BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(54), 2007, Pages 69–80
ISSN 1024–7696

Isomorphism of the factor powers of finite groups

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Abstract. All factor powers of an arbitrary finite group are classified up to isomorphism.

Mathematics subject classification: 20B05.
Keywords and phrases: Factor-power, classification, power set, group action.

1 Introduction

The set $\mathcal{P}(G)$ of all subsets of a group G is called a *global supersemigroup* of that group. The operation on $\mathcal{P}(G)$ is defined in a natural way:

$$H \cdot F = \{xy \colon x \in H,\ y \in F\} \quad \text{for all} \quad H,\, F \subseteq G.$$

If the group G acts on a set M, we can define an induced action of a semigroup $\mathcal{P}(G)$ on the power set of the set M:

$$N^F = \{a^x \colon a \in N,\, x \in F\} \quad \text{for all} \quad N \subseteq M,\, F \in \mathcal{P}(G).$$

There exists a natural congruence $\sim$ of the semigroup $\mathcal{P}(G)$, related to this action: $H \sim F$ iff H and F act identically on $\mathfrak{B}(M)$. Note that this definition is equivalent to the following: $H \sim F$ iff $\{m\}^H = \{m\}^F$ (shortly: $m^H = m^F$) for all $m \in M$.

The corresponding factor semigroup $\mathcal{FP}(G, M) = \mathcal{P}(G)/{\sim}$ is called a *factor power* of the action (G, M). $\mathcal{FP}(G, M)$ is called also a factor power of the *group* G; note that a group can have several factor powers. The congruence $\sim$ is said to *define* the factor power $\mathcal{FP}(G, M)$.

The factor powers were introduced in [1] for arbitrary action semigroups; the further results are stated in [2, 3]. In [4] the classification of factor powers of finite groups is given for the case of cyclic groups.

From now on, we consider only *right* group actions, so $m^{hf} = (m^h)^f$ for all $m \in M$, $h, f \in G$.

If $|M| = 1$, then $|\mathcal{FP}(G, M)| = 2$, so the factor power $\mathcal{FP}(G, M)$ in this case is called *trivial*. If the action (G, M) is regular, global supersemigroup $\mathcal{P}(G)$ is isomorphic to factor power $\mathcal{FP}(G, M)$, which is called then a *regular* factor power.

From now on, we consider a finite group G and its different factor powers.

2 Preliminaries

For an arbitrary subgroup $H \subseteq G$ we introduce the notation:

$$\{x_1, \ldots, x_k\} \equiv \{y_1, \ldots, y_m\} \pmod H$$
$$\overset{\text{def}}{\Longleftrightarrow} \{x_1H, \ldots, x_kH\} = \{y_1H, \ldots, y_mH\},$$
$$x \in \{y_1, \ldots, y_m\} \pmod H \overset{\text{def}}{\Longleftrightarrow} xH \in \{y_1H, \ldots, y_mH\},$$

where $\{x_1, \ldots, x_k\}$, $\{y_1, \ldots, y_m\}$ are subsets of G, $x \in G$.

If $X \equiv Y \pmod H$, we say that the sets X and Y are *equivalent* with respect to the subgroup H.

Consider the collection of point stabilizers of the action (G, M). This collection is partitioned into equivalency classes by conjugacy relation. If two points m_1, m_2 belong to the same orbit, there exists an element $x \in G$ such that $m_1^x = m_2$. Clearly, $\mathrm{St}(m_2) = x^{-1}\,\mathrm{St}(m_1)x$, so the two stabilizers are conjugate. Furthermore, a conjugate subgroup to the stabilizer of some point m_1 is itself a stabilizer of some point m_2 from the same orbit as m_1. Indeed, the subgroup $x^{-1}\,\mathrm{St}(m_1)x$ is the stabilizer for the point $m_2 = m_1^x$: $\mathrm{St}(m_2) = \{y : m_2^y = m_2\} = \{y : m_1^{xy} = m_1^x\} = \{y : m_1^{xyx^{-1}} = m_1\} = x^{-1}\,\mathrm{St}(m_1)x$.

If the stabilizer of point m is a superset of the stabilizer of another point m', the orbit $\mathrm{Orb}(m)$ will be called *neglectable*. As stabilizers of points belonging to the same orbit are conjugate, any stabilizer of point from $\mathrm{Orb}(m)$ is a subset of some stabilizer of point from $\mathrm{Orb}(m')$. All (conjugate) stabilizers of points from $\mathrm{Orb}(m)$ will be called neglectable as well.

Lemma 1. *Let "$\sim$" be the congruence defining a factor power $\mathcal{FP}(G)$ of a finite group G, X, $Y \in \mathcal{P}(G)$. Then $X \sim Y$ iff $X \equiv Y \pmod N$ for each stabilizer N.*

Proof. Suppose that $X \sim Y$. Let N be a stabilizer of some point $m \in M$. We have: $m^X = m^Y$, which can be rewritten as $\{m^x : x \in X\} = \{m^y : y \in Y\}$. Therefore for each x from X there exists such a y from Y that $m^x = m^y$, which is equivalent to $m^{xy^{-1}} = m$, $xy^{-1} \in \mathrm{St}(m) = N$, or $xN = yN$.

Thus for each $X \in X$ there exists such $y \in Y$ that $xN = yN$, that is, $\{xH : x \in X\} \subseteq \{yH : y \in Y\}$. The opposite inclusion can be proved in the similar way. We obtain: $\{xH : x \in X\} = \{yH : y \in Y\}$, that is, $X \equiv Y \pmod N$.

