BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(54), 2007, Pages 43–54
ISSN 1024–7696

On definitions of groupoids closely connected with quasigroups

V.A. Shcherbacov

Abstract. Both "existential" and "equational" definitions of binary quasigroups and groupoids closely connected with quasigroups are given. It is proved that a groupoid $(Q, \cdot)$ is a quasigroup if and only if all middle translations of $(Q, \cdot)$ are bijective maps of the set Q.

Mathematics subject classification: 20N05.
Keywords and phrases: Quasigroup, left quasigroup, right quasigroup, division groupoid, cancellation groupoid, translation.

1 Introduction

Historically one of the first definitions of binary quasigroups, left binary quasigroups and right binary quasigroups have been given by R. Moufang in language of the existence of solutions of some equations in [19] (see Definition 12). These "existencial" definitions are used at present since these definitions are quite short and convenient. See also article of W. Dörnte [7].

Later T. Evans has given an "equational" definition of a quasigroup [11] (see Definition 16). Evans' definition is usually used by the study of universal algebraic questions of quasigroup theory.

In this paper we give some new both "existential" and "equational" definitions of quasigroups and groupoids closely connected with quasigroups. Some other conditions when a groupoid is a quasigroup can be found in [6, 15, 17, 20].

For convenience of readers we start with some definitions which can be found in [1, 4, 12, 21].

A binary operation defined on a non-empty set Q is a map $A : Q^2 \longrightarrow Q$ such that $D(A) = Q^2$, i.e. this map is defined for any element of the set $Q \times Q$. Often are used more traditional symbols for the binary operation, for example, the expression $A(x, y) = z$ can be written in the form $x \cdot y = z$.

Definition 1. *A binary groupoid (G, A) is understood to be a non-empty set G together with a binary operation A.*

As usual the product of mappings is their consecutive realization. We shall use the following (left) order of multiplication of maps: if μ, ν are some maps, then $(\mu\nu)(x) = \mu(\nu(x))$. We recall the following definitions [12].

Definition 2. *A mapping $f : S \to T$ is onto or surjective if every $t \in T$ is the image under f of some $s \in S$; that is, if and only if, for given $t \in T$, there exists an $s \in S$ such that $t = f(s)$.*

Definition 3. *A mapping $f : S \to T$ is said to be one-to-one (written 1-1) or injective if for $s_1 \neq s_2$ in S, $f(s_1) \neq f(s_2)$ in T. Equivalently, f is 1-1 if $f(s_1) = f(s_2)$ implies $s_1 = s_2$.*

Definition 4. *A mapping $f : S \to T$ is said to be 1-1 correspondence or a bijection if f is both 1-1 and onto (i.e. f is an injective and surjective map).*

Let $(Q, \cdot)$ be a groupoid. As usual, the map $L_a : Q \to Q$, $L_a x = a \cdot x$ for all $x \in Q$, is a left translation of the groupoid $(Q, \cdot)$ relative to a fixed element $a \in Q$, the map $R_a : Q \to Q$, $R_a x = x \cdot a$, is a right translation.

We give the following definitions [1, 4, 13, 14, 21].

Definition 5. *A groupoid $(G, \cdot)$ is called a left cancellation groupoid, if the following implication is fulfilled: $a \cdot x = a \cdot y \Rightarrow x = y$ for all $a, x, y \in G$, i.e. the translation L_a is an injective map for any $a \in G$.*

Definition 6. *A groupoid $(G, \cdot)$ is called a right cancellation groupoid if the following implication fulfilled: $x \cdot a = y \cdot a \Rightarrow x = y$ for all $a, x, y \in G$, i.e. the translation R_a is an injective map for any $a \in G$.*

Definition 7. *A groupoid $(G, \cdot)$ is called a cancellation groupoid if it is both a left and a right cancellation groupoid.*

Example 1. Let $x \circ y = 2x + 3y$ for all $x, y \in Z$, where $(Z, +, \cdot)$ is the ring of integers. It is possible to check that $(Z, \circ)$ is a cancellation groupoid.

