Isomorphism of the factor powers of finite groups

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Abstract. All factor powers of an arbitrary finite group are classified up to isomorphism.

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1 Introduction

The set \( \mathcal{P}(G) \) of all subsets of a group \( G \) is called a global supersemigroup of that group. The operation on \( \mathcal{P}(G) \) is defined in a natural way:

\[
H \cdot F = \{ xy : x \in H, y \in F \} \quad \text{for all } H, F \subseteq G.
\]

If the group \( G \) acts on a set \( M \), we can define an induced action of a semigroup \( \mathcal{P}(G) \) on the power set of the set \( M \):

\[
N^F = \{ a^x : a \in N, x \in F \} \quad \text{for all } N \subseteq M, F \in \mathcal{P}(G).
\]

There exists a natural congruence \( \sim \) of the semigroup \( \mathcal{P}(G) \), related to this action: \( H \sim F \) iff \( H \) and \( F \) act identically on \( \mathcal{B}(M) \). Note that this definition is equivalent to the following: \( H \sim F \) iff \( \{m\}^H = \{m\}^F \) (shortly: \( m^H = m^F \)) for all \( m \in M \).

The corresponding factor semigroup \( \mathcal{FP}(G,M) = \mathcal{P}(G)/\sim \) is called a factor power of the action \( (G,M) \). \( \mathcal{FP}(G,M) \) is called also a factor power of the group \( G \); note that a group can have several factor powers. The congruence \( \sim \) is said to define the factor power \( \mathcal{FP}(G,M) \).

The factor powers were introduced in [1] for arbitrary action semigroups; the further results are stated in [2,3]. In [4] the classification of factor powers of finite groups is given for the case of cyclic groups.

From now on, we consider only right group actions, so \( m^{hf} = (m^h)^f \) for all \( m \in M, h, f \in G \).

If \( |M| = 1 \), then \( |\mathcal{FP}(G,M)| = 2 \), so the factor power \( \mathcal{FP}(G,M) \) in this case is called trivial. If the action \( (G,M) \) is regular, global supersemigroup \( \mathcal{P}(G) \) is isomorphic to factor power \( \mathcal{FP}(G,M) \), which is called then a regular factor power.

From now on, we consider a finite group \( G \) and its different factor powers.
2 Preliminaries

For an arbitrary subgroup $H \subseteq G$ we introduce the notation:

$$\{x_1, \ldots, x_k\} \equiv \{y_1, \ldots, y_m\} \pmod{H}$$

where $\{x_1, \ldots, x_k\}, \{y_1, \ldots, y_m\}$ are subsets of $G, x \in G$.

If $X \equiv Y \pmod{H}$, we say that the sets $X$ and $Y$ are equivalent with respect to the subgroup $H$.

Consider the collection of point stabilizers of the action $(G, M)$. This collection is partitioned into equivalency classes by conjugacy relation. If two points $m_1, m_2$ belong to the same orbit, there exists an element $x \in G$ such that $m_1^x = m_2$. Clearly, $\text{St}(m_2) = x^{-1} \text{St}(m_1)x$, so the two stabilizers are conjugate. Furthermore, a conjugate subgroup to the stabilizer of some point $m_1$ is itself a stabilizer of some point $m_2$ from the same orbit as $m_1$. Indeed, the subgroup $x^{-1} \text{St}(m_1)x$ is the stabilizer for the point $m_2 = m_1^x$: $\text{St}(m_2) = \{y : m_2^y = m_2\} = \{y : m_1^{xy^{-1}} = m_1\} = \{y : m_1^{xy^{-1}} = m_1\}$.

If the stabilizer of point $m$ is a superset of the stabilizer of another point $m'$, the orbit $\text{Orb}(m)$ will be called neglectable. As stabilizers of points belonging to the same orbit are conjugate, any stabilizer of point from $\text{Orb}(m)$ is a subset of some stabilizer of point from $\text{Orb}(m')$. All (conjugate) stabilizers of points from $\text{Orb}(m)$ will be called neglectable as well.

**Lemma 1.** Let “∼” be the congruence defining a factor power $FP(G)$ of a finite group $G$, $X, Y \in P(G)$. Then $X \sim Y$ iff $X \equiv Y \pmod{N}$ for each stabilizer $N$.

**Proof.** Suppose that $X \sim Y$. Let $N$ be a stabilizer of some point $m \in M$. We have: $m^X = m^Y$, which can be rewritten as $\{m^x : x \in X\} = \{m^y : y \in Y\}$. Therefore for each $x$ from $X$ there exists such a $y$ from $Y$ that $m^x = m^y$, which is equivalent to $m^{xy^{-1}} = m$, $xy^{-1} \in \text{St}(m) = N$, or $xN = yN$.

Thus for each $X \in X$ there exists such $y \in Y$ that $xN = yN$, that is, $\{xH : x \in X\} \subseteq \{yH : y \in Y\}$. The opposite inclusion can be proved in the similar way. We obtain: $\{xH : x \in X\} = \{yH : y \in Y\}$, that is, $X \equiv Y \pmod{N}$.