The sufficiency is proved in a similar way. □

Remark 1.1. As point stabilizers are subgroups, the notion of equivalency, which was introduced at the beginning of this section, is applicable to them. Equivalency w. r. t. non-neglectable stabilizers implies equivalency w. r. t. neglectable ones, therefore in Lemma 1 $X \sim Y$ if $X \equiv Y \pmod N$ for every *non-neglectable* stabilizer N.

Lemma 2. *Factor power will not be changed if we remove from M all points of neglectable orbits, and leave only one of several non-neglectable equal ones.*

Proof. This is a corollary of Lemma 1. □

The set of all non-neglectable stabilizers defines the congruence $\sim$, therefore defines a factor power $\mathcal{FP}(G) = \mathcal{P}(G)/\sim$.

If $\{N_1, \ldots, N_s\} = \mathcal{N}$ is the set of all non-neglectable stabilizers, denote the corresponding factor power as $\mathcal{FP}_{\{N_1,\ldots,N_s\}}(G) = \mathcal{FP}_{\mathcal{N}}(G)$. The set $\mathcal{N}$ is called a *signature* of factor power $\mathcal{FP}_{\mathcal{N}}(G)$.

Lemma 3. *Let $\mathcal{N}$ be a collection of subgroups of a finite group G such that*

(1) if a subgroup N belongs to $\mathcal{N}$, all conjugate subgroups belong to $\mathcal{N}$ as well;

(2) no subgroup from $\mathcal{N}$ is a subset of another subgroup from $\mathcal{N}$.

Then $\mathcal{N}$ is a signature of some factor power $\mathcal{FP}(G)$.

Proof. Indeed, let M be the set of all right cosets of subgroups from $\mathcal{N}$ in G. The group G acts on M in the natural way. It is easy to check that the signature of this factor power is exactly $\mathcal{N}$. □

3 Properties of factor powers

For an element $X = \{x_1, \ldots, x_k\} \in \mathcal{P}(G)$ denote $\langle x_1, \ldots, x_k\rangle$ its projection to the factor power. For an element $\boldsymbol{x}$ of the factor power $\mathcal{FP}_{\mathcal{N}}(G)$ denote an inverse projection image $[\boldsymbol{x}] = \bigcup\limits_{\langle Y\rangle=\boldsymbol{x}} Y \in \mathcal{P}(G)$.

Obviously $\langle[\boldsymbol{x}]\rangle = \boldsymbol{x}$.

For a factor power $S = \mathcal{FP}_{\mathcal{N}}(G)$, denote $N^\circ = \bigcap\limits_{N\in\mathcal{N}} N$. Obviously N° is the kernel of the group action and therefore a normal subgroup. N° is called the *kernel* of the signature $\mathcal{N}$.

Lemma 4. *The maximal subgroup of factor power $\mathcal{FP}_{\mathcal{N}}(G)$ of a finite group G is isomorphic to G/N°.*

Proof. I. The equality $\langle x_1, \ldots, x_k\rangle = \langle e\rangle$ is held iff on every orbit the element $\boldsymbol{x} = \{x_1, \ldots, x_k\}$ doesn't move points. This is equivalent to the fact that every element x_i belongs to each stabilizer, that is, to their intersection N°. Therefore $\langle x_1, \ldots, x_k\rangle = \langle e\rangle \iff \{x_1, \ldots, x_k\} \subseteq N^\circ$.

II. Let all x_i belong to the same conjugacy class x_0N°. Then $\{x_0^{-1}\} \cdot \{x_1, \ldots, x_k\} \subseteq N^\circ$, or $\left\langle x_0^{-1}\right\rangle \cdot \langle x_1, \ldots, x_k\rangle = \langle e\rangle$, so $\langle x_1, \ldots, x_k\rangle$ is a group element.

III. Let now $\langle x_1, \ldots, x_k\rangle$ be a group element. There exist an inverse element $\langle y_1, \ldots, y_m\rangle$, $\langle x_1, \ldots, x_k\rangle \cdot \langle y_1, \ldots, y_m\rangle = \langle e\rangle$. Taking any nonequal x_i, x_j, we obtain: $x_iy_1 \in N^\circ$, $x_jy_1 \in N^\circ$, therefore $x_i = ny_1^{-1}$, $x_j = n'y_1^{-1}$ for some $n, n' \in N^\circ$. We have: $x_jx_i^{-1} = n'y_1^{-1}y_1n^{-1} = n'n \in N^\circ$, therefore all x_i belong to the same conjugacy class. Moreover, if $\langle z_1, \ldots, z_l\rangle = \langle x_1, \ldots, x_k\rangle$, then similarly $z_1y_1 \in N^\circ$, $z_1x_1^{-1} \in N^\circ$, so all z_i belong ot the same conjugacy class as x_i.

IV. Putting the conjugacy class that contains element x_1N° as correspondent for the group element $\langle x_1, \ldots, x_k\rangle$, we obtain an isomorphism. Indeed, III shows that it is well-defined, injectivity follows from the uniqueness of inverse element, surjectivity is proved in II. □

Next proof requires the following technical lemma.