Definition 8. *A groupoid $(G, \cdot)$ is said to be a left (right) division groupoid if the mapping L_x (R_x) is surjective for every $x \in G$.*

Definition 9. *A groupoid $(G, \cdot)$ is said to be a division groupoid if it is simultaneously a left and right division groupoid.*

Example 2. Let $x \circ y = x^2 \cdot y^3$ for all $x, y \in \mathbb{C}$, where $(\mathbb{C}, +, \cdot)$ is the field of complex numbers. It is possible to check that $(\mathbb{C}, \circ)$ is a division groupoid.

Definition 10. *A groupoid $(Q, \circ)$ is called a right quasigroup (a left quasigroup) if, for all $a, b \in Q$, there exists a unique solution $x \in Q$ to the equation $x \circ a = b$ ($a \circ x = b$), i.e. in this case any right (left) translation of the groupoid $(Q, \circ)$ is a bijective map of the set Q.*

Definition 11. *A left and right quasigroup is called a quasigroup.*

Definition 12. *A binary groupoid (Q, A) with a binary operation A such that in the equality $A(x_1, x_2) = x_3$ the knowledge of any 2 elements of x_1, x_2, x_3 uniquely specifies the remaining one is called a binary quasigroup [3, 19].*

From Definition 12 it follows that with any quasigroup (Q, A) it is possible to associate $(3!-1) = 5$ more quasigroups, so-called parastrophes of quasigroup (Q, A): $A(x_1, x_2) = x_3 \Leftrightarrow A^{(12)}(x_2, x_1) = x_3 \Leftrightarrow A^{(13)}(x_3, x_2) = x_1 \Leftrightarrow A^{(23)}(x_1, x_3) = x_2 \Leftrightarrow A^{(123)}(x_2, x_3) = x_1 \Leftrightarrow A^{(132)}(x_3, x_1) = x_2$.

We shall denote:

the operation of (12)-parastrophe of a quasigroup $(Q, \cdot)$ by $*$;

the operation of (13)-parastrophe of a quasigroup $(Q, \cdot)$ by $/$;

the operation of (23)-parastrophe of a quasigroup $(Q, \cdot)$ by $\backslash$;

the operation of (123)-parastrophe of a quasigroup $(Q, \cdot)$ by $//$;

the operation of (132)-parastrophe of a quasigroup $(Q, \cdot)$ by $\backslash\backslash$.

2 Ternary relations, their translations and parastrophes

We have defined left and right translations of a groupoid and, therefore, of a quasigroup. But for quasigroups it is possible to define also the third kind of translations, namely, the map $P_a : Q \longrightarrow Q$, $x \cdot P_a x = a$ for all $x \in Q$ [2].

Unfortunately, in general, for groupoids a "middle" translation is not even a map. This fact and the wish to define "middle" translations of groupoids forces us to use by the study of translations of groupoids more or less standard language of ternary relations [6, 8]. Binary relations were used by the study of quasigroup translations in [22, 24].

It is known that any binary groupoid (Q, A) can be defined as the set $\mathfrak{T}(Q, A)$ of ordered triplets $(a_1, a_2, A(a_1, a_2))$, where $a_1, a_2 \in Q$. It is clear that the set $\mathfrak{T}(Q, A)$ is a ternary relation on the set Q. Binary groupoids (Q, A) and (Q, B) are equal if and only if $\mathfrak{T}(Q, A) = \mathfrak{T}(Q, B)$, where $\mathfrak{T}(Q, B)$ is the set of triplets of the groupoid (Q, B) [15, 16].

We shall denote an element from the set $\mathfrak{T}(Q, A)$ as $A(x_1, x_2, x_3)$, if we need to show which groupoid operation was used by obtaining the third component of a groupoid triplet.

We recall that an n-ary relation on a set Q is a subset of Q^n [6, 8].

