U'$ is similar;

Case $u \times u' \in (U \cap V) \times U' = (U \times U') \cap (V \times U')$

$$((f \cup g) \times f')(u \times u') = (f \cup g)(u) \times f'(u') = (f(u) \cup g(u)) \times f'(u') =$$
$$= (f(u) \times f'(u')) \cup (g(u) \times f'(u')) = (f \times f')(u \times u') \cup (g \times f')(u \times u') =$$
$$= ((f \times f') \cup (g \times f'))(u \times u').$$

□

9 Parallel connection

Definition 28. *The parallel connection of* f *with* $f_1' : U_1' \to P^*(S^{(n')}), U_1' \in P^*(S^{(m)})$ *is* $(f, f_1') : U \cap U_1' \to P^*(S^{(n+n')})$,

$$\forall u \in U \cap U_1', (f, f_1')(u) = (f \times f_1')(u \times u).$$

Remark 10. The parallel connection models two circuits that are not interconnected and run under the same input.

Theorem 21. *We consider the systems* $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $f_1' : U_1' \to P^*(S^{(n')})$, *with* $U, V, U_1' \in P^*(S^{(m)})$. *We presume that* $\exists u \in U \cap V \cap U_1'$ *so that* $f(u) \cap g(u) \neq \emptyset$. *Then the set*

$$W' = \{u | u \in U \cap V \cap U_1', f(u) \cap g(u) \neq \emptyset\}$$

is the domain of the systems $(f \cap g, f_1')$, $(f, f_1') \cap (g, f_1')$ *and the next equality holds*

$$(f \cap g, f_1') = (f, f_1') \cap (g, f_1').$$

Proof. We observe that W' is non-empty, it is the domain of $(f \cap g, f_1')$ and we show that it is also the domain of $(f, f_1') \cap (g, f_1')$. We denote by

$$W" = \{u | u \in (U \cap U_1') \cap (V \cap U_1'), (f, f_1')(u) \cap (g, f_1')(u) \neq \emptyset\}$$

the domain of $(f, f_1') \cap (g, f_1')$ for which we have

$$W" = \{u | u \in U \cap V \cap U_1', (f(u) \times f_1'(u)) \cap (g(u) \times f_1'(u)) \neq \emptyset\} =$$
$$= \{u | u \in U \cap V \cap U_1', (f(u) \cap g(u)) \times f_1'(u) \neq \emptyset\} =$$

$$= \{u|u \in U \cap V \cap U_1', f(u) \cap g(u) \neq \emptyset\}$$

thus $W" = W'$. For any $u \in W'$ we have:

$$(f \cap g, f_1')(u) = ((f \cap g) \times f_1')(u \times u) \overset{Theorem\ 19}{=} ((f \times f_1') \cap (g \times f_1'))(u \times u) =$$

$$= (f \times f_1')(u \times u) \cap (g \times f_1')(u \times u) = (f, f_1')(u) \cap (g, f_1')(u) = ((f, f_1') \cap (g, f_1'))(u).$$

□

Remark 11. A similar result with the one from Theorem 19 states the truth of the formula

$$f \times (f' \cap g') = (f \times f') \cap (f \times g')$$

and then from Theorem 19 we get the next property

$$(f \cap g) \times (f' \cap g') = (f \times f') \cap (f \times g') \cap (g \times f') \cap (g \times g').$$

Like in Theorem 21 we can prove that

$$(f, f_1' \cap g_1') = (f, f_1') \cap (f, g_1')$$

is true and then from Theorem 21 we obtain

$$(f \cap g, f_1' \cap g_1') = (f, f_1') \cap (f, g_1') \cap (g, f_1') \cap (g, g_1').$$

