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# Radicals of Morita rings revisited

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**Abstract.** The radical of a Morita ring has been determined explicitly in terms of the radicals of the underlying base rings for radical classes which satisfy certain conditions. Here we again look at the radicals of Morita rings. But, in order to describe the radical of such a ring in terms of the underlying base rings, we rather exploit certain structural properties of Morita rings and weaken the requirements on the radical class.

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## 1 Introduction

Matrix rings play an important role in ring theory – both in structural considerations as well as a source of readily available and easy accessible examples demonstrating a wide variety of ring properties (or lack thereof). This example base has been extended to structural matrix rings (eg. upper triangular matrix rings) and also to the more general Morita rings.

Since the development of a general radical theory of rings in the early fifties by Kurosh and Amitsur, each of these ring constructions has received its due attention. This resulted in strong and satisfying results describing the radical of each of these ring constructions in terms of the underlying base ring(s).

As the ring construction becomes more general (matrix ring →structural matrix ring → Morita ring), the conditions imposed on the radical become more demanding. The weakest condition, common to all three cases, is the Matrix Extension Property. But even this requirement on the radical could be too restrictive. For example, it is not known if the nilradical satisfies the Matrix Extension Property (Köthe Conjecture); hence none of the results mentioned above can be used to describe the nilradical of the $2 \times 2$ upper triangular matrix ring $\begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$ over a ring $A$. However, structural properties of this matrix ring enable one to easily determine its radical.

This is true in general – inherent structural properties of a Morita ring may enable us to describe its radical without imposing too many conditions on the radical. And that is our purpose here: To describe the radicals of certain Morita rings exploiting inherent structral properties for fairly general radical classes. Typically this will be for Morita rings $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ in which $GH$ (or $HG$) degenerates. Our starting point is always that the radicals of the underlying base rings $L$ and $R$ are known and we want to use this to describe the radical of $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$. As is often the case, it will be necessary to distinguish between hypernilpotent radical classes and hypoidempotent radical classes.

## 2 Morita rings and their ideals

Ring will always mean associative ring, not necessarily commutative and not necessarily with an identity. For a ring $A$, $A^0$ will denote the ring with zero multiplication on $A^+$, the underlying group of the ring $A$. A Morita ring $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ is a $2 \times 2$ matrix ring given by

$$\begin{bmatrix} L & G \\ H & R \end{bmatrix} = \left\{ \begin{bmatrix} l & g \\ h & r \end{bmatrix} \mid l \in L, g \in G, h \in H \text{ and } r \in R \right\}$$

where $L$ and $R$ are rings, $G$ is an $L-R$-bimodule, $H$ is a $R-L$-bimodule and two products $G \times H \to L$ and $H \times G \to R$, are given with all products associative and distributive over addition (from both sides) whenever defined. The operations on the Morita ring are the usual matrix addition and multiplication.

For a ring $A$ and integer $n \geq 1$, $M_n(A)$ will denote the ring of all $n \times n$ matrices over $A$ and $M_n(A,\rho)$ will denote the $n \times n$ structural matrix ring over $A$ determined by $\rho$. This means $\rho$ is a transitive relation on the set $I_n = \{1, 2, 3, ..., n\}$ and $M_n(A,\rho) = \{[a_{ij}] \in M_n(A) \mid a_{ij} \in A, a_{ij} = 0 \text{ for all } (i,j) \notin \rho\}$. If $\rho_s := \{(i,j) \in \rho \mid (j,i) \in \rho\}$ and $\rho_a := \rho \backslash \rho_s$, then $M_n(A,\rho) = M_n(A,\rho_s) + M_n(A,\rho_a)$.

Any matrix ring is a structural matrix ring and any structural matrix ring can be regarded as a Morita ring in many different ways. For example, if $M_3(A,\rho) = \begin{bmatrix} A & A & A \\ 0 & A & A \\ 0 & 0 & A \end{bmatrix}$, then $M_3(A,\rho) = \begin{bmatrix} L & G \\ H & R \end{bmatrix} = \begin{bmatrix} L' & G' \\ H' & R' \end{bmatrix}$ where $L = A$, $G = \begin{bmatrix} A & A \end{bmatrix}$, $H = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $R = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$ and $L' = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$, $G' = \begin{bmatrix} A \\ A \end{bmatrix}$, $H' = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and $R' = A$.

More information on structural matrix rings and Morita rings rings can be found in the references relating to their radicals given in Section 3.

As is usual in ring theory, for two subsets $S$ and $T$ of a ring $A$, $ST$ will denote all finite sums of products with factors from $S$ and $T$, i.e. $ST = \{ \sum^{finite} s_i t_i \mid s_i \in S,$