Definition 13. *If $\mathfrak{T}(Q)$ is a ternary relation, then the first projection of relation $\mathfrak{T}(Q)$ is the following set $pr_1\mathfrak{T}(Q) = \{x_1 \mid (x_1, x_2, x_3) \in \mathfrak{T}(Q)\}$. By analogy we define the second and third projection of relation $\mathfrak{T}(Q)$: $pr_2\mathfrak{T}(Q) = \{x_2 \mid (x_1, x_2, x_3) \in \mathfrak{T}(Q)\}$, $pr_3\mathfrak{T}(Q) = \{x_3 \mid (x_1, x_2, x_3) \in \mathfrak{T}(Q)\}$ [8].*

Definition 14. *If $\mathfrak{T}(Q)$ is a ternary relation, then (12)-projection of relation $\mathfrak{T}(Q)$ is the following set $pr_{(12)}\mathfrak{T}(Q) = \{(x_1, x_2) \mid (x_1, x_2, x_3) \in \mathfrak{T}(Q)\}$. By analogy we define (13)- and (23)-projection of relation $\mathfrak{T}(Q)$: $pr_{(13)}\mathfrak{T}(Q) = \{(x_1, x_3) \mid (x_1, x_2, x_3) \in \mathfrak{T}(Q)\}$, $pr_{(23)}\mathfrak{T}(Q) = \{(x_2, x_3) \mid (x_1, x_2, x_3) \in \mathfrak{T}(Q)\}$.*

Theorem 1.
1. *A ternary relation $\mathfrak{T}(Q)$ defines a groupoid on the set Q if and only if there exist bijections between the sets $\mathfrak{T}(Q)$, $pr_{(12)}\mathfrak{T}(Q)$ and $Q \times Q$.*
2. *A ternary relation $\mathfrak{T}(Q)$ defines a left quasigroup on the set Q if and only if there exist bijections between the sets $\mathfrak{T}(Q)$, $pr_{(12)}\mathfrak{T}(Q)$, $pr_{(13)}\mathfrak{T}(Q)$ and $Q \times Q$.*

3. *A ternary relation $\mathfrak{T}(Q)$ defines a right quasigroup on the set Q if and only if there exist bijections between the sets $\mathfrak{T}(Q)$, $pr_{(12)}\mathfrak{T}(Q)$, $pr_{(23)}\mathfrak{T}(Q)$ and $Q\times Q$.*

4. *A ternary relation $\mathfrak{T}(Q)$ defines a quasigroup on the set Q if and only if there exist bijections between the sets $\mathfrak{T}(Q)$, $pr_{(12)}\mathfrak{T}(Q)$, $pr_{(13)}\mathfrak{T}(Q)$, $pr_{(23)}\mathfrak{T}(Q)$ and $Q \times Q$.*

Proof. *Case 1.* The existence of a bijection between the sets $pr_{(12)}\mathfrak{T}(Q)$ and $Q \times Q$ means that any pair $(x_1, x_2) \in Q \times Q$ lies in the set $pr_{(12)}\mathfrak{T}(Q)$. The existence of a bijection between the sets $\mathfrak{T}(Q)$ and $pr_{(12)}\mathfrak{T}(Q)$ means that there exists a unique triple in the set $\mathfrak{T}(Q)$ of the form (x_1, x_2, a) for any fixed pair $(x_1, x_2) \in pr_{(12)}\mathfrak{T}(Q)$.

Thus the existence of bijections between the sets $\mathfrak{T}(Q)$, $pr_{(12)}\mathfrak{T}(Q)$ and $Q \times Q$ means that for any element $(x, y) \in pr_{(12)}\mathfrak{T}(Q)$ there exists a unique element $a \in Q$ such that $(x, y, a) \in \mathfrak{T}(Q)$. Therefore the relation $\mathfrak{T}(Q)$ defines on the set Q a binary operation, i.e. by Definition 1 $\mathfrak{T}(Q)$ defines a groupoid.

Cases 2–4 are proved in the similar way.

Let $\mathfrak{T}(Q, A)$ be a set of triplets of a groupoid (Q, A). A ternary relation

$$\mathcal{L}_a = \{(a, x, y) \mid (a, x, y) \in \mathfrak{T}(Q, A)\},$$

where a is a fixed element of the set Q, will be called a *left relation translation of groupoid* (Q, A). Sometimes we shall use denotation $\mathcal{L}_a(Q, A)$ instead of $\mathcal{L}_a$.

