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On torsionfree LCA groups with commutative rings of continuous endomorphisms

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Abstract. Let $\mathcal{L}$ be the class of locally compact abelian (LCA) groups. For certain subclasses $\mathcal{S}$ of $\mathcal{L}$, we obtain information about the groups $X \in \mathcal{S}$ such that the ring $E(X)$ of continuous endomorphisms of X is commutative. The main results concern torsionfree groups, groups with splitting torsion subgroups and their duals.

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1 Introduction

This paper continues the study of LCA groups with commutative rings of continuous endomorphisms, started in [14] and [15]. Our work was motivated by some results available for discrete torsionfree groups. Although the classification of all discrete torsionfree abelian groups with commutative endomorphism rings seems to be impossible, for certain restricted subclasses of such groups there exist complete characterizations. Thus, J. Zelmanowitz disposed in [18] of the case of $\mathbb{Z}$-dualizable groups. Further, C. Murley and Ph. Schultz [16] settled the case of reduced groups X with the property that, for all $p \in \mathbb{P}$, either $X = pX$ or $X/pX \cong \mathbb{Z}(p)$. Also, L.C.A. van Leeuwen solved in [7] the case of completely decomposable groups, and in [8] the case of irreducible groups of finite rank ρ, such that ρ is squarefree. Later, A. Mader and Ph. Schultz [10] extended the result of L.C.A. van Leeuwen on completely decomposable groups to the case of almost completely decomposable groups. On the other hand, T. Szele and J. Szendrei [17] have obtained useful information in the case of mixed groups with splitting torsion subgroups, which is closely related to the case of torsionfree groups because the torsion groups with commutative endomorphism rings are described completely [17].

The purpose of the present paper is to exhibit analogues of these results in the more general context of all LCA groups.

2 Notation

Throughout, we shall freely use the notation and terminology introduced in [14] and [15]. In addition, if $p \in \mathbb{P}$, we let J_p be the group of p-adic integers taken discrete. The direct product of all groups J_p, taken with the discrete topology too, is denoted by $\prod_{p\in\mathbb{P}}^{loc}(J_p;\{0\})$. Given any $X \in \mathcal{L}$, we redefine the symbol $S(X)$ by setting $S(X) = \{p \in \mathbb{P} \mid \big(\big(c(X)+k(X)\big)/c(X)\big)_p \neq \{0\}\}$. Also, we let $X_\omega = \bigcap_{n\in\mathbb{N}_0} \overline{nX}$, and denote by $D(X)$ the minimal divisible extension of X, topologized in such a way that X becomes open in $D(X)$. If X is torsionfree and A is a subgroup of X, we denote by A_* the smallest pure subgroup of X containing A. If f is a homomorphism from X into another group $Y \in \mathcal{L}$, then $f|A$ stands for the restriction of f to A.

3 $\mathbb{Z}$-dualizable and topologically purely indecomposable groups

As we mentioned in Introduction, a description of discrete, torsionfree groups in $\mathcal{L}$ having commutative endomorphism rings is improbable. A reason for such an assertion is given in [16, §4]. There are, however, several interesting special cases which admit complete solutions. In this section, we extend two of them to the more general framework of LCA groups.

Definition 3.1. *A group $X \in \mathcal{L}$ is said to be $\mathbb{Z}$-dualizable if $H(X,\mathbb{Z}) \neq \{0\}$.*

In [18, Corollary 2.3], J. Zelmanowitz mentions that a discrete, torsionfree, $\mathbb{Z}$-dualizable group $X \in \mathcal{L}$ has a commutative ring $E(X)$ if and only if $X \cong \mathbb{Z}$.

As a generalization, we have

Theorem 3.2. *Let X be a torsionfree $\mathbb{Z}$-dualizable group in $\mathcal{L}$. The ring $E(X)$ is commutative if and only if $X \cong \mathbb{Z}$.*

Proof. Assume $E(X)$ is commutative. By hypothesis, there exist $f \in H(X,\mathbb{Z})$ and $a \in X$ such that $f(a) \neq 0$. Since $\mathbb{Z}$ is discrete, f is open. Therefore, since the nonzero subgroups of $\mathbb{Z}$ are isomorphic to $\mathbb{Z}$, it is sufficient to show that f is injective. For $x \in X$, let $g_x \in H(\mathbb{Z},X)$ be given by $g_x(1) = x$. If there existed a nonzero $b \in \ker(f)$, then $u = g_a \circ f$ and $v = g_b \circ f$ would be elements of $E(X)$ satisfying $u \circ v = 0$ and $(v \circ u)(a) = f(a)^2 b$, which contradicts the commutativity of $E(X)$. Thus f is injective, and hence $X \cong \mathbb{Z}$.

The converse is clear. □

We continue with

Corollary 3.3. *Let X be a densely divisible group in $\mathcal{L}$ containing a copy of $\mathbb{T}$. The ring $E(X)$ is commutative if and only if $X \cong \mathbb{T}$.*

Proof. It is easy to see that X contains a copy of $\mathbb{T}$ if and only if $H(\mathbb{T},X) \neq \{0\}$. Since $H(\mathbb{T},X) \cong H(X^*,\mathbb{Z})$ [12, Ch. II, Theorem 2.8, Corollary 2], the assertion follows from Theorem 3.2 and duality. □

C. Murley and Ph. Schultz [16] showed that every discrete, reduced, torsionfree group $X \in \mathcal{L}$ with cyclic p-basic subgroups for all $p \in \mathbb{P}$ has a commutative endomorphism ring. It is well known that such a group X is isomorphic to a pure subgroup of $\prod_{p\in\mathbb{P}}^{loc}(J_p;\{0\})$. On the other hand, Ph. Griffith showed in [5] that a discrete, reduced, torsionfree, purely indecomposable group in $\mathcal{L}$ is isomorphic to a subgroup of $\prod_{p\in\mathbb{P}}^{loc}(J_p;\{0\})$, and gave a characterization of those pure subgroups of $\prod_{p\in\mathbb{P}}^{loc}(J_p;\{0\})$, which are purely indecomposable.

Next we introduce a generalization of purely indecomposable groups and show that some of these new groups have commutative rings of continuous endomorphisms.

Definition 3.4. *A group $X \in \mathcal{L}$ is said to be topologically purely indecomposable if every closed, pure subgroup of X is topologically indecomposable.*

Theorem 3.5. *Let X be a torsionfree, topologically purely indecomposable group in $\mathcal{L}$. Suppose also that if X is discrete, then for each $p \in \mathbb{P}$ the p-basic subgroups of X are cyclic. Then X is topologically isomorphic either to one of the groups $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}^*$, $\mathbb{Q}_p$, or $\mathbb{Z}_p$, where $p \in \mathbb{P}$, or to a pure subgroup of $\prod_{p\in\mathbb{P}}^{loc}(J_p;\{0\})$. In particular, $E(X)$ is commutative.*

Proof. Assume the stated condition. If $c(X) \neq \{0\}$, we must have $c(X) = X$. For, if $c(X)$ were a proper subgroup of X, we could write $X = c(X) \oplus Y$ for some nonzero closed subgroup Y of X [6, (25.30)(c)], in contradiction with the fact that every topologically purely indecomposable group in $\mathcal{L}$ is topologically indecomposable. Thus $c(X) = X$. Now, as is well known, every connected torsionfree group in $\mathcal{L}$ is topologically isomorphic to a group of the form $\mathbb{R}^d \times (\mathbb{Q}^*)^\alpha$, where $d \in \mathbb{N}$ and α is a cardinal number [6, see (9.14) and (25.8)]. By topological indecomposability of X, we conclude that in this case either $X \cong \mathbb{R}$ or $X \cong \mathbb{Q}^*$.

Next suppose $c(X) = \{0\}$. In particular, $X \in \mathcal{L}_0$ and hence $k(X)$ is open in X. Consequently, if $k(X) = \{0\}$, X is discrete. As every discrete group in $\mathcal{L}$ can be written as a direct sum of a divisible group with a reduced group [6, (A.8)], we conclude that X is either divisible or reduced. Therefore, as every discrete, divisible, torsionfree group in $\mathcal{L}$ is isomorphic to $\mathbb{Q}^{(\beta)}$ for some cardinal number β [6, (A.14)], we deduce that in the former case $X \cong \mathbb{Q}$. Further, in the second case we see that, for each $p \in \mathbb{P}$, every p-basic subgroup of X is cyclic, and so X is isomorphic to a pure subgroup of $\prod_{p\in\mathbb{P}}^{loc}(J_p;\{0\})$ [5, p. 740].