$t_i \in T\}$. We will follow this convention also in relation to a Morita ring $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$, for example $G'H' = \{\sum^{finite} g_i h_i \mid g_i \in G', h_i \in H'\}$ for subsets $G' \subseteq G$ and $H' \subseteq H$. It can be verified that $GH$ is an ideal of the ring $L$ and $HG$ is an ideal of the ring $R$. Note that if $GH$ is a nilpotent ideal of $L$, then $HG$ is a nilpotent ideal of $R$ and conversely. To simplify terminology, an $L-R$-sub-bimodule of an $L-R$-bimodule will just be called an ideal and we will rely on the context within which it is used to clear any ambiguity. Ideals will be denoted by $\lhd$. For the Morita ring $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$, let $I \subseteq L, P \subseteq G, Q \subseteq H$ and $J \subseteq R$. Then $\begin{bmatrix} I & P \\ Q & J \end{bmatrix} \lhd \begin{bmatrix} L & G \\ H & R \end{bmatrix}$ if and only if $I \lhd L, P \lhd G, Q \lhd H$ and $J \lhd R$ with $GQ + PH \subseteq I, GJ + IG \subseteq P, HI + JH \subseteq Q$ and $HP + QG \subseteq J$. If $\begin{bmatrix} I & P \\ Q & J \end{bmatrix} \lhd \begin{bmatrix} L & G \\ H & R \end{bmatrix}$, then the quotient $\begin{bmatrix} L & G \\ H & R \end{bmatrix} \Big/ \begin{bmatrix} I & P \\ Q & J \end{bmatrix}$ is isomorphic to $\begin{bmatrix} L/I & G/P \\ H/Q & R/J \end{bmatrix}$ where all actions in the latter Morita ring are the canonical ones, eg. $L/I \times G/P \to G/P$ is given by $(l+L)(g+P) \mapsto lg+P$. Two particular cases will be important for our considerations: If $I \lhd L$ and $J \lhd R$, then $\begin{bmatrix} I & G \\ H & J \end{bmatrix} \lhd \begin{bmatrix} L & G \\ H & R \end{bmatrix}$ if and only if $GH \subseteq I$ and $HG \subseteq J$. In such a case,

$$\begin{bmatrix} L & G \\ H & R \end{bmatrix} \Big/ \begin{bmatrix} I & G \\ H & J \end{bmatrix} \cong \begin{bmatrix} L/I & 0 \\ 0 & R/J \end{bmatrix} \cong L/I \oplus R/J.$$

Secondly,

$$\begin{bmatrix} L & G \\ H & R \end{bmatrix} \Big/ \begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \cong L/GH \oplus R/HG.$$

An ideal $K$ of a Morita ring $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ is called *homogeneous* if it is of the form $K = \begin{bmatrix} I & P \\ Q & J \end{bmatrix}$ for $I, P, Q$ and $J$ as above. Not all ideals of a Morita ring need to be homogeneous (look, for example, at ideals in matrix rings). But there are instances when it will be the case. A Morita ring is called *unital* if both the rings $L$ and $R$ have an identity and all modules are unital. Then the Morita ring has an identity $\begin{bmatrix} 1_L & 0 \\ 0 & 1_R \end{bmatrix}$. As for matrix rings, all the ideals in a unital Morita ring are homogeneous.

Any Morita ring can be embedded as an ideal in a unital Morita ring. Indeed, let $L'$ and $R'$ be the canonical unital extensions (= Dorroh extension) of the rings $L$ and $R$ respectively. Then $\begin{bmatrix} L' & G \\ H & R' \end{bmatrix}$ is a unital Morita ring (all actions are the canonical ones). For example, $L' = (L, \mathbb{Z}) := \{(l, n) \mid l \in L, n \in \mathbb{Z}\}$ with componentwise addition and multiplication given by $(a,n)(b,m) = (ab+nb+ma, nm)$. Then $L \cong (L,0) := \{(l,0) \mid l \in L\} \lhd L'$ and $L' \times G \to G$ is defined by $(l,n)g := lg + ng$. It can be verified that

$$\begin{bmatrix} L & G \\ H & R \end{bmatrix} \lhd \begin{bmatrix} L' & G \\ H & R' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} L' & G \\ H & R' \end{bmatrix} \Big/ \begin{bmatrix} L & G \\ H & R \end{bmatrix} \cong L'/L \oplus R'/R \cong \mathbb{Z} \oplus \mathbb{Z}.$$

For a subset $X$ of $L, G, H$ or $R$ and each of $Y$ and $W$ one of $L, G, H$ or $R$, we define $Y^{-1}X := \{w \in W \mid Yw \subseteq X\}$ whenever the product $YW$ makes sense and is contained in $X$. Likewise for $XY^{-1} := \{w \in W \mid wY \subseteq X\}$. It can be shown that if $X$ is an ideal, then so are both $Y^{-1}X$ and $XY^{-1}$. For example, if $I$ is an ideal of the ring $L$, then $IH^{-1} := \{g \in G \mid gH \subseteq I\}$ is an ideal of $G$ and $G^{-1}I := \{h \in H \mid Gh \subseteq I\}$ is an ideal of $H$.

## 3 General radical theory

For completeness, we recall the most important radical theoretic definitions and results we will require. For a more comprehensive account of the radical theory of associative rings, Gardner and Wiegandt [2] can be consulted.