Theorem 22. *Let $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $f_1' : U_1' \to P^*(S^{(n')})$ be three systems with $U, V, U_1' \in P^*(S^{(m)})$. If $U \cap U_1' \neq \emptyset$, $V \cap U_1' \neq \emptyset$, then the common domain of the systems $(f \cup g, f_1')$, $(f, f_1') \cup (g, f_1')$ is $(U \cup V) \cap U_1' = (U \cap U_1') \cup (V \cap U_1')$ and we have*

$$(f \cup g, f_1') = (f, f_1') \cup (g, f_1').$$

Remark 12. We observe the truth of the formulas

$$f \times (f' \cup g') = (f \times f') \cup (f \times g'),$$

$$(f \cup g) \times (f' \cup g') = (f \times f') \cup (f \times g') \cup (g \times f') \cup (g \times g')$$

and respectively of the formulas

$$(f, f_1' \cup g_1') = (f, f_1') \cup (f, g_1'),$$

$$(f \cup g, f_1' \cup g_1') = (f, f_1') \cup (f, g_1') \cup (g, f_1') \cup (g, g_1').$$

10 Serial connection

Definition 29. *The serial connection of* $h : X \to P^*(S^{(p)}), X \in P^*(S^{(n)})$ *with* $f : U \to P^*(S^{(n)})$, $U \in P^*(S^{(m)})$ *is defined whenever* $\bigcup_{u\in U} f(u) \subset X$ *by*[2] $h \circ f : U \to P^*(S^{(p)})$,

$$\forall u \in U, (h \circ f)(u) = \bigcup_{x\in f(u)} h(x).$$

Remark 13. The serial connection of the systems models two circuits connected in cascade and it coincides with the usual composition of the multi-valued functions.

Theorem 23. *We consider the systems* $f : U \to P^*(S^{(n)})$, $g : V \to P^*(S^{(n)})$, $U, V \in P^*(S^{(m)})$ *and* $h : X \to P^*(S^{(p)})$, $h_1 : X_1 \to P^*(S^{(p)})$, $X, X_1 \in P^*(S^{(n)})$.

a) If $\bigcup_{u\in U} f(u) \subset X$, $\bigcup_{u\in V} g(u) \subset X$ *and* $\exists u \in U \cap V, f(u) \cap g(u) \neq \emptyset$ *then the sets*

$$W = \{u|u \in U \cap V, f(u) \cap g(u) \neq \emptyset\},$$

$$W_1 = \{u|u \in U \cap V, \bigcup_{x\in f(u)} h(x) \cap \bigcup_{x\in g(u)} h(x) \neq \emptyset\}$$

are non-empty and represent the domains of the systems $h \circ (f \cap g)$, $(h \circ f) \cap (h \circ g)$. *We have*

$$h \circ (f \cap g) \subset (h \circ f) \cap (h \circ g);$$

b) We ask that $\bigcup_{u\in U} f(u) \subset \{x|x \in X \cap X_1, h(x) \cap h_1(x) \neq \emptyset\}$. U *is the domain of the systems* $(h \cap h_1) \circ f$, $(h \circ f) \cap (h_1 \circ f)$ *and the next inclusion is true:*

$$(h \cap h_1) \circ f \subset (h \circ f) \cap (h_1 \circ f)$$

Proof. a) From the hypothesis $f \cap g$ is defined and has the domain W. As

$$\bigcup_{u\in W} (f \cap g)(u) \subset \bigcup_{u\in W} f(u) \subset \bigcup_{u\in U} f(u) \subset X$$

we have obtained that $h \circ (f \cap g)$ is defined and has the domain W.

From the same hypothesis $h \circ f$ and $h \circ g$ are defined and have the domains U, V. Because $\emptyset \neq W \subset W_1$, the system $(h \circ f) \cap (h \circ g)$ is defined and has the domain W_1.