Next suppose $k(X) \neq \{0\}$. Since $c(X) = \{0\}$, it follows that $k(X)$ is a topological torsion group. Now, since the topological primary components of a topological torsion group in $\mathcal{L}$ split topologically from that group [1, Theorem 3.13] and since $k(X)$ is pure in X, we conclude that there exists $p \in \mathbb{P}$ such that $k(X) = k_p(X)$, and hence $k(X)$ is a topological p-primary group. We assert that if $k(X)$ is nonreduced, then $X \cong \mathbb{Q}_p$. Indeed, every nonreduced, torsionfree, topological p-primary group in $\mathcal{L}$ contains copies of $\mathbb{Q}_p$ [1, Theorem 4.23]. Since $\mathbb{Q}_p$ is splitting in the class of torsionfree LCA groups [1, Proposition 6.23], we conclude by topological indecomposability of X that $X \cong \mathbb{Q}_p$.

In the remaining case when $k(X)$ is reduced, we consider an arbitrary compact open subgroup U of $k(X)$. By [6, (25.8)], $U \cong \mathbb{Z}_p^\gamma$ for some cardinal number γ. We first show that $\gamma = 1$. To see this, assume the contrary, and write $U = A \oplus B$, where $A \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and B is a closed subgroup of U. Our first task is to show that A_* is closed in X. This is clear in case $B = \{0\}$, since then $A_* \supset U$. In the other case when $B \neq \{0\}$, define $f \in H(U, D(B))$ by setting $f = \eta_B \circ \pi_B$, where π_B is the canonical projection of U onto B and η_B is the canonical injection of B into $D(B)$. Since $D(B)$ is divisible and U is open in $k(X)$, f extends to a homomorphism $f_0 \in H(k(X), D(B))$. In particular, $\ker(f_0)$ is closed in $k(X)$. Further, since $X/\ker(f_0)$ is algebraically isomorphic to a subgroup of $D(B)$ and $D(B)$ is torsionfree, $X/\ker(f_0)$ is torsionfree too, and hence $\ker(f_0)$ is pure in $k(X)$. It is also clear that $A \subset \ker(f_0)$, so that $A_* \subset \ker(f_0)$. To show the reverse inclusion, pick an arbitrary $y \in \ker(f_0)$. For sufficiently large n, we have $p^n y \in U$, so that $p^n y = a + b$ for some $a \in A$ and $b \in B$. It follows that $b = p^n y - a \in \ker(f_0)$, and hence $b = 0$ because $f_0|B = f$ is injective. Then $p^n y \in A$, so $y \in A_*$, and since $y \in \ker(f_0)$ was arbitrary, $A_* = \ker(f_0)$. Consequently, in both cases A_* is closed in $k(X)$.

We next proceed to show that A_* is topologically decomposable. Let π_A be the canonical projection of U onto A, φ a topological isomorphism of A onto $\mathbb{Z}_p \times \mathbb{Z}_p$, and j the canonical inclusion of $\mathbb{Z}_p \times \mathbb{Z}_p$ into $\mathbb{Q}_p \times \mathbb{Q}_p$. Extend $g = j \circ \varphi \circ \pi_A \in H(U, \mathbb{Q}_p \times \mathbb{Q}_p)$ to a homomorphism $g_0 \in H(k(X), \mathbb{Q}_p \times \mathbb{Q}_p)$. Since $g_0(A) = j(\mathbb{Z}_p \times \mathbb{Z}_p)$ and since $j(\mathbb{Z}_p \times \mathbb{Z}_p)$ is open in $\mathbb{Q}_p \times \mathbb{Q}_p$, g_0 is an open mapping. It is then clear that $g_0|A_*$ establishes a topological isomorphism between A_* and the open subgroup $g_0(A_*)$ of $\mathbb{Q}_p \times \mathbb{Q}_p$. Since $k(X)$ is reduced, we conclude that $g_0(A_*)$ is compact in $\mathbb{Q}_p \times \mathbb{Q}_p$ [2, Ch.VII, §1, Exercise 17(d)], so that $g_0(A_*)$ and hence A_* is topologically isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ [2, Ch.VII, §1, Exercise 17(c)]. This contradiction shows that $\gamma = 1$.

Now, define $h \in H(U, \mathbb{Q}_p)$ by setting $h = i \circ \psi$, where ψ is a topological isomorphism of U onto $\mathbb{Z}_p$ and i is the canonical injection of $\mathbb{Z}_p$ into $\mathbb{Q}_p$. Then h extends to a topological isomorphism of $k(X)$ onto an open subgroup of $\mathbb{Q}_p$. By the reduceness of $k(X)$, we deduce that $k(X) \cong \mathbb{Z}_p$. Finally, we claim that $k(X) = X$. For, otherwise $k(X)$ would be a nontrivial algebraical direct summand of X because $k(X)$ is compact and pure in X [6, (25.21)]. But then $k(X)$ would be a nontrivial topological direct summand of X because $k(X)$ is open in X [1, Corollary 6.8], a contradiction. Thus $X = k(X)$, and hence $X \cong \mathbb{Z}_p$. □

In order to dualize Theorem 3.5, we need some definitions.

Definition 3.6. *Let S be a subset of $\mathbb{P}$. A group $X \in \mathcal{L}$ is called S-torsionfree if $\sum_{p \in S} t_p(X) = \{0\}$. In case $S = \{p\}$, we say X is p-torsionfree.*

Definition 3.7. *Let X be a compact group in $\mathcal{L}$. Given any $p \in \mathbb{P}$, a subgroup A of X is said to be a p-cobasic subgroup of X if A is closed, p-pure and p-torsionfree, and X/A is topologically isomorphic to a group of the form $\mathbb{T}^\alpha \times \prod_{n \in \mathbb{N}_0} \mathbb{Z}(p^n)^{\alpha_n}$, where α and the α_n's are cardinal numbers.*

We have

Corollary 3.8. *Let X be a densely divisible group in $\mathcal{L}$ such that every its quotient by a closed, densely divisible subgroup is topologically indecomposable. Suppose also that if X is compact, then for each $p \in \mathbb{P}$ its quotients by p-cobasic subgroups are topologically isomorphic to $\mathbb{T}$. Then X is topologically isomorphic either to $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Q}^*$, $\mathbb{Q}_p$, or $\mathbb{Z}(p^\infty)$, or to a quotient of $\left(\prod^{loc}_{p \in \mathbb{P}}(j_p; \{0\})\right)^*$ by a closed subgroup. In particular, $E(X)$ is commutative.*

4 Topologically completely decomposable groups and their generalizations

We recall that a type is an isomorphism class of discrete, torsionfree, abelian groups of rank one, and that the set of all types admits a partial ordering:

$\tau_1 \leq \tau_2$ if and only if there exist $A \in \tau_1$ and $B \in \tau_2$ such that $H(A, B) \neq \{0\}$.

L.C.A. van Leeuwen proved the following

Theorem 4.1 ([7], Theorem 3)**.** *A discrete, torsionfree, completely decomposable abelian group $X = \bigoplus_{i\in I} X_i$ has a commutative endomorphism ring if and only if the types of the rank one components X_i are pairwise incomparable.*

Let X be a discrete, torsionfree group in $\mathcal{L}$. If $x \in X$, the type of x, written *type*(x), is the type containing the subgroup $\langle x\rangle_*$ of X. Given a type τ, let $X(\tau)$ denote the subgroup of X consisting of those elements $x \in X$ for which *type*$(x) \geq \tau$, and $X^\natural(\tau)$ the pure subgroup of X generated by all $X(\sigma)$ for $\sigma > \tau$. The set $T_{cr}(X)$ of types τ such that $X(\tau) \neq X^\natural(\tau)$ is called the set of critical types of X.

Definition 4.2. *A discrete, torsionfree group in $\mathcal{L}$ is said to be*

(i) *almost completely decomposable if X is of finite rank and contains a completely decomposable subgroup of finite index;*

(ii) *rigid if for every $\tau \in T_{cr}(X)$, $X(\tau)/X^\natural(\tau)$ is of rank one, and any distinct $\tau_1, \tau_2 \in T_{cr}(X)$ are incomparable.*

The following theorem is due to A. Mader and Ph. Schultz.

Theorem 4.3 ([10], Proposition 3.3)**.** *Let $X \in \mathcal{L}$ be almost completely decomposable. The ring $E(X)$ is commutative if and only if X is rigid.*

This section will be concerned with extending to general LCA groups the results of Theorems 4.1 and 4.3.

We begin with a lemma which will be frequently used in the sequel.

Lemma 4.4. *Let X be a group in $\mathcal{L}$ such that $E(X)$ is commutative. If $X = D \oplus Y$, where D is divisible and $k(Y)$ is open in Y, then $Y = k(Y)$.*

Proof. Assume by way of contradiction that $Y \neq k(Y)$, and pick any $a \in Y \setminus k(Y)$ and any nonzero $b \in D$. Letting $\pi : Y \to Y/k(Y)$ denote the canonical projection, define $f \in H(\langle\pi(a)\rangle, D)$ by setting $f(\pi(a)) = b$. Since D is divisible, f extends to a homomorphism $f_0 \in H(Y/k(Y), D)$ [6, (A.7)]. Then applying [14, Lemma 3.5] with $\omega = 1_X$ and $h = f_0 \circ \pi$, we obtain the required contradiction. Consequently, we must have $Y = k(Y)$. □

Definition 4.5. *A group $X \in \mathcal{L}$ is said to be residual if $d(X) \subset k(X)$ and $c(X) \subset m(X)$.*

The following theorem reduces the study of general LCA groups with commutative rings of continuous endomorphisms to some more special groups.