All radicals under discussion will be radical classes in the sense of Kurosh and Amitsur. This means a class of rings $\alpha$ is a *radical class* if and only if it is homomorphically closed, for every ring $A$ there is an ideal $\alpha(A)$ of $A$, called the *radical* of $A$, such that $\alpha(A) \in \alpha$ and if $I$ is any ideal of $A$ with $I \in \alpha$, then $I \subseteq \alpha(A)$ and lastly, $\alpha(\frac{A}{\alpha(A)}) = 0$ for all rings $A$. We will not distinguish between a radical class $\alpha$ and the mapping which assigns to each ring its radical $\alpha(A)$, i.e. $\alpha = \{$rings $A \mid \alpha(A) = A\}$. Any radical class $\alpha$ is *closed under extensions*, i.e. $I \lhd A$ with $I \in \alpha$ and $\frac{A}{I} \in \alpha$ implies $A \in \alpha$. All radical classes $\alpha$ have the *ADS property*, namely for any $I \lhd A$, it is always the case that $\alpha(I) \lhd A$. Associated with any radical class $\alpha$ is its *semisimple class* $S\alpha$ defined by $S\alpha := \{$rings $A \mid \alpha(A) = 0\}$. Semisimple classes are always *hereditary* (i.e. $I \lhd A \in S\alpha$ implies $I \in S\alpha$) and also closed under extensions. A property we frequently use is the following: If $\alpha$ is a radical class and $I \lhd A$ with $\frac{A}{I} \in S\alpha$, then $\alpha(A) \subseteq I$.

A radical class $\alpha$ is *hypernilpotent* if all nilpotent rings are radical, i.e. $\alpha(A) = A$ for any nilpotent ring $A$. A hereditary hypernilpotent radical class is called a *supernilpotent radical* class. A radical $\alpha$ is called *hypoidempotent* if all the nilpotent rings are semisimple, i.e. $\alpha(A) = 0$ for any nilpotent ring $A$. Concerning the latter radical classes, our interest will really be with the hereditary hypoidempotent radical classes; these radical classes are called *subidempotent radicals*.

All the well-known radical classes fall into one of these groups, but there are radical classes which are neither of these two types. Amongst the supernilpotent radical classes one will find the Brown-McCoy radical class, the Jacobson radical class, the nil radical class, the Levitzki radical class (= locally nilpotent radical class), the prime radical class, the strongly prime radical class, the completely prime radical class, etc. The class of von Neumann regular rings is a subidempotent radical class, as is the class of f-regular rings (= Blair radical class). The class of all idempotent rings and the class of all $\lambda-$regular rings are examples of non-hereditary hypoidempotent radicals.

A radical class $\alpha$ is called an $N-radical$ if it is hypernilpotent, left hereditary ($I$ a left ideal of $A \in \alpha$ implies $I \in \alpha$) and left strong ($I$ a left ideal of $A$ with $I \in \alpha$ implies $I \subseteq \alpha(A)$).

A radical $\alpha$ has the *Matrix Extension Property* if it satisfies: $A \in \alpha$ if and only if $M_n(A) \in \alpha$ for all $n \geq 1$. If $\alpha$ is a radical with this propoerty, then $\alpha(M_n(A)) = M_n(\alpha(A))$ for all rings $A$ and $n \geq 1$ (Amitsur [1], Snider [7], Propes [4]). For the radical theory of structural matrix rings, see van Wyk [8], Sands [6] and Veldsman [9]. But we recall: For a radical $\alpha$, the class $\alpha^+ := \{A \mid A^0 \in \alpha\}$ is a radical class. In fact, $\alpha^+$ is an $A$-radical (cf. Gardner and Wiegandt [2]) and $\alpha^+(A) = \alpha(A)$ for all nilpotent rings $A$. Sands [6] has shown that if $\alpha$ is a radical with the Matrix Extension Property and with $\alpha \subseteq \alpha^+$, then $\alpha(M_n(A,\rho)) = M_n(\alpha(A),\rho_s) + M_n(\alpha^+(A),\rho_a)$. In particular, if $\alpha$ is hypernilpotent with the Matrix Extension Property, then $\alpha(M_n(A,\rho)) = M_n(\alpha(A),\rho_s) + M_n(A,\rho_a)$.

The radicals of Morita rings for $N$-radicals and the more general normal radicals have been determined by Sands [5] and Jaegermann [3]. From the latter we know that $\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right)$ is a homogenous ideal of $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ for any radical $\alpha$. In the sequel, we will denote $\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right)$ by $\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} L_\alpha & G_\alpha \\ H_\alpha & R_\alpha \end{bmatrix}$. This means $L_\alpha, G_\alpha, H_\alpha$ and $R_\alpha$ are ideals of $L, G, H$ and $R$ respectively with

$$LL_\alpha + GH_\alpha \subseteq L_\alpha, \quad L_\alpha L + G_\alpha H \subseteq L_\alpha,$$

$$LG_\alpha + GR_\alpha \subseteq G_\alpha, \quad L_\alpha G + G_\alpha R \subseteq G_\alpha,$$

$$HL_\alpha + RH_\alpha \subseteq H_\alpha, \quad H_\alpha L + R_\alpha H \subseteq H_\alpha,$$

$$HG_\alpha + RR_\alpha \subseteq R_\alpha, \quad H_\alpha G + R_\alpha R \subseteq R_\alpha.$$