Similarly a ternary relation

$$\mathcal{R}_a = \{(x, a, y) \mid (x, a, y) \in \mathfrak{T}(Q, A)\},$$

where a is a fixed element of the set Q, will be called a *right relation translation of groupoid* (Q, A); a ternary relation

$$\mathcal{P}_a = \{(x, y, a) \mid (x, y, a) \in \mathfrak{T}(Q, A)\},$$

where a is a fixed element of the set Q, will be called a *middle relation translation of groupoid* (Q, A).

Further we define inverse relations to the relations $\mathcal{L}_a$, $\mathcal{R}_a$ and $\mathcal{P}_a$ in the following way:

$$\begin{aligned} \mathcal{L}_a^{-1} &= \{(a, y, x) \mid (a, x, y) \in \mathcal{L}_a\}, \\ \mathcal{R}_a^{-1} &= \{(y, a, x) \mid (x, a, y) \in \mathcal{R}_a\}, \\ \mathcal{P}_a^{-1} &= \{(y, x, a) \mid (x, y, a) \in \mathcal{P}_a\}. \end{aligned}$$

We notice that only if a groupoid translation is a surjective or injective map, then this translation has an inverse map. See below Propositions 2 and 3.

We define the following correspondence between "usual" translations of a groupoid $(Q,\cdot)$ and its relation translations:

$$\begin{aligned}
&L_a x = y \Longleftrightarrow (a,x,y)\in \mathcal{L}_a,\\
&R_a x = y \Longleftrightarrow (x,a,y)\in \mathcal{R}_a,\\
&P_a x = y \Longleftrightarrow (x,y,a)\in \mathcal{P}_a\\
&L_a^{-1} x = y \Longleftrightarrow (a,x,y)\in \mathcal{L}_a^{-1},\\
&R_a^{-1} x = y \Longleftrightarrow (x,a,y)\in \mathcal{R}_a^{-1},\\
&P_a^{-1} x = y \Longleftrightarrow (x,y,a)\in \mathcal{P}_a^{-1}.
\end{aligned}$$

The sentence "a relation translation of a groupoid $(Q,\cdot)$ defines a map" will mean that the corresponding "usual" translation of $(Q,\cdot)$ is a map. We remember that usually a groupoid translation is a map, therefore the last four groupoid translations from previous table do not exist for a "general" groupoid. But all six kinds of the above mentioned translations are well defined in any quasigroup.

Lemma 1. *There exists a bijection between the set of all left translations $LT(Q,A)$ and the set of all left relation translations $LRT(Q,A)$ of a groupoid (Q,A).*

There exists a bijection between the set of all right translations $RT(Q,A)$ and the set of all right relation translations $RRT(Q,A)$ of a groupoid (Q,A).

Proof. 1. The correspondence $L_a x = y \Longleftrightarrow A(a,x) = y \Longleftrightarrow (a,x,y)\in\mathcal{L}_a$ for any fixed $a\in Q$ defines a bijection between the sets $LT(Q,A)$ and $LRT(Q,A)$.

2. The correspondence $R_a x = y \Longleftrightarrow (x,a,y)\in\mathcal{R}_a$ for any fixed $a\in Q$ defines a bijection between the sets $RT(Q,A)$ and $RRT(Q,A)$.

For a quasigroup there exist four more analogs of Lemma 1.

Using the "relation" approach we define for groupoids analogs of quasigroup parastrophes.

Definition 15. *With a ternary relation $\mathfrak{T}$ we associate the following ternary relations, so-called parastrophes of the relation $\mathfrak{T}$:*

$$(x_1,x_2,x_3)\in\mathfrak{T}\Leftrightarrow(x_2,x_1,x_3)\in\mathfrak{T}^{(12)}\Leftrightarrow(x_3,x_2,x_1)\in\mathfrak{T}^{(13)}\Leftrightarrow$$
$$(x_1,x_3,x_2)\in\mathfrak{T}^{(23)}\Leftrightarrow(x_2,x_3,x_1)\in\mathfrak{T}^{(123)}\Leftrightarrow(x_3,x_1,x_2)\in\mathfrak{T}^{(132)}.$$

For example, $(x_3,x_1,x_2)\in\mathfrak{T}^{(132)}\Leftrightarrow(x_1,x_2,x_3)\in\mathfrak{T}$: that is, $(x_{(132)1},\ x_{(132)2},\ x_{(132)3})\in\mathfrak{T}^{(132)}\Leftrightarrow(x_1,x_2,x_3)\in\mathfrak{T}$.