Theorem 4.6. *Let X be a group in $\mathcal{L}$ such that $E(X)$ is commutative. Then either X is residual or it is topologically isomorphic with one of the following groups:*

(i) $\mathbb{R} \times \prod_{p \in S(X)}(A_p; U_p)$, *where every A_p is a topological p-primary group and U_p is a compact open subgroup of A_p;*

(ii) $\mathbb{Q} \times \prod_{p \in S(X)}(B_p; V_p)$, *where every B_p is a reduced, topological p-primary group and V_p is a compact open subgroup of B_p;*

(iii) $\mathbb{Q}^* \times \prod_{p \in S(X)}(C_p; W_p)$, *where every C_p is a topological p-primary group with $m(C_p) = C_p$ and W_p is a compact open subgroup of C_p.*

Proof. First suppose that $X \notin \mathcal{L}_0$. As is well known, X can then be written in the form $X = D \oplus A$, where D, A are closed subgroups of X such that $D \cong \mathbb{R}^d$ for some $d \in \mathbb{N}_0$ and $A \in \mathcal{L}_0$ [6, (24.30)]. It is clear from [14, Lemma 3.2] that $E(D)$ is commutative, so that we must have $d = 1$. Further, since D is divisible and $k(A)$ is open in A, it follows from Lemma 4.4 that $k(A) = A$. We must also have $c(A) = \{0\}$, since otherwise it would follow from [6, (25.20)] that there exists a nonzero $h \in H(D, A)$, and an application of [14, Lemma 3.5] to h and $w = 1_X$ would provide a contradiction. In conclusion, A is a topological torsion group, and an appeal to [1, Theorem 3.13] shows that in this case X satisfies (i).

Next suppose that $X \in \mathcal{L}_0$. Taking account of [1, Theorem 9.3], we conclude that either X is residual or X is topologically isomorphic either to a group of the form $\mathbb{Q} \times B$ or to a group of the form $\mathbb{Q}^* \times C$, where $B, C \in \mathcal{L}_0$. Assume $X \cong \mathbb{Q} \times B$, and so $c(B)$ is compact. In view of Lemma 4.4, we must have $k(B) = B$. On the other hand, $H(\mathbb{Q}, B) = \{0\}$ by [14, Lemma 3.5]. We claim that $d(B) = \{0\}$. Indeed, if we had $d(B) \neq \{0\}$, then extending any nonzero homomorphism $f : \mathbb{Z} \to d(B)$ to a homomorphism $f_0 : \mathbb{Q} \to d(B)$ [6, (A.7)] and combining f_0 with the canonical inclusion of $d(B)$ into B, we would obtain a contradiction. Thus $d(B) = \{0\}$, and hence $c(B) = \{0\}$ because $c(X) \subset d(X)$ by [6, (24.25)]. Consequently, B is a reduced topological torsion group. Appealing again to [1, Theorem 3.13], we see that in this case X satisfies (ii). Finally, if $X \cong \mathbb{Q}^* \times C$, then $X^* \cong \mathbb{Q} \times C^*$. Since $C^* \in \mathcal{L}_0$ and $E(X^*)$ is commutative, we conclude that X^* satisfies (ii), so that in this case X must satisfy (iii). □

We recall the following definition due to V. Charin.

Definition 4.7. *A topological group X is said to be a group of finite special rank in case there exists a natural number r such that every finite subset F of X topologically generates a subgroup with no more than r topological generators, i.e., $\overline{\langle F \rangle} = \overline{\langle x_1, \ldots, x_k \rangle}$ for some $x_1, \ldots, x_k \in X$ and $k \leq r$. The smallest r with this property is called the special rank of X and is denoted by $\rho_s(X)$. In case no such r exists, X is said to have infinite special rank.*

It is easy to see that for discrete abelian groups the introduced notion of rank coincides with the usual general rank for abstract groups.

Definition 4.8. *Let $S \subset \mathbb{P}$. A group $X \in \mathcal{L}$ is said to be S-divisible if $pX = X$ for all $p \in S$. We say X is densely S-divisible if it contains a dense S-divisible subgroup.*

Definition 4.9. *A group $X \in \mathcal{L}$ is said to be topologically completely decomposable in case X is topologically isomorphic to a group of the form $\prod_{i\in I}\big(X(i);U(i)\big)$, where, for each $i \in I$, $X(i)$ is a group in $\mathcal{L}$ of special rank one and $U(i)$ is a compact open subgroup of $X(i)$.*

We can generalize Theorem 4.1 as follows.

Theorem 4.10. *Let $X \in \mathcal{L}$ be torsionfree and topologically completely decomposable. The ring $E(X)$ is commutative if and only if X is topologically isomorphic with one of the groups $\mathbb{Q}^*$, $\prod_{p\in S}(\mathbb{Q}_p;\mathbb{Z}_p)\times\prod_{p\in S(X)\setminus S}(\mathbb{Z}_p;p^{n_p}\mathbb{Z}_p)$, or $L\times\prod_{p\in S(X)}(\mathbb{Z}_p;p^{n_p}\mathbb{Z}_p)$, where S is a nonempty subset of $S(X)$, the n_p's are natural numbers, and L is a discrete, $S(X)$-divisible, torsionfree, completely decomposable abelian group such that the rank one components of its decomposition have pairwise incomparable types.*

Proof. We may assume that $X = \prod_{i\in I}\big(X(i);U(i)\big)$, where, for each $i \in I$, $X(i)$ is a group in $\mathcal{L}$ of special rank one and $U(i)$ is a compact open subgroup of $X(i)$.

Let $E(X)$ be commutative. By [11, Lemma 2], a group in $\mathcal{L}$ having special rank one is either compact and connected, or else discrete or topologically torsion. Write $I = I_1 \cup I_2 \cup I_3$, where

$$\begin{aligned} & I_1 = \{i \in I \mid X(i) \text{ is compact and connected}\},\\ & I_2 = \{i \in I \mid X(i) \text{ is discrete}\},\\ \text{and}\qquad & I_3 = \{i \in I \mid X(i) \text{ is topologically torsion}\}. \end{aligned}$$

Letting $K = \prod_{i\in I_1}\big(X(i);U(i)\big)$, $L = \prod_{i\in I_2}\big(X(i);U(i)\big)$, and $M = \prod_{i\in I_3}\big(X(i);U(i)\big)$, we have $X = K \oplus L \oplus M$. Now, if $I_1 \neq \varnothing$, then $K \cong (\mathbb{Q}^*)^\alpha$ for some nonzero cardinal number α because X is torsionfree [6, (25.8)]. Since X is then nonresidual, we deduce from Theorem 4.6 that $X \cong \mathbb{Q}^*$. Next suppose $I_1 = \varnothing$, and so $X = L \oplus M$. Clearly, M is topologically torsion, so that by [1, Theorem 3.13]

$$M \cong \prod_{p\in S(M)} (M_p;V_p), \tag{4.1}$$

where, for each $p \in S(M)$, V_p is a compact open subgroup of M_p. It is also clear that $S(M) = \bigcup_{i\in I_3} S\big(X(i)\big)$. Pick any distinct $i, j \in I_3$. We shall show that

$$S\big(X(i)\big) \cap S\big(X(j)\big) = \varnothing.$$

Assume the contrary, and fix any $p \in S\big(X(i)\big) \cap S\big(X(j)\big)$. Since $X(i)$ and $X(j)$ split topologically from M, and since the topological p-primary component of a topological torsion group in $\mathcal{L}$ splits topologically from that group, we can write

$X = L \oplus M' \oplus X(i)_p \oplus X(j)_p$ for some closed subgroup M' of M. It follows from [14, Lemma 3.2] that $E\big(X(i) \oplus X(j)\big)$ is commutative, and hence $H\big(X(i), X(j)\big)$ and $H\big(X(j), X(i)\big)$ must coincide with the zero group by [14, Lemma 3.5]. But, as follows from [4, Corollary, p.92], $X(i)_p$ and $X(j)_p$ are topologically isomorphic with either $\mathbb{Q}_p$ or $\mathbb{Z}_p$. Therefore $H\big(X(i), X(j)\big)$ and $H\big(X(j), X(i)\big)$ cannot be both zero, a contradiction. This proves that $S\big(X(i)\big) \cap S\big(X(j)\big) = \varnothing$, so that for each $p \in S(M)$, $M_p \cong X(i)_p$ for some $i \in I$, and hence either $M_p \cong \mathbb{Z}_p$ or $M_p \cong \mathbb{Q}_p$ [4, Corollary, p.92]. Let

$$S_1 = \{p \in S(M) | M_p \cong \mathbb{Z}_p\} \qquad \text{and} \qquad S_2 = \{p \in S(M) | M_p \cong \mathbb{Q}_p\}.$$