From this it can be shown that

$$GR_\alpha H \subseteq L_\alpha \quad \text{and} \quad HL_\alpha G \subseteq R_\alpha,$$

$$G_\alpha \subseteq L_\alpha H^{-1} \cap H^{-1} R_\alpha, \quad H_\alpha \subseteq R_\alpha G^{-1} \cap G^{-1} L_\alpha \ \text{ and}$$

$$L_\alpha / G_\alpha H_\alpha \in \alpha, \quad \text{and} \quad R_\alpha / H_\alpha G_\alpha \in \alpha.$$

**Proposition 1.** *Let $\alpha$ be a radical class. Let $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ be a Morita ring with $GH \subseteq \alpha(L)$ and $HG \subseteq \alpha(R)$. Then*

$$\begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix} \lhd \begin{bmatrix} L & G \\ H & R \end{bmatrix} \quad \textit{and} \quad \alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) \subseteq \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix}.$$

*If* $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \in \alpha$, *then* $\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix}$.

*Conversely, let* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *be a Morita ring with*

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix}.$$

*Then* $GH \subseteq \alpha(L), HG \subseteq \alpha(R)$ *and if* $\alpha$ *is hereditary, then* $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \in \alpha$.

**Proof.** Since $GH \subseteq \alpha(L)$ and $HG \subseteq \alpha(R)$, $\begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix} \lhd \begin{bmatrix} L & G \\ H & R \end{bmatrix}$ and $\begin{bmatrix} L & G \\ H & R \end{bmatrix} / \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix} \cong L/\alpha(L) \oplus R/\alpha(R) \in S\alpha$. Hence

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) \subseteq \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix}.$$

Suppose $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \in \alpha$. From $\begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix} \Big/ \begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \cong \alpha(L)/GH \oplus \alpha(R)/HG \in \alpha$ and the fact that radical classes are closed under extensions, we get $\begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix} \in \alpha$. Thus $\begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix} \subseteq \alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right)$ and hence $\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix}$.

Conversely, $\begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix} = \alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) \lhd \left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right)$ implies $GH \subseteq \alpha(L)$ and $HG \subseteq \alpha(R)$. Then $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \subseteq \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix} = \alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) \in \alpha$ and if $\alpha$ is hereditary, then $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \in \alpha$. □

## 4 Hypernilpotent radicals

We start with the main result of this section for hypernilpotent radicals:

**Proposition 2.** *Let* $\alpha$ *be a radical class. Then the following are equivalent:*

*(a)* $\alpha$ *is a hypernilpotent radical class.*

*(b) For all Morita rings* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *with* $GH$ *a nilpotent ideal of* $L$, $\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix}$.

*(c) For all Morita rings* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *with* $HG$ *a nilpotent ideal of* $R$, $\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix}$.

**Proof.** (a) $\Rightarrow$(b). Let $\alpha$ be a hypernilpotent radical class and let $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ be a Morita ring with $GH$ a nilpotent ideal of $L$. Then also $HG$ is a nilpotent ideal of $R$ and thus $GH \subseteq \alpha(L)$ and $HG \subseteq \alpha(R)$. For any $n \geq 1$, it can be shown that $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix}^{2n} \subseteq \begin{bmatrix} (GH)^n & (GH)^n G \\ H(GH)^n & (HG)^n \end{bmatrix}$. This means, if $(GH)^k = 0$, then $(HG)^{k+1} = 0$ and $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix}^{2(k+1)} = 0$. Hence $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \in \alpha$ and by Proposition 1 we get $\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix}$.

(b) $\Rightarrow$ (a). Let $A$ be a ring with $A^2 = 0$. Then $\begin{bmatrix} L & G \\ H & R \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}$ is a Morita ring with $GH = 0$. By (b) we have $A \cong \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} = \alpha\left(\begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix}\right) \in \alpha$. □

**Corollary 1.** *Let* $\alpha$ *be a hypernilpotent radical class and let* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *be a Morita ring with* $GH$ *nilpotent. Then:*

*(a)* $\begin{bmatrix} L & G \\ H & R \end{bmatrix} \in \alpha$ *if and only if* $L$ *and* $R$ *are in* $\alpha$,

*(b)* $\begin{bmatrix} L & G \\ H & R \end{bmatrix} \in S\alpha$ *if and only if* $L$ *and* $R$ *are in* $S\alpha$ *and* $G = H = 0$.

It can easily be verified that any one of the conditions mentioned in the corollary below implies that $HG$ is nilpotent; hence the stated result will follow.

**Corollary 2.** *Let* $\alpha$ *be a hypernilpotent radical class and let* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *be a Morita ring such that any one of the following conditions hold: For some* $n \geq 1$, $L^n G = 0$ *or* $GR^n = 0$ *or* $R^n H = 0$ *or* $HL^n = 0$. *Then* $\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix}$.

The next result describes the necessary interaction between the four components of the hypernilpotent radical of a Morita ring. We will also see that if one component is known, others may be determined from it. For example, if $L_\alpha$ is known, then so are $G_\alpha$ and $H_\alpha$.