The following table contains connections between different kinds of relation translations in different parastrophes of a ternary relation $\mathfrak{T}(Q,\cdot)$. This table generalizes the corresponding table for translations of quasigroup parastrophes from [9, 23].

Table 1

$Kinds$	$\varepsilon = \cdot$	$(12) = *$	$(13) = /$	$(23) = \backslash$	$(123) = //$	$(132) = \backslash\backslash$
$\mathcal{R}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{R}^{-1}$	$\mathcal{P}$	$\mathcal{P}^{-1}$	$\mathcal{L}^{-1}$
$\mathcal{L}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{P}^{-1}$	$\mathcal{L}^{-1}$	$\mathcal{R}^{-1}$	$\mathcal{P}$
$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}^{-1}$	$\mathcal{L}^{-1}$	$\mathcal{R}$	$\mathcal{L}$	$\mathcal{R}^{-1}$
$\mathcal{R}^{-1}$	$\mathcal{R}^{-1}$	$\mathcal{L}^{-1}$	$\mathcal{R}$	$\mathcal{P}^{-1}$	$\mathcal{P}$	$\mathcal{L}$
$\mathcal{L}^{-1}$	$\mathcal{L}^{-1}$	$\mathcal{R}^{-1}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}$	$\mathcal{P}^{-1}$
$\mathcal{P}^{-1}$	$\mathcal{P}^{-1}$	$\mathcal{P}$	$\mathcal{L}$	$\mathcal{R}^{-1}$	$\mathcal{L}^{-1}$	$\mathcal{R}$

In Table 1, for example, $\mathcal{R}^{(23)} = \mathcal{P}^{(\cdot)}$.

Theorem 2. *1. A groupoid $(Q, \cdot)$ is a quasigroup if on the set Q there exist binary operations $/$ and $\backslash$ such that in the algebra $(Q, \cdot, /, \backslash)$ the following equivalences hold $x \cdot y = z \Longleftrightarrow z/y = x \Longleftrightarrow x\backslash z = y$ for all suitable $x, y, z \in Q$.*

2. A groupoid $(Q, \cdot)$ is a quasigroup if on the set Q there exist binary operations $//$ and $\backslash$ such that in the algebra $(Q, \cdot, //, \backslash)$ the following equivalences hold $x \cdot y = z \Longleftrightarrow y//z = x \Longleftrightarrow x\backslash z = y$ for all suitable $x, y, z \in Q$.

3. A groupoid $(Q, \cdot)$ is a quasigroup if on the set Q there exist binary operations $/$ and $\backslash\backslash$ such that in the algebra $(Q, \cdot, /, \backslash\backslash)$ the following equivalences hold $x \cdot y = z \Longleftrightarrow z/y = x \Longleftrightarrow z\backslash\backslash x = y$ for all suitable $x, y, z \in Q$.

Proof. Case 1. Since $(Q, \cdot)$ is a groupoid, then from Theorem 1 follows the existence of bijections between the sets $\mathfrak{T}(Q, \cdot)$, $pr_{(12)}\mathfrak{T}(Q, \cdot)$ and $Q \times Q$.

Since $(Q, /)$ is a groupoid and it is (13)-parastrophe of groupoid $(Q, \cdot)$, then there exists a bijection between the sets $pr_{(12)}\mathfrak{T}(Q, \cdot)$ and $pr_{(32)}\mathfrak{T}(Q, \cdot)$.

Since $(Q, \backslash)$ is a groupoid and it is (23)-parastrophe of groupoid $(Q, \cdot)$, then there exists a bijection between the sets $pr_{(12)}\mathfrak{T}(Q, \cdot)$ and $pr_{(13)}\mathfrak{T}(Q, \cdot)$.

Therefore, by Theorem 1 the groupoid $(Q, \cdot)$ is a quasigroup.

Cases 2, 3 are proved in the similar way.