[2] We show a more general definition of the serial connection that was used in previous works: the request $\bigcup_{u\in U} f(u) \subset X$ is replaced by $\exists u \in U, f(u) \cap X \neq \emptyset$ and $h \circ f : Z \to P^*(S^{(p)})$ is defined by

$$Z = \{u|u \in U, f(u) \cap X \neq \emptyset\},$$

$$\forall u \in Z, (h \circ f)(u) = \bigcup_{x\in f(u)\cap X} h(x).$$

$\forall u\in W$ we get

$$\bigcup_{x\in f(u)\cap g(u)}h(x)\subset\bigcup_{x\in f(u)}h(x),\quad\bigcup_{x\in f(u)\cap g(u)}h(x)\subset\bigcup_{x\in g(u)}h(x)$$

from where

$$\bigcup_{x\in f(u)\cap g(u)}h(x)\subset\bigcup_{x\in f(u)}h(x)\cap\bigcup_{x\in g(u)}h(x)$$

and we conclude that $\forall u\in W$,

$$(h\circ(f\cap g))(u)=\bigcup_{x\in f(u)\cap g(u)}h(x)\subset\bigcup_{x\in f(u)}h(x)\cap\bigcup_{x\in g(u)}h(x)=$$

$$=(h\circ f)(u)\cap(h\circ g)(u)=((h\circ f)\cap(h\circ g))(u).$$

b) The hypothesis $\bigcup\limits_{u\in U}f(u)\subset\{x|x\in X\cap X_1,h(x)\cap h_1(x)\neq\emptyset\}$ states that the domain $\{x|x\in X\cap X_1,h(x)\cap h_1(x)\neq\emptyset\}$ of $h\cap h_1$ is non-empty and that $(h\cap h_1)\circ f$ is defined. From the hypothesis we infer that $\bigcup\limits_{u\in U}f(u)\subset X$, $\bigcup\limits_{u\in U}f(u)\subset X_1$ and $h\circ f,h_1\circ f$ are defined themselves. The domain of $(h\cap h_1)\circ f$ is U. Moreover from $\forall u\in U,\forall x\in f(u),h(x)\cap h_1(x)\neq\emptyset$ we conclude that the domain $\{u|u\in U,\bigcup\limits_{x\in f(u)}h(x)\cap\bigcup\limits_{x\in f(u)}h_1(x)\neq\emptyset\}$ of $(h\circ f)\cap(h_1\circ f)$ is equal with U too.

Let $u\in U$ be arbitrary and fixed. From

$$\bigcup_{x\in f(u)}(h\cap h_1)(x)\subset\bigcup_{x\in f(u)}h(x),\quad\bigcup_{x\in f(u)}(h\cap h_1)(x)\subset\bigcup_{x\in f(u)}h_1(x)$$

we get

$$\bigcup_{x\in f(u)}(h\cap h_1)(x)\subset\bigcup_{x\in f(u)}h(x)\cap\bigcup_{x\in f(u)}h_1(x)$$

and eventually we obtain

$$((h\cap h_1)\circ f)(u)=\bigcup_{x\in f(u)}(h\cap h_1)(x)\subset$$

$$\subset\bigcup_{x\in f(u)}h(x)\cap\bigcup_{x\in f(u)}h_1(x)=(h\circ f)(u)\cap(h_1\circ f)(u)=((h\circ f)\cap(h_1\circ f))(u).$$

□

Theorem 24. *We have the systems* $f:U\to P^*(S^{(n)})$, $g:V\to P^*(S^{(n)})$, $U,V\in P^*(S^{(m)})$ *and* $h:X\to P^*(S^{(p)})$, $h_1:X_1\to P^*(S^{(p)})$, $X,X_1\in P^*(S^{(n)})$.

a) We presume that $\bigcup\limits_{u\in U}f(u)\subset X$, $\bigcup\limits_{u\in V}g(u)\subset X$; *the set* $U\cup V$ *is the common domain of* $h\circ(f\cup g)$, $(h\circ f)\cup(h\circ g)$ *and the next equality is true*

$$h\circ(f\cup g)=(h\circ f)\cup(h\circ g).$$

b) If $\bigcup_{u\in U} f(u)\subset X$, $\bigcup_{u\in U} f(u)\subset X_1$ *then* $(h\cup h_1)\circ f$, $(h\circ f)\cup(h_1\circ f)$ *have the domain* U *and*

$$(h\cup h_1)\circ f=(h\circ f)\cup(h_1\circ f).$$

Proof. a) The systems $h\circ f$ and $h\circ g$ are defined from the hypothesis and because (see the proof of Theorem 18)

$$\bigcup_{u\in U\cup V}(f\cup g)(u)=\bigcup_{u\in U} f(u)\cup\bigcup_{u\in V} g(u)\subset X$$

we infer that $h\circ(f\cup g)$ is defined too. The common domain of $h\circ(f\cup g)$ and $(h\circ f)\cup(h\circ g)$ is $U\cup V$.