**Proposition 3.** *Let $\alpha$ be a hypernilpotent radical class and let*

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} L_\alpha & G_\alpha \\ H_\alpha & R_\alpha \end{bmatrix}.$$

*Then:*

(a) *$L_\alpha$ and $R_\alpha$ are semiprime ideals of $L$ and $R$ respectively,*
(b) *$GR_\alpha H \subseteq L_\alpha$ and $HL_\alpha G \subseteq R_\alpha$,*
(c) *$G_\alpha = L_\alpha H^{-1} = H^{-1}R_\alpha$ and $H_\alpha = R_\alpha G^{-1} = G^{-1}L_\alpha$,*
(d) *$L_\alpha L^{-1} = L_\alpha = L^{-1}L_\alpha$ and $R_\alpha R^{-1} = R_\alpha = R^{-1}R_\alpha$,*
(e) *if $HG = R$, then $R_\alpha = \{r \in R \mid GrH \subseteq L_\alpha\}$ and if $GH = L$, then $L_\alpha = \{l \in R \mid HlG \subseteq R_\alpha\}$.*

**Proof.** Since $\alpha$ is a hypernilpotent radical,

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} L_\alpha & G_\alpha \\ H_\alpha & R_\alpha \end{bmatrix}$$

is a semiprime ideal of $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$. Let $x \in L$ with $xLx \subseteq L_\alpha$. Then

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} L & G \\ H & R \end{bmatrix}\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \subseteq \begin{bmatrix} L_\alpha & G_\alpha \\ H_\alpha & R_\alpha \end{bmatrix}$$

and thus

$$\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} L_\alpha & G_\alpha \\ H_\alpha & R_\alpha \end{bmatrix}.$$

Hence $x \in L_\alpha$. Likewise $R_\alpha$ is a semiprime ideal. From Section 3 we know $GR_\alpha H \subseteq L_\alpha$, $HL_\alpha G \subseteq R_\alpha$, $G_\alpha \subseteq L_\alpha H^{-1} \cap H^{-1}R_\alpha$ and $H_\alpha \subseteq= R_\alpha G^{-1} \cap G^{-1}L_\alpha$. We show $G_\alpha = L_\alpha H^{-1}$, the other equalities can be verified similarly. Let $x \in L_\alpha H^{-1}$. Then $xH \subseteq L_\alpha$ and so $xHx \subseteq L_\alpha x \subseteq L_\alpha G \subseteq$ . Thus

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} L & G \\ H & R \end{bmatrix}\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \subseteq \begin{bmatrix} L_\alpha & G_\alpha \\ H_\alpha & R_\alpha \end{bmatrix}$$

and so

$$\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} L_\alpha & G_\alpha \\ H_\alpha & R_\alpha \end{bmatrix}, \quad \text{i.e. } x \in G_\alpha.$$

Lastly we verify the second statement of (d): Suppose $GH = L$. From Section 3 we know $L_\alpha \subseteq \{l \in L \mid HlG \subseteq R_\alpha\}$. Let $T := \{l \in L \mid HlG \subseteq R_\alpha\}$. Then $T \lhd L$ and $T^3 \subseteq LTL = GHTGH \subseteq GR_\alpha H \subseteq L_\alpha$. Since $L_\alpha$ is a semiprime ideal of $L$, we have $T \subseteq L_\alpha$. Hence $L_\alpha = \{l \in L \mid HlG \subseteq R_\alpha\}$. □

It is known that $H^{-1}\alpha(R)$ is an $L - R$-ideal of $G$. This means

$$L(H^{-1}\alpha(R)) \subseteq H^{-1}\alpha(R) \quad \text{and} \quad (H^{-1}\alpha(R))R \subseteq H^{-1}\alpha(R).$$

Of course, $H(H^{-1}\alpha(R)) \subseteq \alpha(R)$ and the only remaining product $(H^{-1}\alpha(R))H$ still has to be described. Similar considerations hold for $\alpha(R)G^{-1}, \alpha(L)H^{-1}$ and $G^{-1}\alpha(L)$. When $\alpha$ is hypernilpotent, then $H^{-1}R_\alpha = L_\alpha H^{-1}$, so one would expect $(H^{-1}\alpha(R))H \subseteq \alpha(L)$. This, and more, is contained in:

**Proposition 4.** *Let $\alpha$ be a hypernilpotent radical class and let* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *be a Morita ring.*

*(a) The following five conditions are equivalent:*

*(1) $G\alpha(R)H \subseteq \alpha(L)$,*

*(2) $(H^{-1}\alpha(R))H \subseteq \alpha(L)$,*

*(3) $H^{-1}\alpha(R) \subseteq \alpha(L)H^{-1}$,*

*(4) $G(\alpha(R)G^{-1}) \subseteq \alpha(L)$,*

*(5) $\alpha(R)G^{-1} \subseteq G^{-1}\alpha(L)$;*

*(b) The following five conditions are equivalent:*

*(1) $H\alpha(L)G \subseteq \alpha(R)$,*

*(2) $H(\alpha(L)H^{-1}) \subseteq \alpha(R)$,*

*(3) $\alpha(L)H^{-1} \subseteq H^{-1}\alpha(R)$,*

*(4) $(G^{-1}\alpha(L))G \subseteq \alpha(R)$,*

*(5) $G^{-1}\alpha(L) \subseteq \alpha(R)G^{-1}$.*

**Proof.** We only verify (a); the equivalences in (b) can be shown likewise.

(1) $\Rightarrow$ (2). Suppose $G\alpha(R)H \subseteq \alpha(L)$. Let $g \in H^{-1}\alpha(R)$. Then $Hg \subseteq \alpha(R)$. For any $x \in L$ and $h \in H$, $(gh)x(gh) \in GHLgH \subseteq GHgH \subseteq G\alpha(R)H \subseteq \alpha(L)$. Since $\alpha(L)$ is a semiprime ideal of $L$, we get $gH \subseteq \alpha(L)$. Hence $(H^{-1}\alpha(R))H \subseteq \alpha(L)$.