Remark 1. In algebra $(Q, \cdot, \backslash, /)$ the equivalence $x \cdot y = z \Longleftrightarrow z/y = x$ is equivalent to the following quasiidentities $x \cdot y = z \Longrightarrow z/y = x$ and $z/y = x \Longrightarrow x \cdot y = z$. Other definition of quasigroup with the help of quasiidentities can be found in [17].

Theorem 3. *A groupoid $(Q, \cdot)$ is a quasigroup if and only if all middle translations of $(Q, \cdot)$ are bijective maps of the set Q.*

Proof. Let a, b be a pair of fixed elements of the set Q. Since the translation P_b of the groupoid $(Q, \cdot)$ is a bijective map, then we have that for the element a of set Q there exists a unique element $x \in Q$ such that $P_b a = x$, i.e. $a \cdot x = b$. Since all middle translations of $(Q, \cdot)$ are bijective maps, then for any fixed elements $a, b \in Q$ there exists a unique element x such that the equation $a \cdot x = b$ has a unique solution.

If a translation P_b is a bijective map, then the translation P_b^{-1} also is a bijective map. Further we have that for any fixed element a of the set Q there exists a unique element $y \in Q$ such that $P_b^{-1}a = y$, i.e. $y \cdot a = b$. Therefore the equation $y \cdot a = b$ has a unique solution in $(Q, \cdot)$ for any fixed elements $a, b \in Q$.

Converse. It is known that in a quasigroup all its middle translations are bijective mappings [2].

Here a square means Cayley table of a finite groupoid without bordering row and column. A Latin square is the inner part of Cayley table of a finite quasigroup. We notice that in [20] for squares an analog of Theorem 3 is proved.

Theorem 4. *A square $S(Q)$ defines an m-tuple of permutations of the kind p if and only if this square is a Latin square.*

3 Equational definitions of some classes of groupoids

Trevor Evans [11] defined a binary quasigroup as an algebra $(Q, \cdot, /, \backslash)$ with three binary operations. He has defined the following identities:

$$x \cdot (x \backslash y) = y, \tag{1}$$

$$(y/x) \cdot x = y, \tag{2}$$

$$x \backslash (x \cdot y) = y, \tag{3}$$

$$(y \cdot x)/x = y. \tag{4}$$

Definition 16. *An algebra $(Q, \cdot, \backslash, /)$ with identities (1), (2), (3) and (4) is called a quasigroup [1, 4–6, 11, 21].*

In [5] an algebra $(Q, \cdot, \backslash, /)$ with identities (1)–(4) is called an *e-quasigroup* (equational quasigroup). A.I. Mal'tsev [17, 18] has called this algebra a *primitive quasigroup*. The equivalence of Definitions 11 and 16 is well known [1, 5, 6, 18].

Using Table 1 it is easy to check that: if $(Q, \cdot)$ is a quasigroup, then in the algebra $(Q, \cdot, //)$ the following identities are fulfilled:

$$(x//y) \cdot x = y, \tag{5}$$

$$x//(y \cdot x) = y; \tag{6}$$

if $(Q, \cdot)$ is a quasigroup, then in the algebra $(Q, \cdot, \backslash\backslash)$ the following identities are fulfilled:

$$x \cdot (y \backslash\backslash x) = y, \tag{7}$$

$$(x \cdot y) \backslash\backslash x = y. \tag{8}$$

It is possible to rewrite identities (1)–(8) in the language of translations of the corresponding groupoids:

$$x \cdot (x \backslash y) = y \Longleftrightarrow L_x^{(\cdot)} L_x^{(\backslash)} y = y, \tag{9}$$

$$(y/x) \cdot x = y \Longleftrightarrow R_x^{(\cdot)} R_x^{(/)} y = y, \tag{10}$$

$$x\backslash(x\cdot y)=y \Longleftrightarrow L_x^{(\backslash)}L_x^{(\cdot)}y=y, \tag{11}$$
$$(y\cdot x)/x=y \Longleftrightarrow R_x^{(/)}R_x^{(\cdot)}y=y, \tag{12}$$
$$(x//y)\cdot x=y \Longleftrightarrow R_x^{(\cdot)}L_x^{(//)}y=y, \tag{13}$$
$$x//(y\cdot x)=y \Longleftrightarrow L_x^{(//)}R_x^{(\cdot)}y=y, \tag{14}$$
$$x\cdot(y\backslash\backslash x)=y \Longleftrightarrow L_x^{(\cdot)}R_x^{(\backslash\backslash)}y=y, \tag{15}$$
$$(x\cdot y)\backslash\backslash x=y \Longleftrightarrow R_x^{(\backslash\backslash)}L_x^{(\cdot)}y=y. \tag{16}$$

The situation with the existence and the uniqueness of a solution of an equation in a groupoid will be more clear if we give the following lemmas.