The sufficiency is proved in a similar way. \qed

**Remark 1.1.** As point stabilizers are subgroups, the notion of equivalency, which was introduced at the beginning of this section, is applicable to them. Equivalency w.r.t. non-neglectable stabilizers implies equivalency w.r.t. neglectable ones, therefore in Lemma 1 $X \sim Y$ if $X \equiv Y \pmod{N}$ for every non-neglectable stabilizer $N$.

**Lemma 2.** Factor power will not be changed if we remove from $M$ all points of neglectable orbits, and leave only one of several non-neglectable equal ones.
Proof. This is a corollary of Lemma 1. □

The set of all non-neglectable stabilizers defines the congruence ∼, therefore defines a factor power $\mathcal{FP}(G) = \mathcal{P}(G)/\sim$.

If $\{N_1, \ldots, N_s\} = N$ is the set of all non-neglectable stabilizers, denote the corresponding factor power as $\mathcal{FP}_{N_1, \ldots, N_s}(G) = \mathcal{FP}_N(G)$. The set $N$ is called a signature of factor power $\mathcal{FP}_N(G)$.

Lemma 3. Let $\mathcal{N}$ be a collection of subgroups of a finite group $G$ such that

1. if a subgroup $N$ belongs to $\mathcal{N}$, all conjugate subgroups belong to $\mathcal{N}$ as well;
2. no subgroup from $\mathcal{N}$ is a subset of another subgroup from $\mathcal{N}$.

Then $\mathcal{N}$ is a signature of some factor power $\mathcal{FP}(G)$.

Proof. Indeed, let $M$ be the set of all right cosets of subgroups from $\mathcal{N}$ in $G$. The group $G$ acts on $M$ in the natural way. It is easy to check that the signature of this factor power is exactly $\mathcal{N}$. □

3 Properties of factor powers

For an element $X = \{x_1, \ldots, x_k\} \in \mathcal{P}(G)$ denote $\langle x_1, \ldots, x_k \rangle$ its projection to the factor power. For an element $x$ of the factor power $\mathcal{FP}_N(G)$ denote an inverse projection image $[x] = \bigcup_{(Y) = x} Y \in \mathcal{P}(G)$.

Obviously $\langle [x] \rangle = x$.

For a factor power $S = \mathcal{FP}_N(G)$, denote $N^0 = \bigcap_{N \in \mathcal{N}} N$. Obviously $N^0$ is the kernel of the group action and therefore a normal subgroup. $N^0$ is called the kernel of the signature $\mathcal{N}$.

Lemma 4. The maximal subgroup of factor power $\mathcal{FP}_N(G)$ of a finite group $G$ is isomorphic to $G/N^0$.

Proof. I. The equality $\langle x_1, \ldots, x_k \rangle = \langle e \rangle$ is held iff on every orbit the element $x = \{x_1, \ldots, x_k\}$ doesn’t move points. This is equivalent to the fact that every element $x_i$ belongs to each stabilizer, that is, to their intersection $N^0$. Therefore $\langle x_1, \ldots, x_k \rangle = \langle e \rangle \iff \{x_1, \ldots, x_k\} \subseteq N^0$.

II. Let all $x_i$ belong to the same conjugacy class $x_0 N^0$. Then $\{x_0^{-1}\} \cdot \{x_1, \ldots, x_k\} \subseteq N^0$, or $\langle x_0^{-1} \rangle \cdot \langle x_1, \ldots, x_k \rangle = \langle e \rangle$, so $\langle x_1, \ldots, x_k \rangle$ is a group element.

III. Let now $\langle x_1, \ldots, x_k \rangle$ be a group element. There exist an inverse element $\langle y_1, \ldots, y_m \rangle$, $\langle x_1, \ldots, x_k \rangle \cdot \langle y_1, \ldots, y_m \rangle = \langle e \rangle$. Taking any nonequal $x_i$, $x_j$, we obtain: $x_i y_i \in N^0$, $x_j y_1 \in N^0$, therefore $x_i = n y_i^{-1}$, $x_j = n' y_1^{-1}$ for some $n, n' \in N^0$. We have: $x_j x_i^{-1} = n' y_1^{-1} y n^{-1} = n n' \in N^0$, therefore all $x_i$ belong to the same conjugacy class. Moreover, if $\langle z_1, \ldots, z_l \rangle = \langle x_1, \ldots, x_k \rangle$, then similarly $z_1 y_1 \in N^0$, $z_1 x_1^{-1} \in N^0$, so all $z_i$ belong to the same conjugacy class as $x_i$. 


IV. Putting the conjugacy class that contains element $x_1N^o$ as correspondent for the group element $\langle x_1, \ldots, x_k \rangle$, we obtain an isomorphism. Indeed, III shows that it is well-defined, injectivity follows from the uniqueness of inverse element, surjectivity is proved in II.

Next proof requires the following technical lemma.

**Lemma 5.** Let $FP(G) = P(G)/\sim$ be a factor power of a finite group $G$, and $X, Y \in P(G)$. Then $X \sim Y$ iff for each point stabilizer $N$ the equality $XN =YN$ is held.