It follows from (4.1) that

$$\begin{aligned} M &\cong \prod_{p \in S_1} (M_p; V_p) \times \prod_{p \in S_2} (M_p; V_p) \\ &\cong \prod_{p \in S_1} (\mathbb{Z}_p; p^{n_p}\mathbb{Z}_p) \times \prod_{p \in S_2} (\mathbb{Q}_p; \mathbb{Z}_p), \end{aligned}$$

where $n_p \in \mathbb{N}$ for all $p \in S_1$. Now, if $S_2 \neq \varnothing$, we conclude from Lemma 4.4 that $L = \{0\}$, so $X \cong \prod_{p \in S_1}(\mathbb{Z}_p; p^{n_p}\mathbb{Z}_p) \times \prod_{p \in S_2}(\mathbb{Q}_p; \mathbb{Z}_p)$. Next suppose $S_2 = \varnothing$, and hence $S_1 = S(X)$. Since, for each $i \in I_2$, $X(i)$ is discrete, it is clear that L is discrete too, so that

$$L = \bigoplus_{i \in I_2} X(i). \tag{4.2}$$

As $E(L)$ is commutative, we conclude from Theorem 4.1 that the components $X(i)$ of decomposition (4.2) must have pairwise incomparable types. Moreover, by [14, Lemma 3.5], we also must have $H(L, M) = \{0\}$, so $H(M^*, L^*) = \{0\}$ because $H(M^*, L^*) \cong H(L, M)$ [12, Ch. II, Theorem 2.8, Corollary 2]. Since $M^* \cong \prod_{p \in S(X)}\big(\mathbb{Z}(p^\infty); \mathbb{Z}(p^\infty)[p^{n_p}]\big)$ [6, (23.33)], and since L^* is connected hence divisible, the equality $H(M^*, L^*) = \{0\}$ can occur only if L^* is $S(X)$-torsionfree, or equivalently L is $S(X)$-divisible [13, (1.5)].

The converse is clear. $\square$

As is well known [10], every discrete, almost completely decomposable group $X \in \mathcal{L}$ contains completely decomposable subgroups of minimal index, called regulating subgroups of X. The intersection of all regulating subgroups of X is a completely decomposable, fully invariant subgroup of X having finite index.

This suggests the following definition.

Definition 4.11. *A group $X \in \mathcal{L}$ is said to be topologically completely decomposable-by-bounded order if X has a closed, topologically fully invariant subgroup A such that A is topologically completely decomposable and X/A is a group of bounded order.*

Our next theorem generalizes Theorem 4.3. Its proof is inspired by that of Theorem 4.3.

Theorem 4.12. *Let $X \in \mathcal{L}$ be torsionfree and topologically completely decomposable-by-bounded order. The following statements are equivalent:*

(i) *$E(X)$ is commutative.*

(ii) *Either X is topologically isomorphic to one of the groups $\mathbb{Q}^*$ or $\prod_{p\in S}(\mathbb{Q}_p;\mathbb{Z}_p)\times\prod_{p\in S(X)\setminus S}(\mathbb{Z}_p;p^{n_p}\mathbb{Z}_p)$, or else X contains a closed, topologically fully invariant subgroup A such that $A\cong L\times\prod_{p\in S(X)}(\mathbb{Z}_p;p^{n_p}\mathbb{Z}_p)$ and X/A is of bounded order, where S is a nonempty subset of $S(X)$, the n_p's are natural numbers, and L is a discrete, $S(X)$-divisible, torsionfree, completely decomposable abelian group whose rank one components have pairwise incomparable types.*

Proof. Assume (i). By hypothesis, there exists a closed, topologically fully invariant subgroup A of X such that A is topologically completely decomposable and X/A is of bounded order. Pick any $u,v\in E(A)$. Letting $\eta\in H(A,X)$ denote the canonical injection of A into X and fixing any $n\in\mathbb{N}_0$ such that $nX\subset A$, we have $\eta\circ u\circ n1_X,\eta\circ v\circ n1_X\in E(X)$, so that

$$(\eta\circ u\circ n1_X)\circ(\eta\circ v\circ n1_X)-(\eta\circ v\circ n1_X)\circ(\eta\circ u\circ n1_X)=0.$$

It follows that for each $x\in A$, $n^2\big[(u\circ v)(x)-(v\circ u)(x)\big]=0$, and hence $(u\circ v)(x)-(v\circ u)(x)=0$ because A is torsionfree. This proves that $E(A)$ is commutative. Consequently, by Theorem 4.10, A is topologically isomorphic to one of the groups

$$\mathbb{Q}^*,\qquad \prod_{p\in S}(\mathbb{Q}_p;\mathbb{Z}_p)\times\prod_{p\in S(X)\setminus S}(\mathbb{Z}_p;p^{n_p}\mathbb{Z}_p),\qquad\text{or}\qquad L\times\prod_{p\in S(X)}(\mathbb{Z}_p;p^{n_p}\mathbb{Z}_p),$$

where S is a nonempty subset of $S(X)$, the n_p's are natural numbers, and L is a discrete, $S(X)$-divisible, completely decomposable abelian group whose rank one components have pairwise incomparable types. Suppose $A\cong\mathbb{Q}^*$. Since $\mathbb{Q}^*$ is splitting in the class of torsionfree groups in $\mathcal{L}$ [1, Proposition 6.20], we have $X\cong A\times\big(X/A\big)$, and hence $X\cong A$ because X is torsionfree and X/A is of bounded order. Next suppose that $A\cong\prod_{p\in S}(\mathbb{Q}_p;\mathbb{Z}_p)\times\prod_{p\in S(X)\setminus S}(\mathbb{Z}_p;p^{n_p}\mathbb{Z}_p)$. Since $S\neq\varnothing$ and X is torsionfree, we can write $X=D\oplus Y$, where $D\cong\mathbb{Q}_p$ for some $p\in S$ [1, Proposition 6.23]. It follows from Lemma 4.4 that $X=k(X)$. Further, since every connected group in $\mathcal{L}$ is divisible [6, (9.14) and (24.25)], we also have

$$c(X)\subset n\cdot c(X)\subset nX\subset A,$$

and hence $c(X)=\{0\}$ because A is totally disconnected. We claim that $S(X)=S(A)$. Indeed, since A is a closed subgroup of X, we clearly have $S(A)\subset S(X)$. In order to establish the reverse inclusion, pick any $s\in S(X)$ and any nonzero $x\in X_s$. Then $\lim_{k\to\infty}s^kx=0$, so $\lim_{k\to\infty}s^k(nx)=0$, and hence nx is a nonzero element of A_s. This proves that $s\in S(A)$, so $S(X)=S(A)$. Now, let $p\in S$, so that $A_p\cong\mathbb{Q}_p$. Since X_p splits topologically from X [1, Theorem 3.13], $E(X_p)$ is commutative by [14, Lemma 3.2]. Since X_p is nonreduced, we deduce from [14, Theorem 5.12] that in this case $X_p \cong \mathbb{Q}_p$. Further, let $p \in S(X) \setminus S$, so that $A_p \cong \mathbb{Z}_p$. Pick a compact open subgroup U of X. We clearly have $nU_p \subset A_p$. Since X is torsionfree, $n1_X|U_p$ establishes a topological isomorphism between U_p and a closed and hence open subgroup V_p of A_p. Letting $j : V_p \to D(A_p)$ denote the canonical inclusion, extend $j \circ n1_X|U_p$ to a homomorphism $f \in H(X_p, \mathbb{Q}_p)$. It is easy to see that f is a topological isomorphism from X_p onto an open subgroup of $\mathbb{Q}_p$. We cannot have $f(X_p) = \mathbb{Q}_p$, because then X/A could not be of bounded order. Thus $X_p \cong \mathbb{Z}_p$, and hence

$$X \cong \prod_{p\in S}(\mathbb{Q}_p; \mathbb{Z}_p) \times \prod_{p\in S(X)\setminus S}(\mathbb{Z}_p; p^{m_p}\mathbb{Z}_p),$$

where $m_p \in \mathbb{N}_0$ for all $p \in S(X) \setminus S$. Consequently, (i) implies (ii).

Assume (ii). In view of Theorem 4.10, we need only to consider the case when X has a closed, topologically fully invariant subgroup A such that $A \cong L \times \prod_{p\in S(X)}(\mathbb{Z}_p; p^{n_p}\mathbb{Z}_p)$ and X/A is of bounded order. Now, since A is topologically fully invariant in X, the restriction map $\rho : E(X) \to E(A)$ is well defined. It is clear that ρ is a ring homomorphism. Fix any $n \in \mathbb{N}_0$ such that $nX \subset A$. If $f \in \ker(\rho)$, we have

$$nf(X) = f(nX) \subset f(A) = \{0\},$$

so $f = 0$, and hence $E(X)$ is isomorphic to a subring of $E(A)$. Since $E(A)$ is commutative by Theorem 4.10, $E(X)$ is commutative too. Hence (ii) implies (i). □

In order to dualize Theorem 4.12, we need the notion of pattern due to P. Loth.