Lemma 2. *Let $f : A \to B$ be a map of non-empty sets. Then*

(i) the map f is surjective if and only if there exists a map $g : B \to A$ such that $gf = 1_A$;

(ii) the map f is injective if and only if there exists a map $g : B \to A$ such that $fg = 1_B$;

(iii) the map f is bijective if and only if there exists a map $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$ ([12], 1. Proposition.)

Let $f : A \to B$ be a map. A left inverse map to the map f is a map $g : B \to A$ such that $gf = 1_A$. A right inverse map to the map f is a map $h : B \to A$ such that $fh = 1_B$. A left and right inverse map is called an inverse map. It is known that every map has a unique inverse map. In general this is not true for left or right inverse maps.

Lemma 3. *Let $f : A \to B$ be a map of non-empty sets. Then*

(i) f is injective if and only if f has a left inverse map;

(ii) f is surjective if and only if f has a right inverse map;

(iii) f is bijective if and only if f has an inverse map. ([12], 2. Proposition.)

Lemma 4. *Let μ and ν be some maps of a non-empty set Q into itself. The product of mappings μ and ν is the identity map 1_Q of the set Q, i.e $\mu\nu = 1_Q$, if and only if the map ν is an injective map and the map μ is a surjective map.*

Proof. This lemma is a corollary of Lemma 2 and Lemma 3.

Lemma 5.
1. *In an algebra $(Q, \cdot, \backslash)$ with identity (1) the groupoid $(Q, \cdot)$ is a left division groupoid, the groupoid $(Q, \backslash)$ is a left cancellation groupoid.*
2. *In an algebra $(Q, \cdot, /)$ with identity (2) the groupoid $(Q, \cdot)$ is a right division groupoid, the groupoid $(Q, /)$ is a right cancellation groupoid.*
3. *In an algebra $(Q, \cdot, \backslash)$ with identity (3) the groupoid $(Q, \cdot)$ is a left cancellation groupoid, the groupoid $(Q, \backslash)$ is a left division groupoid.*
4. *In an algebra $(Q, \cdot, /)$ with identity (4) the groupoid $(Q, \cdot)$ is a right cancellation groupoid, the groupoid $(Q, /)$ is a right division groupoid.*

5. *In an algebra* $(Q,\cdot,//)$ *with identity (5) the groupoid* $(Q,\cdot)$ *is a right division groupoid, the groupoid* $(Q,//)$ *is a left cancellation groupoid.*

6. *In an algebra* $(Q,\cdot,//)$ *with identity (6) the groupoid* $(Q,\cdot)$ *is a right cancellation groupoid, the groupoid* $(Q,//)$ *is a left division groupoid.*

7. *In an algebra* $(Q,\cdot,\backslash\backslash)$ *with identity (7) the groupoid* $(Q,\cdot)$ *is a left division groupoid, the groupoid* $(Q,\backslash\backslash)$ *is a right cancellation groupoid.*

8. *In an algebra* $(Q,\cdot,\backslash\backslash)$ *with identity (8) the groupoid* $(Q,\cdot)$ *is a left cancellation groupoid, the groupoid* $(Q,\backslash\backslash)$ *is a right division groupoid.*

Proof. Case 1. In the language of groupoid translations we can rewrite identity (1) in the following form: $L_x^{(\cdot)}L_x^{(\backslash)}y = y$. From Lemma 4 it follows that the groupoid $(Q,\cdot)$ is a left division groupoid, the groupoid $(Q,\backslash)$ is a left cancellation groupoid.

Cases 2–8 are proved in the similar way.