**Proof.** Sets $X$ and $Y$ are equivalent iff for every stabilizer $N$ $X \equiv Y \pmod{N}$, or, in other words,

$$\{xN: x \in X\} = \{yN: y \in Y\}. \quad (1)$$

**Sufficiency:**

$$XN = \bigcup_{x \in X} xN$$

$$= \bigcup_{y \in Y} yN \quad \text{(due to (1))}$$

$$= YN.$$ 

**Necessity:** let’s take an arbitrary $x \in X$. We want to prove that there exists such $y \in Y$, that $xN = yN$.

As $XN = YN$, for each $n_1 \in N$ there exist such $y \in Y$ and $n_2 \in N$ that $xn_1 = yn_2 = yn_1n_1^{-1}n_2$. For arbitrary $n \in N$ we have $xn = xn_1(n_1^{-1}n) = yn_2(n_1^{-1}n)$. Let now $n$ run over all elements in $N$, then $n_2n_1^{-1}n$ traverses the whole $N$ as well.

Therefore $\{xN: x \in X\} \subseteq \{yN: y \in Y\}$. The counter-inclusion is proved in a similar way. \hfill \square

**Lemma 6** (on reduction). For a finite group $G$, consider a factor power $FP_N(G)$. Let $N^o$ be the kernel of its signature $N$; let $\pi$ be the projection $G \rightarrow G/N^o$. Then $FP\{\pi(N_1), \ldots, \pi(N_s)\}(G) \cong FP\{\pi(N_1), \ldots, \pi(N_s)\}(G/N^o)$.

**Proof.** Define $\tau(\langle x_1, \ldots, x_k \rangle) = \langle x_1N^o, \ldots, x_kN^o \rangle$.

We prove first that $\tau$ is well-defined and injective. Indeed, the following statements are equivalent:

1. $\langle x_1, \ldots, x_k \rangle = \langle y_1, \ldots, y_m \rangle$;
2. $\{x_1N^o, \ldots, x_kN^o\} = \{y_1N^o, \ldots, y_mN^o\}$;
3. for every stabilizer $N$:

$$\{(x_1N^o)(NN^o), \ldots, (x_kN^o)(NN^o)\} = \{(y_1N^o)(NN^o), \ldots, (y_mN^o)(NN^o)\};$$

4. $\langle x_1N^o, \ldots, x_kN^o \rangle = \langle y_1N^o, \ldots, y_mN^o \rangle$.

(Lemma 5 is used.)
Surjectivity is clear. Let’s show that \( \tau \) preserves the semigroup operation. Indeed,

\[
\langle x_1, \ldots, x_k \rangle \langle y_1, \ldots, y_m \rangle = \langle x_i y_j : i = 1, \ldots, k, j = 1, \ldots, m \rangle.
\]

Applying \( \tau \), we obtain:

\[
\langle x_i y_j N^\circ : i = 1, \ldots, k, j = 1, \ldots, m \rangle = \langle x_1 N^\circ, \ldots, x_k N^\circ, y_1 N^\circ, \ldots, y_m N^\circ \rangle.
\]

Hence \( \tau \) is an isomorphism \( \mathcal{FP}_{\{N_1, \ldots, N_s\}}(G) \rightarrow \mathcal{FP}_{\{\pi(N_1), \ldots, \pi(N_s)\}}(G/N^\circ) \).

Notice that the image of a group element under an isomorphism of factor powers is again a group element.

If the kernel of action is trivial, \( N^\circ = \{e\} \), the factor power \( \mathcal{FP}_N(G) \) is called reduced.

**Lemma 7.** Let \( \mathcal{FP}(G) \) be a reduced factor power. There exists a one-to-one correspondence between \( G \) and the maximal subgroup of \( \mathcal{FP}(G) \): an element \( x \in \mathcal{FP}(G) \) is a group element iff \( x = \langle x \rangle \) for some \( x \in G \); moreover, in this case \( [x] = \{x\} \), so the mapping \( x \mapsto \langle x \rangle \) is the needed bijection.

Proof is obvious.

**Lemma 8.** Let \( \mathcal{FP}_N(G) \) be a factor power of a finite group \( G \):

I. If \( X \) is a subgroup of \( G \), \( \langle X \rangle \) is an idempotent in \( \mathcal{FP}_N(G) \);

II. If \( y \) is an idempotent in \( \mathcal{FP}_N(G) \), \( [y] \) is a subgroup;

III. If \( y \) is an idempotent and \( x = \langle x_1, \ldots, x_k \rangle \), the equality \( xy = y \) is held iff all the \( x_i \) belong to the set \([y]\).

**Proof.** I. Since \( X \) is a subgroup, \( \langle X \rangle \langle X \rangle = \langle XX \rangle = \langle X \rangle \).

II. We have: \( \langle [y] \rangle = \langle [y] \rangle \langle [y] \rangle = yy = y \), therefore \([y] [y] \) is an inverse image of \( y \). By definition, \([y] \) is a maximal inverse image, therefore \([y] [y] \subseteq [y] \). Hence the set \([y] \) is closed with respect to multiplication. For a finite group \( G \) this is sufficient to show that \([y] \) is a subgroup.