Definition 4.13. *Two compact connected abelian groups of dimension one are said to be equivalent if they are topologically isomorphic, and the equivalence classes are called patterns.*

We see from [6, (24.28)] and [11, Lemma 3] that a pattern is an equivalence class of topologically isomorphic, compact, connected, abelian groups of special rank one. By duality theory, if τ is a type, then $\tau^* = \{X^*|X \in \tau\}$ is a pattern, and if π is a pattern, then $\pi^* = \{X^*|X \in \pi\}$ is a type. Thus, there is a one-to-one correspondence between the set of types and the set of patterns. In particular, the set of patterns can be ordered by setting: $\pi_1 \leq \pi_2$ if and only if $\pi_1^* \geq \pi_2^*$.

Definition 4.14. *A group $X \in \mathcal{L}$ is said to be bounded order-by-topologically completely decomposable if X has a closed, topologically fully invariant subgroup A such that A is of bounded order and X/A is topologically completely decomposable.*

Corollary 4.15. *Let $X \in \mathcal{L}$ be densely divisible and bounded order-by-topologically completely decomposable. The following statements are equivalent:*

(i) *$E(X)$ is commutative.*

(ii) *Either X is topologically isomorphic to one of the groups $\mathbb{Q}$ or $\prod_{p\in S}(\mathbb{Q}_p; \mathbb{Z}_p) \times \prod_{p\in S(X)\setminus S}(\mathbb{Z}(p^\infty); \mathbb{Z}(p^\infty)[p^{n_p}])$, or else X contains a closed, topologically fully invariant subgroup of bounded order, A, such that*

$$X/A \cong K \times \prod_{p\in S(X)}(\mathbb{Z}(p^\infty); \mathbb{Z}(p^\infty)[p^{n_p}]),$$

where S is a nonempty subset of $S(X)$, the n_p's are natural numbers, and K is a compact, connected, topologically completely decomposable group whose components have pairwise incomparable patterns.

5 Torsionfree groups of finite rank

Motivated by a result of L. C. A. van Leeuwen, we examine here the torsionfree LCA groups of finite special rank having commutative rings of continuous endomorphisms.

Definition 5.1. *A discrete, torsionfree group $X \in \mathcal{L}$ is said to be strongly indecomposable if for any subgroups A, B of X such that $A \cap B = \{0\}$ and $X/(A \oplus B)$ is of bounded order, it follows that either A or B coincides with the zero group.*

L. C. A. van Leeuwen proved

Theorem 5.2 ([8], Theorem 3)**.** *Let $X \in \mathcal{L}$ be a discrete, irreducible, torsionfree group of finite rank ρ such that ρ is square free. Then $E(X)$ is commutative if and only if X is strongly indecomposable.*

The following theorem gives a complete description of topological torsion LCA groups of finite special rank having commutative rings of continuous endomorphisms.

Theorem 5.3. *Let X be a topological torsion group in $\mathcal{L}$. The following statements are equivalent:*

(i) *X is of finite special rank, and $E(X)$ is commutative.*

(ii) *For each $p \in \mathcal{S}(X)$, X_p is topologically isomorphic with one of the groups $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$, $\mathbb{Z}(p^{n_p})$, $\mathbb{Q}_p$, or $\mathbb{Q}_p \times \mathbb{Z}(p^{n_p})$, where $n_p \in \mathbb{N}_0$.*

Proof. Assume (i), and pick any $p \in S(X)$. Since X is of finite special rank, we conclude from the very definition that X_p has finite special rank too. It follows from [4, Theorem 5] that X_p can be written in the form $X_p = G_1 \oplus \cdots \oplus G_{r_p}$, where every G_i, $1 \le i \le r_p$, is topologically isomorphic with one of the groups $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$, or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0$. Now, since in view of [14, Lemma 3.5] we must have

$$H(G_i, G_1 \oplus \cdots \oplus G_{i-1} \oplus G_{i+1} \oplus \cdots \oplus G_{r_p}) = \{0\}$$

for all $i \in \{1, \ldots, r_p\}$, it is clear that X_p must be topologically isomorphic to one of the groups $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$, $\mathbb{Z}(p^{n_p})$, $\mathbb{Q}_p$, or $\mathbb{Q}_p \times \mathbb{Z}(p^{n_p})$, where $n_p \in \mathbb{N}_0$. Thus (ii) holds.

Conversely, if X satisfies (ii), then for each $p \in S(X)$ the special rank of X_p is at most two, and so X is a group of finite special rank by [4, Theorem 6]. □

In the following theorem, we obtain the description of nonresidual LCA groups of finite special rank with a commutative ring of continuous endomorphisms.

Theorem 5.4. *Let $X \in \mathcal{L}$ be a nonresidual group of finite special rank. The ring $E(X)$ is commutative if and only if X is topologically isomorphic with one of the groups:*

(i) $\mathbb{R}\times\prod_{p\in S(X)}(A_p;U_p)$, *where, for each* $p\in S(X), A_p$ *is topologically isomorphic with either* $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$, $\mathbb{Z}(p^{n_p})$, $\mathbb{Q}_p$, *or* $\mathbb{Q}_p\oplus\mathbb{Z}(p^{n_p})$ *for some* $n_p\in\mathbb{N}_0$, *and* U_p *is a compact open subgroup of* A_p;

(ii) $\mathbb{Q}\times\prod_{p\in S(X)}(B_p;V_p)$, *where, for each* $p\in S(X), B_p$ *is topologically isomorphic with either* $\mathbb{Z}_p$ *or* $\mathbb{Z}(p^{n_p})$ *for some* $n_p\in\mathbb{N}_0$, *and* V_p *is a compact open subgroup of* B_p;

(iii) $\mathbb{Q}^*\times\prod_{p\in S(X)}(C_p;W_p)$, *where, for each* $p\in S(X), C_p$ *is topologically isomorphic with either* $\mathbb{Z}(p^\infty)$ *or* $\mathbb{Z}(p^{n_p})$ *for some* $n_p\in\mathbb{N}_0$, *and* W_p *is a compact open subgroup of* C_p.

Proof. Assume $E(X)$ is commutative. By Theorem 4.6, we can write $X=D\oplus Y$, where D and Y are closed subgroups of X for which exactly one of the following situations can happen (we use the notation of Theorem 4.6):

$$\begin{array}{llll} & (a) & D\cong\mathbb{R}, & Y\cong\displaystyle\prod_{p\in S(X)}(A_p;U_p);\\ & (b) & D\cong\mathbb{Q}, & Y\cong\displaystyle\prod_{p\in S(X)}(B_p;V_p);\\ \text{and} & (c) & D\cong\mathbb{Q}^*, & Y\cong\displaystyle\prod_{p\in S(X)}(C_p;W_p).\end{array}$$

Clearly, in each of these cases Y is a topological torsion group having a commutative ring $E(Y)$ [14, Lemma 3.2]. Other properties of Y are described in Theorem 4.6. Now, taking account of Theorem 5.3 and Theorem 4.6, we see that (a) gives us (i), (b) gives us (ii) because every B_p is reduced, and (c) gives us (iii) because $m(C_p)=C_p$ for all $p\in S(X)$.

The converse is clear. $\square$

In the following definition, we introduce topologically irreducible and weakly topologically irreducible groups as generalizations to LCA groups of usual discrete, irreducible, abelian groups. For discrete groups, all these notions coincide.

Definition 5.5. *A group* $X\in\mathcal{L}$ *is said to be (weakly) topologically irreducible in case* X *contains no nontrivial, closed (respectively, open), pure, topologically fully invariant subgroups.*

We shall now restrict attention to (weakly) topologically irreducible groups in $\mathcal{L}$. We have

Theorem 5.6. *Let* $X\in\mathcal{L}_0$ *be a torsionfree group of finite special rank* ρ *such that* ρ *is square free and either* $c(X)\neq\{0\}$ *or* $k(X)\neq X$. *The following statements are equivalent:*

(i) X *is topologically irreducible, and* $E(X)$ *is commutative.*

(ii) X *is weakly topologically irreducible, and* $E(X)$ *is commutative.*

(iii) *Either* $X \cong \mathbb{Q}^*$ *or else* X *is discrete, irreducible, and strongly indecomposable.*

Proof. Clearly, (i) implies (ii).