Lemma 5. *Let $\mathcal{FP}(G) = \mathcal{P}(G)/{\sim}$ be a factor power of a finite group G, and X, $Y \in \mathcal{P}(G)$. Then $X \sim Y$ iff for each point stabilizer N the equality $XN = YN$ is held.*

Proof. Sets X and Y are equivalent iff for every stabilizer N $X \equiv Y \pmod N$, or, in other words,

$$\{xN : x \in X\} = \{yN : y \in Y\}. \tag{1}$$

Sufficiency:

$$\begin{aligned} XN &= \bigcup_{x\in X} xN \\ &= \bigcup_{y\in Y} yN \qquad \text{(due to (1))} \\ &= YN. \end{aligned}$$

Necessity: let's take an arbitrary $x \in X$. We want to prove that there exists such $y \in Y$, that $xN = yN$.

As $XN = YN$, for each $n_1 \in N$ there exist such $y \in Y$ and $n_2 \in N$ that $xn_1 = yn_2 = yn_1n_1^{-1}n_2$. For arbitrary $n \in N$ we have $xn = xn_1(n_1^{-1}n) = yn_2(n_1^{-1}n)$. Let now n run over all elements in N, then $n_2n_1^{-1}n$ traverses the whole N as well.

Therefore $\{xN : x \in X\} \subseteq \{yN : y \in Y\}$. The counter-inclusion is proved in a similar way. □

Lemma 6 (on reduction)**.** *For a finite group G, consider a factor power $\mathcal{FP}_{\mathcal{N}}(G)$. Let N° be the kernel of its signature $\mathcal{N}$; let π be the projection $G \to G/N^\circ$. Then $\mathcal{FP}_{\{N_1,\ldots,N_s\}}(G) \cong \mathcal{FP}_{\{\pi(N_1),\ldots,\pi(N_s)\}}(G/N^\circ)$.*

Proof. Define $\tau(\langle x_1, \ldots, x_k\rangle) = \langle x_1N^\circ, \ldots, x_kN^\circ\rangle$.

We prove first that τ is well-defined and injective. Indeed, the following statements are equivalent:

(1) $\langle x_1, \ldots, x_k\rangle = \langle y_1, \ldots, y_m\rangle$;

(2) $\{x_1N^\circ, \ldots, x_kN^\circ\} = \{y_1N^\circ, \ldots, y_mN^\circ\}$;

(3) for every stabilizer N:

$$\{(x_1N^\circ)(NN^\circ), \ldots, (x_kN^\circ)(NN^\circ)\} = \{(y_1N^\circ)(NN^\circ), \ldots, (y_mN^\circ)(NN^\circ)\};$$

(4) $\langle x_1N^\circ, \ldots, x_kN^\circ\rangle = \langle y_1N^\circ, \ldots, y_mN^\circ\rangle$.

(Lemma 5 is used.)

Surjectivity is clear.

Let's show that τ preserves the semigroup operation. Indeed,

$$\langle x_1, \ldots, x_k\rangle \langle y_1, \ldots, y_m\rangle = \langle x_i y_j : i = 1, \ldots, k,\ j = 1, \ldots, m\rangle .$$

Applying τ, we obtain:

$$\langle x_i y_j N^\circ : i = 1, \ldots, k,\ j = 1, \ldots, m\rangle = \langle x_1 N^\circ, \ldots, x_k N^\circ\rangle \langle y_1 N^\circ, \ldots, y_m N^\circ\rangle .$$

Hence τ is an isomorphism $\mathcal{FP}_{\{N_1,\ldots,N_s\}}(G) \to \mathcal{FP}_{\{\pi(N_1),\ldots,\pi(N_s)\}}(G/N^\circ)$. □

Notice that the image of a group element under an isomorphism of factor powers is again a group element.

If the kernel of action is trivial, $N^\circ = \{e\}$, the factor power $\mathcal{FP}_{\mathcal{N}}(G)$ is called *reduced.*

Lemma 7. *Let $\mathcal{FP}(G)$ be a reduced factor power. There exists a one-to-one correspondence between G and the maximal subgroup of $\mathcal{FP}(G)$: an element $\boldsymbol{x} \in \mathcal{FP}(G)$ is a group element iff $\boldsymbol{x} = \langle x\rangle$ for some $x \in G$; moreover, in this case $[\boldsymbol{x}] = \{x\}$, so the mapping $x \mapsto \langle x\rangle$ is the needed bijection.*

Proof is obvious.

Lemma 8. *Let $\mathcal{FP}_{\mathcal{N}}(G)$ be a factor power of a finite group G:*

I. If X is a subgroup of G, $\langle X\rangle$ is an idempotent in $\mathcal{FP}_{\mathcal{N}}(G)$;

II. If $\boldsymbol{y}$ is an idempotent in $\mathcal{FP}_{\mathcal{N}}(G)$, $[\boldsymbol{y}]$ is a subgroup;

III. If $\boldsymbol{y}$ is an idempotent and $\boldsymbol{x} = \langle x_1, \ldots, x_k\rangle$, the equality $\boldsymbol{xy} = \boldsymbol{y}$ is held iff all the x_i belong to the set $[\boldsymbol{y}]$.

Proof. I. Since X is a subgroup, $\langle X\rangle \langle X\rangle = \langle XX\rangle = \langle X\rangle$.