III. If all \( x_i \) belong to \([y] \), \( xy = \{x_1, \ldots, x_k\} [y] \rangle = \langle [y] \rangle = y \).

If \( xy = y \), we have: \( \langle [y] \rangle = \langle \{x_1, \ldots, x_k\} [y] \rangle = \langle x_i y : i = 1, \ldots, k, y \in [y] \rangle \), therefore \( \{x_i y : i = 1, \ldots, k, y \in [y] \} \subseteq [y] \), hence all elements of the form \( x_i y \) belong to \([y] \). This means that all \( x_i \) belong to \([y] \).

**Remark 8.1.** The definition implies that for each subgroup \( X: [(X)] \supseteq X \).

**Definition 9.** For a given factor power of a finite group \( G \) and a subgroup \( H \) of this group, the subgroup \( [(H)] \) is called closure of the subgroup \( H \). Denote it \( \overline{H} \).

Subgroup \( H \) is called closed, if it is equal to its closure.

Obviously, \( \overline{H} \) is the minimal closed subgroup that includes \( H \).
**Remark 9.1.** Closed subgroups correspond to idempotents of the factor power.

**Remark 9.2.** For a reduced factor power, point stabilizers are closed subgroups. Indeed, let $\mathcal{X}$ be a point stabilizer, $\mathcal{X} = \text{St}(m)$. According to the definition, actions of $\mathcal{X}$ and $\mathcal{X}$ on the points from $M$ must be the same. In particular,

$$m^\mathcal{X} = m^\mathcal{X} = m^{\text{St}(m)} = m,$$

which implies that $\mathcal{X} \subseteq \text{St}(m) = \mathcal{X}$. Taking Remark 8.1 into consideration, we obtain the needed equality.

For a closed subgroup $H$ of the group $G$, denote $U(H)$ the set of all elements $x \in \mathcal{FP}_N(G)$ satisfying condition $x \cdot \langle H \rangle = \langle H \rangle$.

Let $\mathcal{N} = \{N_1, \ldots, N_s\}$ be the signature of some factor power $\mathcal{FP}(G)$, let $H$ be a subgroup of $G$. Denote $\mathcal{N} \cap H$ the set $\{N_1 \cap H, \ldots, N_s \cap H\}$, from which all neglectable elements (i.e., elements which are subsets of other elements) are removed. Notice that according to Lemma 3, $\mathcal{N} \cap H$ is a signature of some factor power of the group $H$. Indeed, it does not contain neglectable elements by definition; if $\mathcal{N} \cap H$ is a non-neglectable element in $\{N_1 \cap H, \ldots, N_s \cap H\}$, $y \in H$, then $y^{-1}N\mathcal{Y}$ belongs to $\mathcal{N}$ (since $\mathcal{N}$ is a signature), and $y^{-1}N\mathcal{Y} \cap H = y^{-1}(N \cap H)y$ (since $y \in H$) belongs to $\{N_1 \cap H, \ldots, N_s \cap H\}$ as well. Furthermore, $y^{-1}(N \cap H)y$ cannot be neglectable, otherwise its conjugate $N \cap H$ would have been neglectable as well.

**Lemma 10.** For a subgroup $H$ of a finite group $G$,

$$U(H) = \mathcal{FP}_{\mathcal{N} \cap \overline{H}}(H).$$

**Proof.** The proof consists mainly of direct checking the definition conditions.

According to Lemma 8 and the definition of closed subgroup, an element $\langle X \rangle$ belongs to $U(H)$ iff the set $X \in \mathcal{P}(G)$ is a subset of $\langle \langle H \rangle \rangle = \overline{H}$. Consider arbitrary subsets $X = \{x_1, \ldots, x_k\}$, $Y = \{y_1, \ldots, y_m\}$ of the set $\overline{H}$. We shall prove that $\{x_1, \ldots, x_k\} \sim \{y_1, \ldots, y_m\}$ is either held or not held in both $\mathcal{FP}(G)$ and $\mathcal{FP}(\overline{H})$ simultaneously. For that it is enough to prove the equivalence of the following two expressions: $X \equiv Y$ (mod $N$) and $X \equiv Y$ (mod $N \cap \overline{H}$) for an arbitrary $N \in \mathcal{N}$.

If $X \equiv Y$ (mod $N \cap \overline{H}$), for every $x$ from $X$ there exist such $y$ from $Y$ that $xy^{-1} \in N \cap \overline{H}$, which implies $xy^{-1} \in N$. Similarly, for every $y$ from $Y$ there exists such $x$ from $X$ that $xy^{-1} \in N$. Therefore $\{xN : x \in X\} = \{yN : y \in Y\}$.