(2) $\Rightarrow$ (3) is straightforward, as is (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(3) $\Rightarrow$ (4). Suppose $H^{-1}\alpha(R)H^{-1}$. Let $h \in \alpha(R)G^{-1}$. Then $hG \subseteq \alpha(R)$ and so $HGhG \subseteq R\alpha(R) \subseteq \alpha(R)$. Hence $GhGH \subseteq$ . Thus, for every $g \in G$ and $x \in L$, $(gh)x(gh) \in GhGH \subseteq \alpha(L)$. Since $\alpha(L)$ is a semiprime ideal, we get $Gh \subseteq \alpha(L)$ as required. □

**Corollary 3.** *Let $\alpha$ be a hypernilpotent radical class. Then*

$$\begin{bmatrix} \alpha(L) & H^{-1}\alpha(R) \\ G^{-1}\alpha(L) & \alpha(R) \end{bmatrix} = \begin{bmatrix} \alpha(L) & \alpha(L)H^{-1} \\ \alpha(R)G^{-1} & \alpha(R) \end{bmatrix} \lhd \begin{bmatrix} L & G \\ H & R \end{bmatrix}$$

*if and only if $H\alpha(L)G \subseteq \alpha(R)$ and $G\alpha(R)H \subseteq \alpha(L)$.*

Note that in general $\alpha(L)H^{-1} \subseteq G$ and $\alpha(L)H^{-1} = G$ if and only if $GH \subseteq L$ if and only if $G^{-1}\alpha(L) = H$. Likewise $H^{-1}\alpha(R) \subseteq G$ and $H^{-1}\alpha(R) = G$ if and only if $HG \subseteq \alpha(R)$ if and only if $H = \alpha(R)G^{-1}$. Because of its relevance here, we mention the following results from Sands [5] and Jaegermann [3]:

**Proposition 5.** *([3, 5]). The following are equivalent for a radical class $\alpha$ :*

*(a) $\alpha$ is an $N-radical$;*

*(b) for every Morita ring $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ with $HG = R$, $\alpha(R) = \{r \in R \mid GrH \subseteq \alpha(L)\}$;*

*(c) for every Morita ring $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$,*

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L) & H^{-1}\alpha(R) \\ G^{-1}\alpha(L) & \alpha(R) \end{bmatrix} = \begin{bmatrix} \alpha(L) & \alpha(L)H^{-1} \\ \alpha(R)G^{-1} & \alpha(R) \end{bmatrix}.$$

We conclude this section with an easy application. Let $\begin{bmatrix} L & G \\ H & R \end{bmatrix} = \begin{bmatrix} A & A \\ I & A \end{bmatrix}$ where $A$ is a ring and $I$ is a nilpotent ideal of $A$. Then $HG \subseteq I$ which is nilpotent and so, if $\alpha$ is a hypernilpotent radical, then

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L) & G \\ H & \alpha(R) \end{bmatrix} = \begin{bmatrix} \alpha(A) & A \\ I & \alpha(A) \end{bmatrix}$$

by Proposition 2. In the same way one can show that

$$\alpha\left(\begin{bmatrix} A & A & A \\ I & A & A \\ I & I & A \end{bmatrix}\right) = \begin{bmatrix} \alpha(A) & A & A \\ I & \alpha(A) & A \\ I & I & \alpha(A) \end{bmatrix}.$$

## 5 Hypoidempotent radicals

We start by fixing some notation. Let $I \lhd L$ where $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ is a Morita ring. Let

$$I(0) := 0_G G^{-1} \cap H^{-1} 0_H \cap I = \{i \in I \mid iG = 0 \text{ and } Hi = 0\}.$$

Likewise, for $J \lhd R$, let

$$J(0) := G^{-1} 0_G \cap 0_H H^{-1} \cap J = \{j \in J \mid Gj = 0 \text{ and } jH = 0\}.$$

It can be verified that $I(0) \lhd L$, $J(0) \lhd R$ and $\begin{bmatrix} I(0) & 0 \\ 0 & J(0) \end{bmatrix} \lhd \begin{bmatrix} L & G \\ H & R \end{bmatrix}$. In particular,

$$\begin{bmatrix} L(0) & 0 \\ 0 & R(0) \end{bmatrix} \lhd \begin{bmatrix} L & G \\ H & R \end{bmatrix}$$

and

$$\begin{bmatrix} L & G \\ H & R \end{bmatrix} \Big/ \begin{bmatrix} L(0) & 0 \\ 0 & R(0) \end{bmatrix} \cong \begin{bmatrix} L/L(0) & G \\ H & R/R(0) \end{bmatrix}.$$

Note that $(L/L(0))(0) = 0 = (R/R(0))(0)$.