Let $u\in U\cup V$ be arbitrary. We can prove the statement of the theorem in the three cases: $u\in(U\setminus V)$, $u\in(V\setminus U)$, $u\in(U\cap V)$. For example in the last case we have:

$$(h\circ(f\cup g))(u)=\bigcup_{x\in(f\cup g)(u)} h(x)=\bigcup_{x\in f(u)\cup g(u)} h(x)=$$

$$=\bigcup_{x\in f(u)} h(x)\cup\bigcup_{x\in g(u)} h(x)=(h\circ f)(u)\cup(h\circ g)(u)=((h\circ f)\cup(h\circ g))(u).$$

b) The hypothesis implies $\bigcup_{u\in U} f(u)\subset X\cup X_1$ thus $(h\cup h_1)\circ f$ is defined and on the other hand $h\circ f$ and $h_1\circ f$ are defined too. The systems $(h\cup h_1)\circ f$, $(h\circ f)\cup(h_1\circ f)$ have the same domain $U=U\cup U$.

For any $u\in U$ fixed, we have

$$f(u)=f(u)\cap(X\cup X_1)=f(u)\cap((X\setminus X_1)\cup(X_1\setminus X)\cup(X\cap X_1))=$$
$$=(f(u)\cap(X\setminus X_1))\cup(f(u)\cap(X_1\setminus X))\cup(f(u)\cap(X\cap X_1))$$

thus

$$((h\cup h_1)\circ f)(u)=\bigcup_{x\in f(u)}(h\cup h_1)(x)=$$

$$=\bigcup_{x\in(f(u)\cap(X\setminus X_1))\cup(f(u)\cap(X_1\setminus X))\cup(f(u)\cap(X\cap X_1))}(h\cup h_1)(x)=$$

$$=\bigcup_{x\in f(u)\cap(X\setminus X_1)}(h\cup h_1)(x)\cup\bigcup_{x\in f(u)\cap(X_1\setminus X)}(h\cup h_1)(x)\cup\bigcup_{x\in f(u)\cap X\cap X_1}(h\cup h_1)(x)=$$

$$=\bigcup_{x\in f(u)\cap(X\setminus X_1)} h(x)\cup\bigcup_{x\in f(u)\cap(X_1\setminus X)} h_1(x)\cup\bigcup_{x\in f(u)\cap X\cap X_1} h(x)\cup\bigcup_{x\in f(u)\cap X\cap X_1} h_1(x)=$$

$$=\bigcup_{x\in(f(u)\cap(X\setminus X_1))\cup(f(u)\cap X\cap X_1)} h(x)\cup\bigcup_{x\in(f(u)\cap(X_1\setminus X))\cup(f(u)\cap X\cap X_1)} h_1(x)=$$

$$=\bigcup_{x\in f(u)\cap X} h(x)\cup\bigcup_{x\in f(u)\cap X_1} h_1(x)=\bigcup_{x\in f(u)} h(x)\cup\bigcup_{x\in f(u)} h_1(x)=$$
$$=(h\circ f)(u)\cup(h_1\circ f)(u)=((h\circ f)\cup(h_1\circ f))(u).$$

□

11 Final remarks

The intersection and the union of the systems are dual concepts and their properties, as expressed by the previous theorems, are similar.

On the other hand, let us remark the roots of our interests in the Romanian mathematical literature represented by the works in schemata with contacts and relays from the 50's and the 60's of Grigore Moisil. Modeling is different there, but the modelled switching phenomena are exactly the same like ours.

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Oradea, Romania
Received September 21, 2006
E-mail: *serban_e_vlad@yahoo.com*