If, in turn, $X \equiv Y$ (mod $N$), for arbitrary $x$ from $X$ there exist such $y$ from $Y$ that $xy^{-1} \in N$. Since $x, y \in \overline{H}$, we have: $xy^{-1} \in N \cap \overline{H}$. Similarly for arbitrary $y$ from $Y$ there exist such $x$ from $X$ that $xy^{-1} \in N \cap \overline{H}$. Therefore $\{x(N \cap \overline{H}) : x \in X\} = \{y(N \cap \overline{H}) : y \in Y\}.$

**Remark 10.1.** If $H$ is a closed subgroup, $U(H) = \mathcal{FP}_{\mathcal{N} \cap \overline{H}}(H)$.

**Remark 10.2.** If the factor power $\mathcal{FP}_N(G)$ is reduced, so is $\mathcal{FP}_{\mathcal{N} \cap \overline{H}}(H)$ as well. Indeed,

$$\bigcap (N_i \cap \overline{H}) = \left(\bigcap N_i\right) \cap \overline{H} = N^\circ \cap \overline{H} = \{e\} \cap \overline{H} = \{e\}.$$
Definition 11. Let $\mathcal{N}$ and $\mathcal{K}$ be signatures. Denote $\mathcal{N} \succ \mathcal{K}$ iff for each $K \in \mathcal{K}$ there exists such $N \in \mathcal{N}$ that $N \subseteq K$.

Lemma 12. Let $\mathcal{N}$ and $\mathcal{K}$ be signatures. Than $\mathcal{N} \prec \mathcal{K} \prec \mathcal{N}$ implies equality of $\mathcal{N}$ and $\mathcal{K}$.

Proof. The Lemma assumption immediately implies that for each $K \in \mathcal{K}$ there exists such $N \in \mathcal{N}$ that is a subset of $K$. Furthermore, there exists such $K' \in \mathcal{K}$ that is a subset of $N$. Hence $K' \subseteq N \subseteq K$. Since neither $\mathcal{N}$ nor $\mathcal{K}$ contains neglectable elements, $K = N = K'$. Thus $\mathcal{K} \subseteq \mathcal{N}$.

Since the Lemma assumption is symmetrical, $\mathcal{K} \prec \mathcal{N} \prec \mathcal{K}$ as well, and we obtain similarly: $\mathcal{N} \subseteq \mathcal{K}$.

Lemma 13 (on local injectivity). Let $S_1 = \mathcal{F} \mathcal{P}_\mathcal{N}(G)$, $S_2 = \mathcal{F} \mathcal{P}_\mathcal{K}(G)$ and $\mathcal{N} \succ \mathcal{K}$. Then $S_1 \not= S_2$.

Proof. Note that $\mathcal{K} \not\succ \mathcal{N}$ (through Lemma 12), hence there exists such $N \in \mathcal{N}$ that is not a subset of $K$ for each $K \in \mathcal{K}$.

Each set $T \subseteq G$ that contains at least one element from $N$ is not equivalent to $G \setminus N$ modulo $N$ (otherwise, if $n$ is that element from $N$, we have: $n = gn'$ for some $g \in G \setminus N$, $n' \in N$, so $g = nn'^{-1} \in N$, which states a contradiction).

In particular, $G \setminus N \not\equiv (G \setminus N) \cup \{e\}$ (mod $N$).

But for every $K \in \mathcal{K}$ we have: $G \setminus N \not\equiv (G \setminus N) \cup \{e\}$ (mod $K$). Indeed, $K \not\subseteq N$, therefore $K \cap (G \setminus N) \not= \emptyset$. For an element $k \in K \cap (G \setminus N)$, $e \equiv k$ (mod $K$), thus, $e \in G \setminus N$.

Consequently, the equivalency w.r.t. congruence $\sim_2$ which defines $S_2$ does not imply the equivalency w.r.t. congruence $\sim_1$ which defines $S_1$. But the equivalency w.r.t. $\sim_1$ implies the equivalency w.r.t. $\sim_2$, since $\mathcal{N} \succ \mathcal{K}$.

4 Isomorphisms of factor powers

Definition 14. Let $S_1$ and $S_2$ be two isomporphic factor powers of a finite group $G$, $\varphi$ be the isomorphism, $N_1^\varphi$ and $N_2^\varphi$ be the corresponding action kernels. We define an adjoint isomorphism $\varphi_m: G/N_1^\varphi \to G/N_2^\varphi$ in the following way: $\varphi_m(X) = [\varphi(\langle X \rangle)]$, where $X$ is a coset of $N_1^\varphi$. Since $\langle X \rangle$ is a group element (see the proof of Lemma 4), $\varphi(\langle X \rangle)$ is therefore a group element in $S_2$, and its inverse image $[\varphi(\langle X \rangle)]$ is a coset of $N_2^\varphi$.

We extend $\varphi_m$ to act on all sets that are unions of cosets by the same formula: $\varphi_m(X) = [\varphi(\langle X \rangle)]$. This can be written as well as follows: $\varphi(\langle x \rangle) = \langle [\varphi_m(x)] \rangle$,
\( \varphi_m([y]) = [\varphi(y)] \). Note that the inverse image of a factor power element is a union of cosets.