**Proposition 6.** *Let* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *be a Morita ring with* $L(0) = 0 = R(0)$. *Then* $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix}$ *is an essential ideal of* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$.

**Proof.** Let $0 \neq K \lhd \begin{bmatrix} L & G \\ H & R \end{bmatrix}$ and suppose $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \cap K = 0$. Firstly note that $K \nsubseteq \begin{bmatrix} L & 0 \\ 0 & R \end{bmatrix}$. Indeed, if $K \subseteq \begin{bmatrix} L & 0 \\ 0 & R \end{bmatrix}$, let $0 \neq \begin{bmatrix} x & 0 \\ 0 & r \end{bmatrix} \in K$. Then

$$\begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & r \end{bmatrix} \subseteq K \cap \begin{bmatrix} GH & G \\ H & HG \end{bmatrix} = 0$$

and likewise

$$\begin{bmatrix} x & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} GH & G \\ H & HG \end{bmatrix} = 0.$$

Thus $x \in L(0) = 0$ and $r \in R(0) = 0$ – a contradiction. Hence $K \nsubseteq \begin{bmatrix} L & 0 \\ 0 & R \end{bmatrix}$, say $\begin{bmatrix} x & g \\ h & r \end{bmatrix} \in K$ with at least $g \neq 0$ or $h \neq 0$. Now

$$\begin{bmatrix} x & g \\ h & r \end{bmatrix} \begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix} \subseteq K \begin{bmatrix} GH & G \\ H & HG \end{bmatrix} = 0$$

and

$$\begin{bmatrix} 0 & G \\ H & 0 \end{bmatrix} \begin{bmatrix} x & g \\ h & r \end{bmatrix} \subseteq K \begin{bmatrix} GH & G \\ H & HG \end{bmatrix} K = 0.$$

Thus $xG = rH = Gr = Hx = 0$ and we conclude that $x = r = 0$. Thus

$$0 \neq \begin{bmatrix} 0 & g \\ h & 0 \end{bmatrix} \in K \cap \begin{bmatrix} GH & G \\ H & HG \end{bmatrix} = 0;$$

a contradiction. Hence $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix} \cap K \neq 0$. □

**Corollary 4.** *Let* $\alpha$ *be a subidempotent radical. Let* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *be a Morita ring with* $L(0) = 0 = R(0)$. *If* $GH$ *is nilpotent (or equivalently,* $HG$ *nilpotent), then* $\begin{bmatrix} L & G \\ H & R \end{bmatrix} \in S\alpha$.

**Proof.** *We have seen earlier that* $GH$ *nilpotent implies* $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix}$ *nilpotent. Hence* $\begin{bmatrix} GH & G \\ H & HG \end{bmatrix}$ *is a semisimple essential ideal of* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$. *Because* $S\alpha$ *is closed under essential extensions, we get* $\begin{bmatrix} L & G \\ H & R \end{bmatrix} \in S\alpha$. □

This brings us to our main result for subidempotent radicals.

**Proposition 7.** *Let* $\alpha$ *be a subidempotent radical and let* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *be a Morita ring with* $GH$ *is nilpotent. Then*

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L)(0) & 0 \\ 0 & \alpha(R)(0) \end{bmatrix}$$

*where* $\alpha(L)(0) = \{x \in \alpha(L) \mid xG = 0 = Hx\}$ *and* $\alpha(R)(0) = \{x \in \alpha(R) \mid Gx = 0 = xH\}$.

**Proof.** $\alpha(L)(0) \lhd L$ implies $\alpha(L)(0) \lhd \alpha(L) \in \alpha$ and so $\alpha(L)(0) \in \alpha$. Likewise $\alpha(R)(0) \in \alpha(R)$ and thus $\alpha(L)(0) \oplus \alpha(R)(0) \in \alpha$. Since

$$\alpha(L)(0) \oplus \alpha(R)(0) \cong \begin{bmatrix} \alpha(L)(0) & 0 \\ 0 & \alpha(R)(0) \end{bmatrix} \lhd \begin{bmatrix} L & G \\ H & R \end{bmatrix},$$

we get

$$\begin{bmatrix} \alpha(L)(0) & 0 \\ 0 & \alpha(R)(0) \end{bmatrix} \subseteq \alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right).$$

As mentioned earlier, $\begin{bmatrix} L/L(0) & G \\ H & R/R(0) \end{bmatrix}$ has $(L/L(0))(0) = 0 = (R/R(0))(0)$ and by Corollary 4 we have

$$\begin{bmatrix} L & G \\ H & R \end{bmatrix} \Big/ \begin{bmatrix} L(0) & 0 \\ 0 & R(0) \end{bmatrix} \cong \begin{bmatrix} L/L(0) & G \\ H & R/R(0) \end{bmatrix} \in S\alpha.$$

Hence

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) \subseteq \begin{bmatrix} L(0) & 0 \\ 0 & R(0) \end{bmatrix}$$

and so

$$\begin{bmatrix} \alpha(L)(0) & 0 \\ 0 & \alpha(R)(0) \end{bmatrix} \subseteq \alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) \subseteq \begin{bmatrix} L(0) & 0 \\ 0 & R(0) \end{bmatrix}.$$