Assume (ii). First we note that since $X \in \mathcal{L}_0$, $c(X)$ is compact and $k(X)$ is open in X. Now, since $k(X)$ is pure in X and X is weakly topologically irreducible, it follows that either $k(X) = \{0\}$ or $k(X) = X$. In the former case, X is discrete, and hence X has to be strongly indecomposable by Theorem 5.2. In the latter case, it follows from hypothesis that $c(X) \neq \{0\}$. We assert that in fact $c(X) = X$. Suppose the contrary. Since X is torsionfree, $c(X)$ splits topologically from X [1, Proposition 6.13], and hence we can write $X = c(X) \oplus D$ for some nonzero, closed subgroup D of X. Pick any $p \in S(D)$, and write $D = D_p \oplus D_p^{\#}$, where $D_p^{\#} = \overline{\sum_{s\in S(D)\setminus\{p\}} D_s}$ [1, Theorem 3.13)]. Fixing an arbitrary compact open subgroup U of D_p, we have $U \cong \mathbb{Z}_p^{\nu}$ for some nonzero cardinal number ν [6, (25.8)], so that we can choose a nonzero $f \in H(U, \mathbb{Z}_p)$. Further, since the topological p-primary component of $c(X)$ is nonzero (being dense in $c(X)$ [1, Corollary 4.18(a)]), we can choose also a nonzero $g \in H(\mathbb{Z}_p, c(X))$. Now, since $c(X)$ is divisible [6, (24.25)], $g \circ f$ extends to a continuous homomorphism $h : D_p \to c(X)$. It follows that the canonical projection of D onto D_p with kernel $D_p^{\#}$ followed by h is a nonzero element of $H(D, c(X))$. But then [14, Lemma 3.5] applies, producing a contradiction to $E(X)$ being commutative. This proves that $X = c(X)$. As every compact, connected, torsionfree group in $\mathcal{L}$ is topologically isomorphic to $(\mathbb{Q}^*)^{\mu}$ for some cardinal number μ [6, (25.8)], we deduce that $X \cong \mathbb{Q}^*$. Hence (ii) implies (iii).

Finally, (iii) implies (i) by Theorem 5.2 and the fact that $\mathbb{Q}^*$ is a pure-simple [1, Proposition 7.11] group of special rank one with a commutative ring $E(\mathbb{Q}^*)$. □

For residual, weakly topologically irreducible groups, we have

Theorem 5.7. *Let* $X \in \mathcal{L}$ *be a residual, torsionfree, weakly topologically irreducible group of finite special rank* ρ *such that* ρ *is squarefree. The ring* $E(X)$ *is commutative if and only if either* X *is discrete and strongly indecomposable or* X *is a topological torsion group such that, for each* $p \in S(X)$, X_p *is topologically isomorphic with either* $\mathbb{Q}_p$ *or* $\mathbb{Z}_p$.

Proof. Assume $E(X)$ is commutative. Since X is residual and torsionfree, $c(X) = \{0\}$, and so $k(X)$ is open in X. Now, since $k(X)$ is pure in X, it follows that either $k(X) = \{0\}$ or $k(X) = X$. In the former case, X is discrete, and hence X has to be strongly indecomposable by Theorem 5.2. In the latter case, X is a topological torsion group, and we see from Theorem 5.3 that for each $p \in S(X)$, X_p is topologically isomorphic with either $\mathbb{Q}_p$ or $\mathbb{Z}_p$.

The converse follows from Theorem 5.2 in case X is discrete, and from [1, Proposition 7.11], [4, Corollary, p. 92] and Theorem 5.3 in case X is topologically torsion. □

We continue with the case of residual, topologically irreducible groups.

Corollary 5.8. *Let* $X \in \mathcal{L}$ *be a residual, torsionfree, topologically irreducible group of finite special rank* ρ *such that* ρ *is squarefree. The ring* $E(X)$ *is commutative if* *and only if either X is discrete and strongly indecomposable or X is topologically isomorphic with one of the groups $\mathbb{Q}_p$ or $\mathbb{Z}_p$ for some $p \in \mathbb{P}$.*

Proof. The fact that (i) implies (ii) follows from Theorem 5.7 by observing that in a topological torsion group the topological p-primary components are closed pure subgroups.

The converse is clear. □

We conclude this section by stating the dual analogues of Theorems 5.6 and 5.7 and Corollary 5.8. First, we introduce a few definitions.

Definition 5.9. *Let $X \in \mathcal{L}$. A closed subgroup C of X is said to satisfy PHH (for "property of Hartman and Hulanicki"), if whenever $\gamma \in C^*$ has order n there exists $\gamma' \in X^*$ such that γ' extends γ and also has order n.*

Definition 5.10. *A group $X \in \mathcal{L}$ is said to be (weakly) topologically coirreducible in case X contains no nontrivial, closed (respectively, compact), topologically fully invariant subgroups satisfying PHH.*

Definition 5.11. *A topological group X is said to be strictly indecomposable if for any two closed subgroups A, B of X such that $A \cap B$ is of bounded order and $A + B = X$, it follows that either A or B is equal to X.*

Corollary 5.12. *Let $X \in \mathcal{L}_0$ be a densely divisible, residual group of finite special rank ρ such that ρ is square free and either $c(X) \neq \{0\}$ or $k(X) \neq X$. The following statements are equivalent:*

(i) *X is topologically coirreducible, and $E(X)$ is commutative.*

(ii) *X is weakly topologically coirreducible, and $E(X)$ is commutative.*

(iii) *Either $X \cong \mathbb{Q}$ or else X is compact, topologically coirreducible, and strictly indecomposable.*

Proof. By [4, Theorem 4], X has finite special rank if and only if X^* does, but, as is shown in [11, Theorem, p. 171], the equality $\rho_s(X) = \rho_s(X^*)$ takes place if and only if none of the groups X or X^* is topologically isomorphic to a group of the form $\mathbb{R}^d \times F$, where $d \in \mathbb{N}_0$ and F is a nonzero, discrete, torsionfree group in $\mathcal{L}$. In our case X is residual, so that neither X nor X^* contains copies of $\mathbb{R}$, and hence $\rho_s(X) = \rho_s(X^*)$. Obviously, X is (weakly) topologically coirreducible if and only if X^* is (weakly) topologically irreducible. It is also clear that X is compact and strictly indecomposable if and only if X^* is discrete and strongly indecomposable. □

Corollary 5.13. *Let $X \in \mathcal{L}$ be a residual, densely divisible, weakly topologically coirreducible group of finite special rank ρ such that ρ is squarefree. The ring $E(X)$ is commutative if and only if either X is compact and strictly indecomposable or X is a topological torsion group such that, for each $p \in S(X)$, X_p is topologically isomorphic with either $\mathbb{Q}_p$ or $\mathbb{Z}(p^\infty)$.*

Corollary 5.14. *Let $X \in \mathcal{L}$ be a residual, densely divisible, topologically coirreducible group of finite special rank ρ such that ρ is squarefree. The ring $E(X)$ is commutative if and only if either X is compact and strictly indecomposable or X is topologically isomorphic with one of the groups $\mathbb{Q}_p$ or $\mathbb{Z}(p^\infty)$ for some $p \in \mathbb{P}$.*

6 Some splitting cases

In [17], T. Szele and J. Szendrei proved among others the following result:

Theorem 6.1 ([17], Theorem 3)**.** *Let X be a discrete, mixed group in $\mathcal{L}$ such that $X = t(X) \oplus W$ for some subgroup W of X. Then $E(X)$ is commutative if and only if the following conditions hold:*

(i) $t(X) \cong \bigoplus_{p \in S(X)} \mathbb{Z}(p^{n_p})$, *where $n_p \in \mathbb{N}_0$ for all $p \in S(X)$;*

(ii) *W is $S(X)$-divisible;*

(iii) *$E(W)$ is commutative.*

Our aim here is to extend this theorem to general LCA groups. By utilizing duality, we will also obtain information about LCA groups with splitting subgroup of elements of infinite topological height, which have commutative rings of continuous endomorphisms.

In the case of compact groups, we have

Corollary 6.2. *Let X be a compact group in $\mathcal{L}$ such that $X \neq c(X) \neq \{0\}$ and $X = c(X) \oplus M$ for some closed subgroup M of X. The ring $E(X)$ is commutative if and only if the following conditions hold:*

(i) $M \cong \prod_{p \in S(X)} \mathbb{Z}(p^{n_p})$, *where $n_p \in \mathbb{N}_0$ for all $p \in S(X)$;*

(ii) *$c(X)$ is $S(X)$-torsionfree;*

(iii) *$E(c(X))$ is commutative.*

Proof. It is clear that X^* is a discrete, mixed group in $\mathcal{L}$ satisfying $X^* = t(X^*) \oplus W$, where $W = A(X^*; M)$. We also have $W^* \cong X/A(X; W) = X/M \cong c(X)$. Letting $p \in S(X)$, it then follows that $c(X)[p] \cong W^*[p] \cong A(W; pW)$. The assertion follows from Theorem 6.1 and duality. □

In extending Theorem 6.1 to general LCA groups, we distinguish two cases. For residual LCA groups, we have

Theorem 6.3. *Let X be a mixed, residual group in $\mathcal{L}$ such that $t(X)$ splits topologically from X. The following statements are equivalent:*

(i) *The ring $E(X)$ is commutative.*

(ii) *X is topologically isomorphic with one of the groups:*

1) $\bigoplus_{p\in S_1}\mathbb{Z}(p^{n_p})\times\bigoplus_{p\in S_2}\mathbb{Z}(p^\infty)\times\prod_{p\in S_3}(\mathbb{Q}_p;\mathbb{Z}_p)\times G$, *where* S_1, S_2 *and* S_3 *are subsets of* $\mathbb{P}$ *satisfying* $S_1\cup S_2\neq\varnothing$ *and* $(S_1\cup S_3)\cap S_2=\varnothing$, $n_p\in\mathbb{N}_0$ *for all* $p\in S_1$, *and* G *is a densely* $(S_1\cup S_2)$*-divisible, reduced, torsionfree, topological torsion group in* $\mathcal{L}$ *with* $S(G)\cap(S_2\cup S_3)=\varnothing$ *and having a commutative ring* $E(G)$.