Consider the case when \( S_1 \) is reduced, then \( S_2 \) is reduced as well (since the maximal subgroup of \( S_1 \) is mapped by \( \varphi \) to the maximal subgroup of \( S_2 \), and due to the fact that the factor power \( FP(G) \) is reduced iff its maximal subgroup is isomorphic to \( G \). For this case \( \varphi_m \) can be naturally represented as a mapping of \( G \) to itself. Obviously \( \varphi_m \) preserves the product; since \( G \) is a finite group, \( \varphi_m \) is an automorphism in this case.

**Lemma 15.** Let \( S_1 = FP_N(G) \) and \( S_2 = FP_{N'}(G) \) be reduced factor powers of a finite group \( G \), \( \varphi \colon S_1 \to S_2 \) be an isomorphism, \( x_0 \) be an idempotent. Then \( \varphi_m([x_0]) = [\varphi(x_0)] \).

**Proof.** We have: \( \varphi(x_0)\varphi(x_0) = \varphi(x_0x_0) = \varphi(x_0) \), therefore \( \varphi(x_0) \) is idempotent.

The next statements are equivalent:

1) Element \( x \) belongs to the inverse image \( [x_0] \).
2) By lemma 8, \( [x_0] \) is an idempotent, thus 1) is equivalent to \( \langle x \rangle x_0 = x_0 \).
3) Since \( \varphi \) is an isomorphism, 2) is equivalent to \( \langle \varphi_m(x) \rangle \varphi(x_0) = \varphi(x_0) \).
4) Using lemma 8 again, we obtain: \( \varphi_m(x) \in [\varphi(x_0)] \).

The statement of lemma follows from the equivalence of 1) and 4). \( \square \)

**Definition 16.** Let \( \mathcal{N} = \{N_1, \ldots, N_s\} \) and \( \mathcal{N}' = \{N'_1, \ldots, N'_s\} \) be signatures on \( G \). \( N^o = \bigcap N_i \) and \( N'^o = \bigcap N'_i \) are the kernels of the corresponding actions. Let \( \tau \) be an isomorphism \( G/N^o \to G/N'^o \). We extend \( \tau \) to unions of cosets in the standard way. Then \( \tau(\mathcal{N}) \stackrel{\text{def}}{=} \{\tau(N_1), \ldots, \tau(N_s)\} \) is obviously a signature as well, and \( \tau \) induces the isomorphism

\[ \hat{\tau} : FP_{\mathcal{N}'}(G) \to FP_{\tau(\mathcal{N})}(G). \]

**Definition 17.** Given any isomorphic factor powers \( S_1 \) and \( S_2 \) of \((G,M)\), be \( N_1^o \) and \( N_2^o \) the respective kernels, isomorphism \( \varphi : S_1 \to S_2 \) induces the isomorphism \( \varphi_m : G/N_1^o \to G/N_2^o \):

\[ \varphi_m([x]) = [\varphi(x)]. \]

If \( \varphi_m \) is an identity transformation, (in particular, \( N_1^o = N_2^o \),) \( \varphi \) is called a direct isomorphism.

**Remark 17.1.** For an isomorphism

\[ \varphi : FP_{\mathcal{N}'}(G) \to FP_{\mathcal{N}''}(G) \]

the mapping

\[ \varphi_m^{-1} \circ \varphi : FP_{\mathcal{N}'}(G) \to FP_{\varphi_m^{-1}(\mathcal{N}'')}(G) \]

is a direct isomorphism.
Definition 18. Let $\pi$ be an epimorphism $G \rightarrow H$, $\mathcal{N} = \{N_1, \ldots, N_s\}$ be a signature on $G$. We define $\pi(\mathcal{N})$ as the set $\{\pi(N_1), \ldots, \pi(N_s)\}$, from which all the neglectable elements (i.e. elements which are subsets of others) are removed.

Remark 18.1. According to Lemma 3, $\pi(\mathcal{N})$ is a signature of some factor power of the group $H$. Indeed, it does not contain neglectable elements by definition; if $\pi(\mathcal{N})$ contains some non-neglectable element in $\pi(N_i)$, then for any $y \in H$ $y^{-1}\pi(N_i)y = \pi(x^{-1}N_ix)$, where $x$ is an arbitrary element, which is mapped to $y$ by $\pi$ (recall that $\pi$ is an epimorphism). Since $\mathcal{N}$ is a signature and $N_i$ belongs to it, $x^{-1}N_ix$ belongs to it as well, therefore $y^{-1}\pi(N_i)y$ belongs to $\{\pi(N_1), \ldots, \pi(N_s)\}$ as well. Furthermore, $y^{-1}\pi(N_i)y$ cannot be neglectable, otherwise its conjugate $\pi(N_i)$ would have been neglectable as well.

Theorem 1. Let factor powers $S_1 = FP_{\mathcal{N}}(G)$ and $S_2 = FP_{\mathcal{N}'}(G)$ of a group $G$ be directly isomorphic. Then $\mathcal{N} = \mathcal{N}'$.

Proof. The proof is by induction on size of the group $G$.

The induction basis (the case of cyclic groups) was proved in [4].

Consider the inductive step.

Factor powers $S_1$ and $S_2$ are both simultaneously reduced or not reduced.