II. We have: $\langle [\boldsymbol{y}]\,[\boldsymbol{y}]\rangle = \langle [\boldsymbol{y}]\rangle \langle [\boldsymbol{y}]\rangle = \boldsymbol{yy} = \boldsymbol{y}$, therefore $[\boldsymbol{y}]\,[\boldsymbol{y}]$ is an inverse image of $\boldsymbol{y}$. By definition, $[\boldsymbol{y}]$ is a maximal inverse image, therefore $[\boldsymbol{y}]\,[\boldsymbol{y}] \subseteq [\boldsymbol{y}]$. Hence the set $[\boldsymbol{y}]$ is closed with respect to multiplication. For a finite group G this is sufficient to show that $[\boldsymbol{y}]$ is a subgroup.

III. If all x_i belong to $[\boldsymbol{y}]$, $\boldsymbol{xy} = \langle \{x_1, \ldots, x_k\}\,[\boldsymbol{y}]\rangle = \langle [\boldsymbol{y}]\rangle = \boldsymbol{y}$.

If $\boldsymbol{xy} = \boldsymbol{y}$, we have: $\langle [\boldsymbol{y}]\rangle = \langle \{x_1, \ldots, x_k\}\,[\boldsymbol{y}]\rangle = \langle x_i y : i = 1, \ldots, k,\ y \in [\boldsymbol{y}]\rangle$, therefore $\{x_i y : i = 1, \ldots, k,\ y \in [\boldsymbol{y}]\} \subseteq [\boldsymbol{y}]$, hence all elements of the form $x_i y$ belong to $[\boldsymbol{y}]$. This means that all x_i belong to $[\boldsymbol{y}]$. □

Remark 8.1. The definition implies that for each subgroup X: $[\langle X\rangle] \supseteq X$.

Definition 9. *For a given factor power of a finite group G and a subgroup H of this group, the subgroup $[\langle H\rangle]$ is called* closure *of the subgroup H. Denote it $\overline{H}$.*

Subgroup H is called closed*, if it is equal to its closure.*

Obviously, $\overline{H}$ is the minimal closed subgroup that includes H.

Remark 9.1. Closed subgroups correspond to idempotents of the factor power.

Remark 9.2. For a reduced factor power, point stabilizers are closed subgroups. Indeed, let X be a point stabilizer, $X = \mathrm{St}(m)$. According to the definition, actions of X and $\overline{X}$ on the points from M must be the same. In particular,

$$m^{\overline{X}} = m^X = m^{\mathrm{St}(m)} = m,$$

which implies that $\overline{X} \subseteq \mathrm{St}(m) = X$. Taking Remark 8.1 into consideration, we obtain the needed equality.

For a closed subgroup H of the group G, denote $U(H)$ the set of all elements $\boldsymbol{x} \in \mathcal{FP}_{\mathcal{N}}(G)$ satisfying condition $\boldsymbol{x} \cdot \langle H \rangle = \langle H \rangle$.

Let $\mathcal{N} = \{N_1, \ldots, N_s\}$ be the signature of some factor power $\mathcal{FP}(G)$, let H be a subgroup of G. Denote $\mathcal{N} \cap H$ the set $\{N_1 \cap H, \ldots, N_s \cap H\}$, from which all neglectable elements (i. e. elements which are subsets of other elements) are removed. Notice that according to Lemma 3, $\mathcal{N} \cap H$ is a signature of some factor power of the group H. Indeed, it does not contain neglectable elements by definition; if $N \cap H$ is a non-neglectable element in $\{N_1 \cap H, \ldots, N_s \cap H\}$, $y \in H$, then $y^{-1}Ny$ belongs to $\mathcal{N}$ (since $\mathcal{N}$ is a signature), and $y^{-1}Ny \cap H = y^{-1}(N \cap H)y$ (since $y \in H$) belongs to $\{N_1 \cap H, \ldots, N_s \cap H\}$ as well. Furthermore, $y^{-1}(N \cap H)y$ cannot be neglectable, otherwise its conjugate $N \cap H$ would have been neglectable as well.

Lemma 10. *For a subgroup H of a finite group G,*

$$U(H) = \mathcal{FP}_{\mathcal{N} \cap \overline{H}}(\overline{H}).$$

Proof. The proof consists mainly of direct checking the definition conditions.

According to Lemma 8 and the definition of closed subgroup, an element $\langle X \rangle$ belongs to $U(H)$ iff the set $X \in \mathcal{P}(G)$ is a subset of $[\langle H \rangle] = \overline{H}$.

Consider arbitrary subsets $X = \{x_1, \ldots, x_k\}$, $Y = \{y_1, \ldots, y_m\}$ of the set $\overline{H}$. We shall prove that $\{x_1, \ldots, x_k\} \sim \{y_1, \ldots, y_m\}$ is either held or not held in both $\mathcal{FP}(G)$ and $\mathcal{FP}(\overline{H})$ simultaneously. For that it is enough to prove the equivalence of the following two expressions: $X \equiv Y \pmod{N}$ and $X \equiv Y \pmod{N \cap \overline{H}}$ for an arbitrary $N \in \mathcal{N}$.

If $X \equiv Y \pmod{N \cap \overline{H}}$, for every x from X there exist such y from Y that $xy^{-1} \in N \cap \overline{H}$, which implies $xy^{-1} \in N$. Similarly, for every y from Y there exists such x from X that $xy^{-1} \in N$. Therefore $\{xN : x \in X\} = \{yN : y \in Y\}$.