2) $\bigoplus_{p\in S}\mathbb{Z}(p^{n_p})\times G$, *where* S *is a nonempty subset of* $\mathbb{P}$, $n_p\in\mathbb{N}_0$ *for all* $p\in S$, *and* G *is a densely* S*-divisible, reduced, torsionfree, totally disconnected group in* $\mathcal{L}$ *such that* $k(G)\neq G$, $S(k(G))\cap S=\varnothing$, *and* $E(G)$ *is commutative.*

Proof. Assume (i). By hypothesis, $t(X)$ splits topologically from X, and hence we can write $X=t(X)\oplus Y$ for some subgroup Y of X. It follows that $t(X)$ is closed in X, so that $m(X)=t(X)$. Since X is residual, we conclude that $c(X)\subset t(X)$, which implies $c(X)=\{0\}$, because $t(X)$ is totally disconnected [6, (24.21)]. It is also clear from [14, Lemma 3.2] that $E(t(X))$ and $E(Y)$ are commutative. In particular, we deduce from [14, Corollary 5.7] that $t(X)\cong\bigoplus_{p\in S_1}\mathbb{Z}(p^{n_p})\times\bigoplus_{p\in S_2}\mathbb{Z}(p^\infty)$, where S_1,S_2 are disjoint subsets of $\mathbb{P}$ and $n_p\in\mathbb{N}_0$ for all $p\in S_1$.

Our first purpose is to show that $k(Y)$ is densely $(S_1\cup S_2)$-divisible. Pick any $p_0\in S_1\cup S_2$. We shall first show that $\overline{p_0Y}=Y$. Assume the contrary, and let φ denote the canonical projection of Y onto $Y/\overline{p_0Y}$. Taking account of [3, Ch. II, §4, Théoréme 2], we can write $Y/\overline{p_0Y}=A\oplus B$, where $A\cong\mathbb{Z}(p_0)$. Let π be the canonical projection of $Y/\overline{p_0Y}$ onto A with kernel B. Choose two arbitrary nonzero elements $a\in A$ and c in the socle of $t_{p_0}(X)$, and define $f\in H(A,t(X))$ by setting $f(a)=c$. Then $f\circ\pi\circ\varphi$ is a nonzero element of $H(Y,t(X))$, and applying [14, Lemma 3.5] with $w=1_X$ and $h=f\circ\pi\circ\varphi$ we obtain a contradiction. Consequently, we must have $\overline{p_0Y}=Y$.

Next we show that $\overline{p_0k(Y)}=k(Y)$. Fix any nonzero $y\in k(Y)$ and any neighborhood V of zero in X. Since X is totally disconnected, we can choose a compact open subgroup W of X such that $y\notin W$ and $W\subset V$ [6, (7.7)]. Further, since $\overline{p_0Y}=Y$, there exists $z\in Y$ such that $p_0z-y\in W$. It follows that $p_0z\neq0$ and $p_0z\in y+W\subset k(Y)$, so that $\overline{\langle p_0z\rangle}$ is compact and nonzero, whence $z\in k(Y)$ [6, (9.1)]. This proves that $\overline{p_0k(Y)}=k(Y)$.

Next we show that $\overline{p_0k_{p_0}(Y)}=k_{p_0}(Y)$. In view of [1, Theorem 3.13], we can write $k(Y)=k_{p_0}(Y)\oplus k_{p_0}(Y)^\#$, where $k_{p_0}(Y)^\#=\overline{\sum_{s\in S(Y)\setminus\{p_0\}}k_s(Y)}$. Since for each $s\in S(k(Y))\setminus\{p_0\}$, we have $\overline{p_0k_s(Y)}=k_s(Y)$ [1, (2.16)], it is clear that $k_s(Y)\subset\overline{p_0k_{p_0}(Y)^\#}$. It follows that $k_{p_0}(Y)^\#\subset\overline{p_0k_{p_0}(Y)^\#}$, and hence $k_{p_0}(Y)^\#=\overline{p_0k_{p_0}(Y)^\#}$. Observe also that, by [15, Lemma 4.4], $\overline{p_0k_{p_0}(Y)}+k_{p_0}(Y)^\#$ is closed in Y. We have

$$\begin{aligned}k_{p_0}(Y)\subset k(Y)&=\overline{p_0k(Y)}\\&\subset\overline{p_0k_{p_0}(Y)+pk_{p_0}(Y)^\#}\\&\subset\overline{p_0k_{p_0}(Y)}+k_{p_0}(Y)^\#.\end{aligned}$$

It follows by modular law that

$$\begin{aligned} k_{p_0}(Y) &= k_{p_0}(Y) \cap \left(\overline{p_0 k_{p_0}(Y)} + k_{p_0}(Y)^{\#}\right) \\ &= \overline{p_0 k_{p_0}(Y)} + k_{p_0}(Y) \cap k_{p_0}(Y)^{\#}. \end{aligned}$$

But $k_{p_0}(Y) \cap k_{p_0}(Y)^{\#} = \{0\}$, so that $\overline{p_0 k_{p_0}(Y)} = k_{p_0}(Y)$.

Now we show that $k_{p_0}(Y)^*$ is torsionfree, and hence $k_{p_0}(Y)$ is densely divisible by [1, Theorem 4.15]. Indeed, $k_{p_0}(Y)^*$ is a topological p_0-primary group. Since

$$k_{p_0}(Y)^*[p_0] \cong \left(k_{p_0}(Y)/\overline{p_0 k_{p_0}(Y)}\right)^*$$

[6, (23.25) and (24.22)], it follows that $k_{p_0}(Y)^*[p] = \{0\}$, so $k_{p_0}(Y)^*$ is torsionfree. As $p_0 \in S_1 \cup S_2$ was arbitrary, we deduce that $k(Y)^*$ is $(S_1 \cup S_2)$-torsionfree, and hence $k(Y)$ is densely $(S_1 \cup S_2)$-divisible [13, (1.5)].

In the following we distinguish two cases in regard to the group Y. First let us suppose that $k(Y) = Y$. By [1, Theorem 3.13], we have $Y \cong \prod_{p \in S(Y)}(Y_p; U_p)$, where, for each $p \in S(Y)$, U_p is a compact open subgroup of Y_p. Letting

$$S_3 = \left\{s \in S(Y) \mid d(Y_s) \neq \{0\}\right\},$$

we can write $Y = D \oplus G$, where $D \cong \prod_{p \in S_3}(Y_p; U_p)$ and $G \cong \prod_{p \in S(Y) \setminus S_3}(Y_p; U_p)$ [3, p. 7]. Clearly, G is reduced, and densely $(S_1 \cup S_2)$-divisible as a continuous homomorphic image of a $(S_1 \cup S_2)$-divisible group. Moreover, $E(D)$ and $E(G)$ are commutative. It follows from [14, Corollary 5.13] that $D \cong \prod_{p \in S_3}(\mathbb{Q}_p; \mathbb{Z}_p)$. Since $\mathbb{Z}(p^\infty)$ is a continuous homomorphic image of $\mathbb{Q}_p$, it is clear from [14, Lemma 3.5] that S_2 and S_3 must be disjoint. Suppose there exists $q \in S_2 \cap S(G)$. Since $U_q \cong \mathbb{Z}_q^\nu$ for some nonzero cardinal number ν, there is a nonzero $f \in H(U_q, \mathbb{Z}(q^\infty))$. Extend f to a homomorphism $\hat{f} \in H(Y_q, \mathbb{Z}(q^\infty))$. Choosing any nonzero $\eta_q \in H(\mathbb{Z}(q^\infty), t(X))$ and letting $\pi_q \in H(Y, Y_q)$ denote the canonical projection, we see that $\eta_q \circ \hat{f} \circ \pi_q$ is a nonzero element of $H(Y, t(X))$, in contradiction with the commutativity of $E(X)$. Thus, in this case X is topologically isomorphic to a group described in 1).