I. Let now them be not reduced. Since the factor powers are directly isomorphic, the two corresponding group actions have a common kernel. Denote this kernel as $N^0$, denote $\pi$ the projection $G \rightarrow G/N^0$. By Lemma 6, $S_1$ is isomorphic to $FP_{\pi(\mathcal{N})}(G/N^0)$, $S_2$ is isomorphic to $FP_{\pi(\mathcal{N}')}\pi(\mathcal{N})(G/N^0)$. Notice that $FP_{\pi(\mathcal{N})}(G/N^0)$ is directly isomorphic to $FP_{\pi(\mathcal{N}')}\pi(\mathcal{N})(G/N^0)$ as factor power of $G/N^0$. Indeed, if $\varphi$ is isomorphism $S_1 \rightarrow S_2$ as factor powers of $G$, by definition $\varphi_m$ is an identity transformation. Since $FP_{\pi(\mathcal{N})}(G/N^0)$ and $FP_{\pi(\mathcal{N}')}\pi(\mathcal{N})(G/N^0)$ are reduced factor powers, their adjacent isomorphism is exactly $\varphi_m$, which is an identity.

So, by induction, $\pi(\mathcal{N}) = \pi(\mathcal{N}')$.

Note that each $N \in \mathcal{N}$ contains $N^0$, and therefore is a union of cosets of $N^0$. The element $\pi(N) \in G/N^0$ is indeed a set of cosets of $N^0$, and $N$ is indeed the union of cosets from $\pi(N)$.

This means in particular that since $\mathcal{N}$ is a signature, for each pair of elements $N_1, N_2 \in \mathcal{N}$ $\pi(N_1)$ does not contain $\pi(N_2)$ (otherwise $N_1$ would have contained $N_2$). Hence $\pi(\mathcal{N}) = \{\pi(N) : N \in \mathcal{N}\}$. The same is true for $\mathcal{N}'$.

Moreover, if some element $N$ belongs to $\mathcal{N}$, its projection $\pi(N)$ belongs to $\pi(\mathcal{N})$ and therefore to $\pi(\mathcal{N}')$, so the corresponding union of cosets belongs to $\mathcal{N}'$.

Hence $\mathcal{N} = \mathcal{N}'$, which completes the proof in this case.

II. Let now both factor powers $S_1$ and $S_2$ be reduced. In this case the kernel of both actions is trivial, and $\varphi_m$ is an identity $G \rightarrow G$.

We claim that for every closed (in either of factor powers) subgroup $H$ of the group $G$

$$\mathcal{N} \cap H = \mathcal{N}' \cap H.$$  \hfill (*)
Indeed, let \( \varphi \) be the direct isomorphism mapping \( S_1 \) to \( S_2 \), \( H \) be a closed subgroup with respect to the first factor power. Then for an arbitrary \( x \) from \( S_1 \):

\[
x \in U(H)_{S_1} \iff x + (H) = (H) \quad \text{(by definition)}
\]

\[
\iff \varphi(x) + \varphi((H)) = \varphi((H)) \quad \text{(since \( \varphi \) is an isomorphism)}
\]

\[
\iff \varphi(x) \in U(\varphi((H))) \quad \text{(by definition)}
\]

\[
\iff \varphi(x) \in U(\varphi_m(H)) \quad \text{(by Lemma 15)}
\]

\[
\iff \varphi(x) \in U(H)_{S_2} \quad \text{(since \( \varphi \) is direct)}.
\]

So, the image of \( U(H) \subseteq S_1 \) is \( U(H) \subseteq S_2 \), therefore \( U(H)_{S_1} \) is directly isomorphic to \( U(H)_{S_2} \). Taking Lemma 10 and Remark ?? into consideration, we obtain that \( \mathcal{FP}_{N \cap H}(H) \) is directly isomorphic to \( \mathcal{FP}_{N' \cap \overline{H}}(\overline{H}) \), where the closure is taken with respect to the second factor power. But according to Remark ??, both factor powers are reduced, thus their maximal subgroups are \( H \) and \( \overline{H} \) respectively. Since maximal subgroups must be isomorphic, and recalling that \( \overline{H} \supseteq H \), we claim that \( \overline{H} = H, H \) is closed with respect to the second factor power as well, so \( \mathcal{FP}_{N \cap H}(H) \) is directly isomorphic to \( \mathcal{FP}_{N' \cap \overline{H}}(\overline{H}) \).

Hence by induction (\( \ast \)) is true.

Let \( N \) be an arbitrary element of \( \mathcal{N} \). We shall prove now that \( N \) belongs to \( \mathcal{N}' \) as well.

According to Remark ??, \( N \) is a closed subgroup. Putting \( H = N \) in (\( \ast \)), we get that there exist some \( N' \) from \( \mathcal{N}' \) which satisfies \( N' \cap H = N \cap H \), which can be written as \( N' \cap N = N, N' \supseteq N \).

Similarly, there exist \( N_1 \) from \( \mathcal{N} \) such that \( N_1 \cap N' = N', N_1 \supseteq N' \).