If, in turn, $X \equiv Y \pmod{N}$, for arbitrary x from X there exist such y from Y that $xy^{-1} \in N$. Since $x, y \in \overline{H}$, we have: $xy^{-1} \in N \cap \overline{H}$. Similarly for arbitrary y from Y there exist such x from X that $xy^{-1} \in N \cap \overline{H}$. Therefore $\{x(N \cap \overline{H}) : x \in X\} = \{y(N \cap \overline{H}) : y \in Y\}$. □

Remark 10.1. If H is a closed subgroup, $U(H) = \mathcal{FP}_{\mathcal{N} \cap H}(H)$.

Remark 10.2. If the factor power $\mathcal{FP}_{\mathcal{N}}(G)$ is reduced, so is $\mathcal{FP}_{\mathcal{N} \cap \overline{H}}(\overline{H})$ as well. Indeed,

$$\bigcap (N_i \cap \overline{H}) = \Big(\bigcap N_i\Big) \cap \overline{H} = N^\circ \cap \overline{H} = \{e\} \cap \overline{H} = \{e\}.$$

Definition 11. *Let $\mathcal{N}$ and $\mathcal{K}$ be signatures. Denote $\mathcal{N} \succ \mathcal{K}$ iff for each $K \in \mathcal{K}$ there exists such $N \in \mathcal{N}$ that $N \subseteq K$.*

Lemma 12. *Let $\mathcal{N}$ and $\mathcal{K}$ be signatures. Than $\mathcal{N} \prec \mathcal{K} \prec \mathcal{N}$ implies equality of $\mathcal{N}$ and $\mathcal{K}$.*

Proof. The Lemma assumption immediately implies that for each $K \in \mathcal{K}$ there exists such $N \in \mathcal{N}$ that is a subset of K. Furthermore, there exists such $K' \in \mathcal{K}$ that is a subset of N. Hence $K' \subseteq N \subseteq K$. Since neither $\mathcal{N}$ nor $\mathcal{K}$ contains neglectable elements, $K = N = K'$. Thus $\mathcal{K} \subseteq \mathcal{N}$.

Since the Lemma assumption is symmetrical, $\mathcal{K} \prec \mathcal{N} \prec \mathcal{K}$ as well, and we obtain similarly: $\mathcal{N} \subseteq \mathcal{K}$. □

Lemma 13 (on local injectivity)**.** *Let*

$$S_1 = \mathcal{FP}_{\mathcal{N}}(G), \quad S_2 = \mathcal{FP}_{\mathcal{K}}(G)$$

and $\mathcal{N} \supsetneq \mathcal{K}$. Then $S_1 \neq S_2$.

Proof. Note that $\mathcal{K} \not\succ \mathcal{N}$ (through Lemma 12), hence there exists such $N \in \mathcal{N}$ that is not a subset of K for each $K \in \mathcal{K}$.

Each set $T \subseteq G$ that contains at least one element from N is not equivalent to $G \setminus N$ modulo N (otherwise, if n is that element from N, we have: $n = gn'$ for some $g \in G \setminus N$, $n' \in N$, so $g = nn'^{-1} \in N$, which states a contradiction).

In particular, $G \setminus N \not\equiv (G \setminus N) \cup \{e\} \pmod{N}$.

But for every $K \in \mathcal{K}$ we have: $G \setminus N \not\equiv (G \setminus N) \cup \{e\} \pmod{K}$. Indeed, $K \nsubseteq N$, therefore $K \cap (G \setminus N) \neq \emptyset$. For an element $k \in K \cap (G \setminus N)$, $e \equiv k \pmod{K}$, thus, $e \in G \setminus N$.

Consequently, the equivalency w. r. t. congruence $\sim_2$ which defines S_2 does not imply the equivalency w. r. t. congruence $\sim_1$ which defines S_1. But the equivalency w. r. t. $\sim_1$ implies the equivalency w. r. t. $\sim_2$, since $\mathcal{N} \succ \mathcal{K}$. □

4 Isomorphisms of factor powers

Definition 14. *Let S_1 and S_2 be two isomporphic factor powers of a finite group G, φ be the isomorphism, N_1° and N_2° be the corresponding action kernels. We define an* adjoint isomorphism *$\varphi_m \colon G/N_1^\circ \to G/N_2^\circ$ in the following way: $\varphi_m(X) = [\varphi(\langle X \rangle)]$, where X is a coset of N_1°. Since $\langle X \rangle$ is a group element (see the proof of Lemma 4), $\varphi(\langle X \rangle)$ is therefore a group element in S_2, and its inverse image $[\varphi(\langle X \rangle)]$ is a coset of N_2°.*

We extend φ_m to act on all sets that are unions of cosets by the same formula: $\varphi_m(X) = [\varphi(\langle X \rangle)]$. This can be written as well as follows: $\varphi(\langle x \rangle) = \langle \varphi_m(x) \rangle$,

$\varphi_m([\boldsymbol{y}]) = [\varphi(\boldsymbol{y})]$. *Note that the inverse image of a factor power element is a union of cosets.*

Consider the case when S_1 is reduced, then S_2 is reduced as well (since the maximal subgroup of S_1 is mapped by φ to the maximal subgroup of S_2, and due to the fact that the factor power $\mathcal{FP}(G)$ is reduced iff its maximal subgroup is isomorphic to G). For this case φ_m can be naturally represented as a mapping of G to itself. Obviously φ_m preserves the product; since G is a finite group, φ_m is an automorphism in this case.