Next we consider the other case when $k(Y) \neq Y$. We assert that X is then reduced. Indeed, it is clear from Lemma 4.4 that $S_2 = \varnothing$. Therefore, if X were nonreduced, it would follow that $d(Y) \neq \{0\}$. Since Y is clearly residual, and hence $d(Y) \subset k(Y)$, this would imply that $d(k_s(Y)) \neq \{0\}$ for some $s \in S(k(Y))$. By [1, Proposition 4.22)], $k_s(Y)$ would then contain a closed subgroup $K \cong \mathbb{Q}_s$. As $\mathbb{Q}_s$ is splitting in the class of torsionfree groups in $\mathcal{L}$, we could write $Y = K \oplus L$ for some closed subgroup L of Y. It would then follow from Lemma 4.4 that $k(L) = L$, and hence $k(Y) = Y$, a contradiction. Thus X must be reduced. In particular, we must have $k_s(Y) = \{0\}$ for all $s \in S_1$ because, as we saw above, every such $k_s(Y)$ is densely divisible. Putting $S = S_1$ and $G = Y$, we see that X is topologically isomorphic to a group described in 2). Consequently, (i) implies (ii).

The converse follows from [14, Lemma 3.4], [17, Theorem 1] and [14, Corollary 5.13]. □

In case of nonresidual LCA groups, we have

Theorem 6.4. *Let X be a mixed, nonresidual group in $\mathcal{L}$ such that $t(X)$ splits topologically from X. The following statements are equivalent:*

(i) *The ring $E(X)$ is commutative.*

(ii) *X is topologically isomorphic with one of the groups:*

1) $\bigoplus_{p\in S_1}\mathbb{Z}(p^{n_p})\times\bigoplus_{p\in S_2}\mathbb{Z}(p^{\infty})\times\prod_{p\in S_3}(\mathbb{Q}_p;\mathbb{Z}_p)\times\mathbb{R}\times G$, *where S_1, S_2 and S_3 are subsets of $\mathbb{P}$ satisfying $S_1\cup S_2\neq\varnothing$ and $(S_1\cup S_3)\cap S_2=\varnothing$, $n_p\in\mathbb{N}_0$ for all $p\in S_1$, and G is a densely $(S_1\cup S_2)$-divisible, reduced, torsionfree, topological torsion group in $\mathcal{L}$ with $S(G)\cap(S_2\cup S_3)=\varnothing$ and having a commutative ring $E(G)$.*

2) $\bigoplus_{p\in S}\mathbb{Z}(p^{n_p})\times\mathbb{Q}\times G$, *where S is a nonempty subset of $\mathbb{P}$, $n_p\in\mathbb{N}_0$ for all $p\in S$, and G is a densely S-divisible, reduced, torsionfree, topological torsion group in $\mathcal{L}$ with a commutative ring $E(G)$.*

3) $\bigoplus_{p\in S_1}\mathbb{Z}(p^{n_p})\times\bigoplus_{p\in S_2}\mathbb{Z}(p^{\infty})\times\mathbb{Q}^*$, *where S_1, S_2 are disjoint subsets of $\mathbb{P}$ such that $S_1\cup S_2\neq\varnothing$, and $n_p\in\mathbb{N}_0$ for all $p\in S_1$.*

Proof. Assume (i). By Theorem 4.6, we can write $X=D\oplus Y$, where D and Y are closed subgroups of X for which exactly one of the following situations can happen (with notation as in Theorem 4.6):

$$
\begin{aligned}
&(a)\quad D\cong\mathbb{R},\qquad Y\cong\prod_{p\in S(X)}(A_p;U_p);\\
&(b)\quad D\cong\mathbb{Q},\qquad Y\cong\prod_{p\in S(X)}(B_p;V_p);\\
\text{and}\quad &(c)\quad D\cong\mathbb{Q}^*,\qquad Y\cong\prod_{p\in S(X)}(C_p;W_p).
\end{aligned}
$$

First note that since in each of these cases Y is a topological torsion group (see Theorem 4.6), it is residual. Next observe that $t(X)\subset Y$. Indeed, given any $x\in t(X)$, we can write $x=x_D+x_Y$ for some $x_D\in D$ and $x_Y\in Y$. Letting $n=o(x)$, we have

$$nx_D=-nx_Y\in D\cap Y=\{0\},$$

so that $x_D=0$ because D is torsionfree. Therefore $x\in Y$, and hence $t(X)\subset Y$.

Next we show that $t(X)$ splits topologically from Y. To this end, write $X=t(X)\oplus M$ for some closed subgroup M of X, and let $\pi\in H(X,t(X))$ denote the canonical projection of X onto $t(X)$. We then have $Y=t(X)+(Y\cap M)$ by the modular law, and $t(X)\cap(Y\cap M)=\{0\}$, so that $Y=t(X)\dotplus(Y\cap M)$. If φ is the canonical projection of Y onto $t(X)$ corresponding to this decomposition, then φ coincides with the restriction of π to Y, so φ is continuous, and hence $Y=t(X)\oplus(Y\cap M)$ [2, Ch. III, §6, Proposition 2]. In conclusion, in any of cases (a), (b) or (c), Y is a mixed, residual group in $\mathcal{L}$ such that $t(Y)$ splits topologically from Y and $E(Y)$ is commutative. It follows that its structure is given by 1) or 2) of Theorem 6.3. But, as in addition Y is a topological torsion group, it is clear that Y cannot be topologically isomorphic to a group described in 2) of Theorem 6.3, and hence Y is topologically isomorphic to a group described in 1) of Theorem 6.3. Taking account of this, in case (a) we are led to 1). Further, in case (b) we have 2), because Y must be reduced (see (ii) of Theorem 4.6). Finally, in case (c) we are led to 3) because, for each $p \in S(Y)$, the topological p-primary component Y_p of Y must satisfy $m(Y_p) = Y_p$ (see (iii) of Theorem 4.6). Thus (i) implies (ii).

The converse is clear. □

Definition 6.5. *A group $X \in \mathcal{L}$ is said to be comixed if $\{0\} \neq X_\omega \neq X$.*

By use of duality, we obtain the following two corollaries:

Corollary 6.6. *Let X be a comixed, residual group in $\mathcal{L}$ such that X/X_ω is compact-by-bounded order. The following statements are equivalent:*

(i) *X_ω splits topologically from X, and $E(X)$ is commutative.*

(ii) *X is topologically isomorphic with one of the groups:*

1) $\prod_{p\in S_1} \mathbb{Z}(p^{n_p}) \times \prod_{p\in S_2} \mathbb{Z}_p \times \prod_{p\in S_3} (\mathbb{Q}_p;\mathbb{Z}_p) \times G$, *where S_1, S_2 and S_3 are subsets of $\mathbb{P}$ satisfying $S_1 \cup S_2 \neq \varnothing$ and $(S_1 \cup S_3) \cap S_2 = \varnothing$, $n_p \in \mathbb{N}_0$ for all $p \in S_1$, and G is a densely divisible, $(S_1 \cup S_2)$-torsionfree, topological torsion group in $\mathcal{L}$ such that $m(G) = G$, $S(G) \cap (S_2 \cup S_3) = \varnothing$, and $E(G)$ is commutative.*

2) $\prod_{p\in S} \mathbb{Z}(p^{n_p}) \times G$, *where S is a nonempty subset of $\mathbb{P}$, $n_p \in \mathbb{N}_0$ for all $p \in S$, and G is a densely divisible, S-torsionfree group in $\mathcal{L}_0$ such that $m(G) = G$, $c(G) \neq \{0\}$, $S(X/c(X)) \cap S = \varnothing$, and $E(G)$ is commutative.*

Corollary 6.7. *Let X be a comixed, nonresidual group in $\mathcal{L}$ such that X/X_ω is compact-by-bounded order. The following statements are equivalent:*

(i) *X_ω splits topologically from X, and $E(X)$ is commutative.*

(ii) *X is topologically isomorphic with one of the groups:*

1) $\prod_{p\in S_1} \mathbb{Z}(p^{n_p}) \times \prod_{p\in S_2} \mathbb{Z}_p \times \prod_{p\in S_3} (\mathbb{Q}_p;\mathbb{Z}_p) \times \mathbb{R} \times G$, *where S_1, S_2 and S_3 are subsets of $\mathbb{P}$ satisfying $S_1 \cup S_2 \neq \varnothing$ and $(S_1 \cup S_3) \cap S_2 = \varnothing$, $n_p \in \mathbb{N}_0$ for all $p \in S_1$, and G is a densely divisible, $(S_1 \cup S_2)$-torsionfree, topological torsion group in $\mathcal{L}$ such that $S(G) \cap (S_2 \cup S_3) = \varnothing$, $m(G) = G$, and $E(G)$ is commutative.*

2) $\prod_{p\in S} \mathbb{Z}(p^{n_p}) \times \mathbb{Q}^* \times G$, *where S is a nonempty subset of $\mathbb{P}$, $n_p \in \mathbb{N}_0$ for all $p \in S$, and G is a densely divisible, S-torsionfree, topological torsion group in $\mathcal{L}$ such that $m(G) = G$ and $E(G)$ is commutative.*

3) $\prod_{p\in S_1} \mathbb{Z}(p^{n_p}) \times \prod_{p\in S_2} \mathbb{Z}_p \times \mathbb{Q}$, *where S_1, S_2 are disjoint subsets of $\mathbb{P}$ such that $S_1 \cup S_2 \neq \varnothing$, and $n_p \in \mathbb{N}_0$ for all $p \in S_1$.*

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