So, \( N \subseteq N' \subseteq N_1 \). As signature does not contain neglectable elements, \( N = N' = N_1 \).

This means that \( \mathcal{N} \) is a subset of \( \mathcal{N}' \). The counter-inclusion is proved in a similar way. \( \square \)

5 Main result

The conjugacy is an equivalence relation on the set of all subgroups of a finite group \( G \). Denote \( \hat{L}(G) \) the quotient set by this relation.

We define a non-strict partial order “\( \geq \)” on \( \hat{L}(G) \) in a straightforward way: \( a \geq b \) iff for every set \( A \in a \) a set \( B \in b \) which is contained in \( A \) can be found.

The relation “\( \geq \)” is indeed a non-strict partial order. It is obviously reflexive. Transitivity can be easily established: if \( a \geq b \geq c \), then for each \( A \in a \) there exists \( B \in b \) such that \( A \supseteq B \); for any such \( B \) there exists \( C \in c \) such that \( B \supseteq C \); therefore for the considered \( a \subseteq c \) contained in \( A \) can be found. Antisymmetry requires a little bit more work. Suppose the opposite: let \( a \geq b \geq a, a \neq b \). Then either some element from \( a \) does not belong to \( b \), or some element of \( b \) does not belong to \( a \). Consider the first case (the second is done in the similar way). Let \( A \in a \), \( A \notin b \). Since \( a \geq b \), there exist \( B \in b \) such that \( A \supseteq B \). Since \( b \geq a \), there exists some \( A' \in a \) such that \( B \supseteq A' \). This yields \( A \supseteq B \supseteq A' \). But \( a \) is a conjugacy
class, therefore $|A| = |A'|$, $A = B = A'$. This contradicts to the assumption that $A \notin b$.

According to Lemma 3, each antichain in $\hat{L}(G)$ corresponds to some factor power of $G$. From the other side, for a factor power $FP_N(G)$ its signature by definition corresponds to an antichain in $\hat{L}(G)$.

Consider now the question: which antichains correspond to the isomorphic factor powers? Let’s fix a factor power $S = FP_N(G)$.

Let $\varphi$ be some non-inner automorphism of $G$. Then $\varphi$ maps $N$ to some another signature $\varphi(G)$. Indeed, if $N \in \varphi(N)$, then $x^{-1}Nx = \varphi((\varphi^{-1}(x))^{-1}N'\varphi^{-1}(x))$, where $N'$ is an element from $N$ which is mapped to $N$ by $\varphi$. Moreover, if $\varphi(N_1) \leq \varphi(N_2)$, then $\varphi^{-1}(\varphi(N_1)) \subseteq \varphi^{-1}(\varphi(N_2))$, i.e. $N_1 \subseteq N_2$, which is impossible since $N$ is a signature.

It’s easy to see that $FP_{\varphi(N)}(G)$ is isomorphic to $FP_N(G)$.

Let now $S' = FP_N'(G)$ be isomorphic to $S$, $\varphi$ be the isomorphism. Then, according to the previous Theorem 1 and Remark ??, $N = \varphi^{-1}(N')$. $\varphi^{-1}$ can be extended to an automorphism of the group $G$.

Consequently, let’s introduce the factor equivalency of signatures and (which is the same) antichains in $\hat{L}(G)$: two elements are factor equivalent iff one can be mapped to another by a mapping corresponding to some element from $Out(G)$.

Factor equivalency is, in turn, an equivalence relation on the set of all antichains on $\hat{L}(G)$. Denote $\hat{L}(G)$ the quotient set by this relation.

Consider the following relation on $\hat{L}$: a set of antichains $\mathfrak{A}$ majorizes set $\mathfrak{B}$ iff for each element $a \in \mathfrak{A}$ there exist such an element $b \in \mathfrak{B}$ that $a \succ b$.

The set $\hat{L}$ equipped with majorization becomes a poset. The proof is identical to the proof that “$\succ$” is a partial order, only instead of using conjugacy class for proving antisymmetry we need to use antichain.

A congruence $\sim_1$ on a semigroup $S$ is called a refinement of a congruence $\sim_2$, if $s_1 \sim_1 s_2$ implies $s_1 \sim_2 s_2$ for each pair of elements $s_1, s_2 \in S$. If moreover $\sim_1 \neq \sim_2$, the refinement is called strict. The factor semigroup $S/\sim_1$ is said to be an extension of the factor semigroup $S/\sim_2$ (denoted: $S/\sim_1 \succ S/\sim_2$), if the congruence $\sim_1$ is a refinement of $\sim_2$. The relation “$\succ$” is a partial order on the set of all factor semigroups of the semigroup $S$.

**Theorem 2.** There exists a one-to-one correspondence between the set of all non-isomorphic factor powers of the finite group $G$ and the set $\hat{L}(G)$ of all factor equivalence classes of antichains from the set $\hat{L}(G)$. This correspondence is consistent with the partial order on $\hat{L}(G)$, that is, factor power $S_1$ is an extension of factor power $S_2$ iff the element of $\hat{L}(G)$ corresponding to $S_1$ majorizes the element corresponding to $S_2$.

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