Lemma 15. *Let $S_1 = \mathcal{FP}_{\mathcal{N}}(G)$ and $S_2 = \mathcal{FP}_{\mathcal{N}'}(G)$ be reduced factor powers of a finite group G, $\varphi\colon S_1 \to S_2$ be an isomorphism, $\boldsymbol{x}_0$ be an idempotent. Then $\varphi_m([\boldsymbol{x}_0]) = [\varphi(\boldsymbol{x}_0)]$.*

Proof. We have: $\varphi(\boldsymbol{x}_0)\varphi(\boldsymbol{x}_0) = \varphi(\boldsymbol{x}_0\boldsymbol{x}_0) = \varphi(\boldsymbol{x}_0)$, therefore $\varphi(\boldsymbol{x}_0)$ is idempotent.

The next statements are equivalent:

1) Element x belongs to the inverse image $[\boldsymbol{x}_0]$.
2) By lemma 8, $[\boldsymbol{x}_0]$ is an idempotent, thus 1) is equivalent to $\langle x\rangle\, \boldsymbol{x}_0 = \boldsymbol{x}_0$.
3) Since φ is an isomorphism, 2) is equivalent to $\langle \varphi_m(x)\rangle\, \varphi(\boldsymbol{x}_0) = \varphi(\boldsymbol{x}_0)$.
4) Using lemma 8 again, we obtain: $\varphi_m(x) \in [\varphi(\boldsymbol{x}_0)]$.

The statement of lemma follows from the equivalence of 1) and 4). □

Definition 16. *Let $\mathcal{N} = \{N_1, \ldots, N_s\}$ and $\mathcal{N}' = \left\{N'_1, \ldots, N'_{s'}\right\}$ be signatures on G. $N^\circ = \bigcap N_i$ and $N^{\circ\prime} = \bigcap N'_i$ are the kernels of the corresponding actions. Let τ be an isomorphism $G/N^\circ \to G/N^{\circ\prime}$. We extend τ to unions of cosets in the standard way. Then $\tau(\mathcal{N}) \stackrel{\text{def}}{=} \{\tau(N_1), \ldots, \tau(N_s)\}$ is obviously a signature as well, and τ induces the isomorphism*

$$\hat{\tau}\colon \mathcal{FP}_{\mathcal{N}}(G) \to \mathcal{FP}_{\tau(\mathcal{N})}(G).$$

Definition 17. *Given any isomorphic factor powers S_1 and S_2 of (G, M), be N_1° and N_2° the respective kernels, isomorphism $\varphi\colon S_1 \to S_2$ induces the isomorphism $\varphi_m\colon G/N_1^\circ \to G/N_2^\circ$:*

$$\varphi_m([\boldsymbol{x}]) = [\varphi(\boldsymbol{x})]\,.$$

If φ_m is an identity transformation, (in particular, $N_1^\circ = N_2^\circ$,) φ is called a direct *isomorphism.*

Remark 17.1. For an isomorphism

$$\varphi\colon \mathcal{FP}_{\mathcal{N}}(G) \to \mathcal{FP}_{\mathcal{N}'}(G)$$

the mapping

$$\hat{\varphi_m}^{-1} \circ \varphi\colon \mathcal{FP}_{\mathcal{N}}(G) \to \mathcal{FP}_{\varphi_m^{-1}(\mathcal{N}')}(G)$$

is a direct isomorphism.

Definition 18. *Let π be an epimorphism $G \to H$, $\mathcal{N} = \{N_1, \dots, N_s\}$ be a signature on G. We define $\pi(\mathcal{N})$ as the set $\{\pi(N_1), \dots, \pi(N_s)\}$, from which all the neglectable elements (i. e. elements which are subsets of others) are removed.*

Remark 18.1. According to Lemma 3, $\pi(\mathcal{N})$ is a signature of some factor power of the group H. Indeed, it does not contain neglectable elements by definition; if $\pi(\mathcal{N})$ contains some non-neglectable element in $\pi(N_i)$, then for any $y \in H$ $y^{-1}\pi(N_i)y = \pi(x^{-1}N_i x)$, where x is an arbitrary element, which is mapped to y by π (recall that π is an epimorphism). Since $\mathcal{N}$ is a signature and N_i belongs to it, $x^{-1}N_i x$ belongs to it as well, therefore $y^{-1}\pi(N_i)y$ belongs to $\{\pi(N_1), \dots, \pi(N_s)\}$ as well. Furthermore, $y^{-1}\pi(N_i)y$ cannot be neglectable, otherwise its conjugate $\pi(N_i)$ would have been neglectable as well.

Theorem 1. *Let factor powers $S_1 = \mathcal{FP}_{\mathcal{N}}(G)$ and $S_2 = \mathcal{FP}_{\mathcal{N}'}(G)$ of a group G be directly isomorphic. Then $\mathcal{N} = \mathcal{N}'$.*

Proof. The proof is by induction on size of the group G.

The induction basis (the case of cyclic groups) was proved in [4].

Consider the inductive step.

Factor powers S_1 and S_2 are both simultaneously reduced or not reduced.