Now $\alpha(L)(0) = L(0) \cap \alpha(L) = \alpha(L(0))$ by the hereditariness of $\alpha$ and likewise $\alpha(R)(0) = \alpha(R(0))$. Thus $L(0)/\alpha(L)(0) \oplus R(0)/\alpha(R)(0) \cong L(0)/\alpha(L(0)) \oplus R(0)/\alpha(R(0)) \in S\alpha$. This gives

$$\begin{bmatrix} L(0) & 0 \\ 0 & R(0) \end{bmatrix} \Big/ \begin{bmatrix} \alpha(L)(0) & 0 \\ 0 & \alpha(R)(0) \end{bmatrix} \cong L(0)/\alpha(L(0)) \oplus R(0)/\alpha(R(0)) \in S\alpha$$

and so

$$\alpha\left(\begin{bmatrix} L(0) & 0 \\ 0 & R(0) \end{bmatrix}\right) \subseteq \begin{bmatrix} \alpha(L)(0) & 0 \\ 0 & \alpha(R)(0) \end{bmatrix}.$$

We thus conclude with

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L)(0) & 0 \\ 0 & \alpha(R)(0) \end{bmatrix}. \qquad \square$$

The converse of this result is also true:

**Proposition 8.** *(a) Let* $\alpha$ *be a radical such that*

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L)(0) & 0 \\ 0 & \alpha(R)(0) \end{bmatrix}$$

*for all Morita rings* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *with* $GH$ *nilpotent. Then* $\alpha$ *is hypoidempotent.*

*(b) Let* $\alpha$ *be a hereditary radical class. Then*

$$\alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L)(0) & 0 \\ 0 & \alpha(R)(0) \end{bmatrix}$$

*for all Morita rings* $\begin{bmatrix} L & G \\ H & R \end{bmatrix}$ *with* $GH$ *nilpotent if and only if* $\alpha$ *is a subidempotent radical.*

**Proof.** We only need to verify the first part. Let $A$ be a ring with $A^2 = 0$. For the Morita ring $\begin{bmatrix} L & G \\ H & R \end{bmatrix} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \cong A$, we have

$$\alpha(A) = \alpha\left(\begin{bmatrix} L & G \\ H & R \end{bmatrix}\right) = \begin{bmatrix} \alpha(L)(0) & 0 \\ 0 & \alpha(R)(0) \end{bmatrix} = 0. \qquad \square$$

In [9] a procedure for describing the subidempotent radical of a structural matrix ring was derived. We conclude by showing that the above result for Morita rings simplifies this procedure considerably. Let $\alpha$ be a subidempotent radical and let

$$\begin{bmatrix} L & G \\ H & R \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A & A \end{bmatrix}$$

where $L = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \cong A$, $G = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $H = \begin{bmatrix} 0 & A \end{bmatrix}$ and $R = A$. Now $\alpha(L) = \begin{bmatrix} \alpha(A) & 0 \\ 0 & 0 \end{bmatrix}$ and $\alpha(R) = \alpha(A)$. Then

$$\alpha(L)(0) = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \,\middle|\, a \in \alpha(L), \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0 = \begin{bmatrix} 0 & A \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right\} = \begin{bmatrix} \alpha(A) & 0 \\ 0 & 0 \end{bmatrix}.$$

Futhermore,

$$\alpha(R)(0) = \left\{ a \in \alpha(A) \mid \begin{bmatrix} 0 \\ 0 \end{bmatrix} a = 0 = a \begin{bmatrix} 0 & A \end{bmatrix} \right\} = \{a \in \alpha(A) \mid aA = 0\}.$$

Since $\alpha(R)(0) \lhd \alpha(R) \in \alpha$, we get $\alpha(R)(0) \in \alpha$. Thus $\alpha(R)(0) = (\alpha(R)(0))^2$ and $\alpha(R)(0) = 0$ follows.

Hence

$$\alpha\left(\begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & A & A \end{bmatrix}\right) = \begin{bmatrix} \alpha(A) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

## References

[1] Amitsur S.A. *A general radical theory of rings, II.* Amer. J. Math., 1954, **76**, p. 100–125.

[2] Gardner B.J., Wiegandt R. *Radical theory of Rings.* Marcel Dekker Inc., USA, 2004.

[3] Jaegermann M. *Normal radicals.* Fund. Math., 1977, **95**, p. 147–155.

[4] Propes R.E. *The radical equation* $P(A_n) = (P(A))_n$,. Proc. Edinburgh Math. Soc., 1974/75, **19**, p. 257–259.

[5] Sands A.D. *Radicals and Morita contexts.* J. Algebra, 1973, **24**, p. 335–345.

[6] Sands A.D. R̂adicals of structural matrix rings. Quaest. Math., 1990, **13**, p. 77–81.

[7] Snider R.L. *Complemented hereditary radicals.* Bull. Austral. Math. Soc., 1971, **4**, p. 307–320.

[8] Van Wyk L. *Special radicals in structural matrix rings.* Comm. Alg., 1988, **16**, p. 421–435.

[9] Veldsman S. *On the radicals of structural matrix rings.* Monatshefte Math., 1996, **122**, p. 227–238.

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