I. Let now them be not reduced. Since the factor powers are directly isomorphic, the two corresponding group actions have a common kernel. Denote this kernel as N°, denote π the projection $G \to G/N^\circ$. By Lemma 6, S_1 is isomorphic to $\mathcal{FP}_{\pi(\mathcal{N})}(G/N^\circ)$, S_2 is isomorphic to $\mathcal{FP}_{\pi(\mathcal{N}')}(G/N^\circ)$. Notice that $\mathcal{FP}_{\pi(\mathcal{N})}(G/N^\circ)$ is directly isomorphic to $\mathcal{FP}_{\pi(\mathcal{N}')}(G/N^\circ)$ as factor power of G/N°. Indeed, if φ is isomorphism $S_1 \to S_2$ as factor powers of G, by definition φ_m is an identity transformation. Since $\mathcal{FP}_{\pi(\mathcal{N})}(G/N^\circ)$ and $\mathcal{FP}_{\pi(\mathcal{N}')}(G/N^\circ)$ are reduced factor powers, their adjacent isomorphism is exactly φ_m, which is an identity.

So, by induction, $\pi(\mathcal{N}) = \pi(\mathcal{N}')$.

Note that each $N \in \mathcal{N}$ contains N°, and therefore is a union of cosets of N°. The element $\pi(N) \in G/N^\circ$ is indeed a set of cosets of N°, and N is indeed the union of cosets from $\pi(N)$.

This means in particular that since $\mathcal{N}$ is a signature, for each pair of elements $N_1, N_2 \in \mathcal{N}$ $\pi(N_1)$ does not contain $\pi(N_2)$ (otherwise N_1 would have contained N_2). Hence $\pi(\mathcal{N}) = \{\pi(N) : N \in \mathcal{N}\}$. The same is true for $\mathcal{N}'$.

Moreover, if some element N belongs to $\mathcal{N}$, its projection $\pi(N)$ belongs to $\pi(\mathcal{N})$ and therefore to $\pi(\mathcal{N}')$, so the corresponding union of cosets belongs to $\mathcal{N}'$.

Hence $\mathcal{N} = \mathcal{N}'$, which completes the proof in this case.

II. Let now both factor powers S_1 and S_2 be reduced. In this case the kernel of both actions is trivial, and φ_m is an identity $G \to G$.

We claim that for every closed (in either of factor powers) subgroup H of the group G

$$\mathcal{N} \cap H = \mathcal{N}' \cap H. \qquad (*)$$

Indeed, let φ be the direct isomorphism mapping S_1 to S_2, H be a closed subgroup with respect to the first factor power. Then for an arbitrary $\boldsymbol{x}$ from S_1:

$$\begin{aligned} x \in U(H)_{S_1} &\iff \boldsymbol{x} + \langle H\rangle = \langle H\rangle && \text{(by definition)} \\ &\iff \varphi(\boldsymbol{x}) + \varphi(\langle H\rangle) = \varphi(\langle H\rangle) && \text{(since } \varphi \text{ is an isomorphism)} \\ &\iff \varphi(\boldsymbol{x}) \in U([\varphi(\langle H\rangle)]) && \text{(by definition)} \\ &\iff \varphi(\boldsymbol{x}) \in U(\varphi_m(H)) && \text{(by Lemma 15)} \\ &\iff \varphi(\boldsymbol{x}) \in U(H)_{S_2} && \text{(since } \varphi \text{ is direct).} \end{aligned}$$

So, the image of $U(H) \subset S_1$ is $U(H) \subset S_2$, therefore $U(H)_{S_1}$ is directly isomorphic to $U(H)_{S_2}$. Taking Lemma 10 and Remark **??** into consideration, we obtain that $\mathcal{FP}_{\mathcal{N}\cap H}(H)$ is directly isomorphic to $\mathcal{FP}_{\mathcal{N}'\cap\overline{H}}(\overline{H})$, where the closure is taken with respect to the second factor power. But according to Remark **??**, both factor powers are reduced, thus their maximal subgroups are H and $\overline{H}$ respectively. Since maximal subgroups must be isomorphic, and recalling that $\overline{H} \supseteq H$, we claim that $\overline{H} = H$, H is closed with respect to the second factor power as well, so $\mathcal{FP}_{\mathcal{N}\cap H}(H)$ is directly isomorphic to $\mathcal{FP}_{\mathcal{N}'\cap H}(H)$.

Hence by induction $(*)$ is true.

Let N be an arbitrary element of $\mathcal{N}$. We shall prove now that N belongs to $\mathcal{N}'$ as well.

According to Remark **??**, N is a closed subgroup. Putting $H = N$ in $(*)$, we get that there exist some N' from $\mathcal{N}'$ which satisfies $N' \cap H = N \cap H$, which can be written as $N' \cap N = N$, $N' \supseteq N$.

Similarly, there exist N_1 from $\mathcal{N}$ such that $N_1 \cap N' = N'$, $N_1 \supseteq N'$.

So, $N \subseteq N' \subseteq N_1$. As signature does not contain neglectable elements, $N = N' = N_1$.

This means that $\mathcal{N}$ is a subset of $\mathcal{N}'$. The counter-inclusion is proved in a similar way. □

5 Main result

The conjugacy is an equivalence relation on the set of all subgroups of a finite group G. Denote $\hat{L}(G)$ the quotient set by this relation.

We define a non-strict partial order "$\succcurlyeq$" on $\hat{L}(G)$ in a straitforward way: $\mathfrak{a} \succcurlyeq \mathfrak{b}$ iff for every set $A \in \mathfrak{a}$ a set $B \in \mathfrak{b}$ which is contained in A can be found.

The relation "$\succcurlyeq$" is indeed a non-strict partial order. It is obviously reflexive. Transitivity can be easily established: if $\mathfrak{a} \succcurlyeq \mathfrak{b} \succcurlyeq \mathfrak{c}$, then for each $A \in \mathfrak{a}$ there exists $B \in \mathfrak{b}$ such that $A \supseteq B$; for any such B there exists $C \in \mathfrak{c}$ such that $B \supseteq C$; therefore for the considered A a $C \in \mathfrak{c}$ contained in A can be found. Antisymmetry requires a little bit more work. Suppose the opposite: let $\mathfrak{a} \succcurlyeq \mathfrak{b} \succcurlyeq \mathfrak{a}$, $\mathfrak{a} \neq \mathfrak{b}$. Then either some element from $\mathfrak{a}$ does not belong to $\mathfrak{b}$, or some element of $\mathfrak{b}$ does not belong to $\mathfrak{a}$. Consider the first case (the second is done in the similar way). Let $A \in \mathfrak{a}$, $A \notin \mathfrak{b}$. Since $\mathfrak{a} \succcurlyeq \mathfrak{b}$, there exist $B \in \mathfrak{b}$ such that $A \supseteq B$. Since $\mathfrak{b} \succcurlyeq \mathfrak{a}$, there exists some $A' \in \mathfrak{a}$ such that $B \supseteq A'$. This yields $A \supseteq B \supseteq A'$. But $\mathfrak{a}$ is a conjugacy class, therefore $|A| = |A'|$, $A = B = A'$. This contradicts to the assumption that $A \notin \mathfrak{b}$.

According to Lemma 3, each antichain in $\hat{L}(G)$ corresponds to some factor power of G. From the other side, for a factor power $\mathcal{FP}_{\mathcal{N}}(G)$ its signature by definition corresponds to an antichain in $\hat{L}(G)$.

Consider now the question: which antichains correspond to the isomorphic factor powers? Let's fix a factor power $S = \mathcal{FP}_{\mathcal{N}}(G)$.

Let φ be some non-inner automorphism of G. Then φ maps $\mathcal{N}$ to some another signature $\varphi(G)$. Indeed, if $N \in \varphi(\mathcal{N})$, then $x^{-1}Nx = \varphi((\varphi^{-1}(x))^{-1}N'\varphi^{-1}(x))$, where N' is an element from $\mathcal{N}$ which is mapped to N by φ. Moreover, if $\varphi(N_1) \subseteq \varphi(N_2)$, then $\varphi^{-1}(\varphi(N_1)) \subseteq \varphi^{-1}(\varphi(N_2))$, i. e. $N_1 \subseteq N_2$, which is impossible since $\mathcal{N}$ is a signature.

It's easy to see that $\mathcal{FP}_{\varphi(\mathcal{N})}(G)$ is isomorphic to $\mathcal{FP}_{\mathcal{N}}(G)$.

Let now $S' = \mathcal{FP}_{\mathcal{N}'}(G)$ be isomorphic to S, φ be the isomorphism. Then, according to the previous Theorem 1 and Remark **??**, $\mathcal{N} = \varphi_m^{-1}(\mathcal{N}')$. φ_m^{-1} can be extended to an automorphism of the group G.

Consequently, let's introduce the *factor equivalency* of signatures and (which is the same) antichains in $\hat{L}(G)$: two elements are factor equivalent iff one can be mapped to another by a mapping corresponding to some element from $\operatorname{Out} G$.

Factor equivalency is, in turn, an equivalence relation on the set of all antichains on $\hat{L}(G)$. Denote $\tilde{L}(G)$ the quotient set by this relation.

Consider the following relation on $\tilde{L}$: a set of antichains $\mathfrak{A}$ *majorizes* set $\mathfrak{B}$ iff for each element $\mathfrak{a} \in \mathfrak{A}$ there exist such an element $\mathfrak{b} \in \mathfrak{B}$ that $\mathfrak{a} \succcurlyeq \mathfrak{b}$.

The set $\tilde{L}$ equipped with majorization becomes a poset. The proof is identical to the proof that "$\succcurlyeq$" is a partial order, only instead of using conjugacy class for proving antisymmetry we need to use antichain.

A congruence $\sim_1$ on a semigroup S is called a *refinement* of a congruence $\sim_2$, if $s_1 \sim_1 s_2$ implies $s_1 \sim_2 s_2$ for each pair of elements $s_1, s_2 \in S$. If moreover $\sim_1 \neq \sim_2$, the refinement is called *strict*. The factor semigroup $S/{\sim_1}$ is said to be an *extension* of the factor semigroup $S/{\sim_2}$ (denoted: $S/{\sim_1} \succ S/{\sim_2}$), if the congruence $\sim_1$ is a refinement of $\sim_2$. The relation "$\succ$" is a partial order on the set of all factor semigroups of the semigroup S.

Theorem 2. *There exists a one-to-one correspondence between the set of all non-isomorphic factor powers of the finite group G and the set $\tilde{L}(G)$ of all factor equivalency classes of antichains from the set $\hat{L}(G)$. This correspondence is consistent with the partial order on $\tilde{L}(G)$, that is, factor power S_1 is an extension of factor power S_2 iff the element of $\tilde{L}(G)$ corresponding to S_1 majorizes the element corresponding to S_2.*

The author is grateful to O.G. Ganyushkin (his scientific advisor) for stating the problem and many fruitful discussions about it, without which the article would have never been written.

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Received January 12, 2007