Theorem 5. *1. A groupoid* $(Q,\cdot)$ *is a left division groupoid if and only if there exists a left cancellation groupoid* $(Q,\backslash)$ *such that in the algebra* $(Q,\cdot,\backslash)$ *identity (1) is fulfilled.*

2. *A groupoid* $(Q,\cdot)$ *is a right division groupoid if and only if there exists a right cancellation groupoid* $(Q,/)$ *such that in the algebra* $(Q,\cdot,/)$ *identity (2) is fulfilled.*

3. *A groupoid* $(Q,\cdot)$ *is a left cancellation groupoid if and only if there exists a left division groupoid* $(Q,\backslash)$ *such that in the algebra* $(Q,\cdot,\backslash)$ *identity (3) is fulfilled.*

4. *A groupoid* $(Q,\cdot)$ *is a right cancellation groupoid if and only if there exists a right division groupoid* $(Q,/)$ *such that in the algebra* $(Q,\cdot,/)$ *identity (4) is fulfilled.*

5. *A groupoid* $(Q,\cdot)$ *is a left quasigroup if and only if there exists a groupoid* $(Q,\backslash)$ *such that in the algebra* $(Q,\cdot,\backslash)$ *identities (1) and (3) are fulfilled.*

6. *A groupoid* $(Q,\cdot)$ *is a right quasigroup if and only if there exists a groupoid* $(Q,/)$ *such that in the algebra* $(Q,\cdot,/)$ *identities (2) and (4) are fulfilled.*

7. *A groupoid* $(Q,\cdot)$ *is a division groupoid if and only if there exist a right cancellation groupoid* $(Q,/)$ *and a left cancellation groupoid* $(Q,\backslash)$ *such that in the algebra* $(Q,\cdot,/,\backslash)$ *identities (1) and (2) are fulfilled.*

8. *A groupoid* $(Q,\cdot)$ *is cancellation groupoid if and only if there exist a right division groupoid* $(Q,/)$ *and a left division groupoid* $(Q,\backslash)$ *such that in the algebra* $(Q,\cdot,/,\backslash)$ *identities (3) and (4) are fulfilled.*

9. *A groupoid* $(Q,\cdot)$ *is a right division groupoid if and only if there exists a left cancellation groupoid* $(Q,//)$ *such that in the algebra* $(Q,\cdot,//)$ *identity (5) is fulfilled.*

10. *A groupoid $(Q, \cdot)$ is a right cancellation groupoid if and only if there exists a left division groupoid $(Q, //)$ such that in the algebra $(Q, \cdot, //)$ identity (6) is fulfilled.*

11. *A groupoid $(Q, \cdot)$ is a left division groupoid if and only if there exists a right cancellation groupoid $(Q, \backslash\backslash)$ such that in the algebra $(Q, \cdot, \backslash\backslash)$ identity (7) is fulfilled.*

12. *A groupoid $(Q, \cdot)$ is a left cancellation groupoid if and only if there exists a right division groupoid $(Q, \backslash\backslash)$ such that in the algebra $(Q, \cdot, \backslash\backslash)$ identity (8) is fulfilled.*

13. *A groupoid $(Q, \cdot)$ is a right quasigroup if and only if there exists a groupoid $(Q, //)$ (a left quasigroup $(Q, //)$) such that in the algebra $(Q, \cdot, //)$ identities (5) and (6) are fulfilled.*

14. *A groupoid $(Q, \cdot)$ is a left quasigroup if and only if there exists a groupoid $(Q, \backslash\backslash)$ such that in the algebra $(Q, \cdot, \backslash\backslash)$ identities (7) and (8) are fulfilled.*

15. *A groupoid $(Q, \cdot)$ is a division groupoid if and only if there exist a left cancellation groupoid $(Q, //)$ and a right cancellation groupoid $(Q, \backslash\backslash)$ such that in the algebra $(Q, \cdot, //, \backslash\backslash)$ identities (5) and (7) are fulfilled.*

16. *A groupoid $(Q, \cdot)$ is cancellation groupoid if and only if there exist a left division groupoid $(Q, //)$ and a right division groupoid $(Q, \backslash\backslash)$ such that in the algebra $(Q, \cdot, //, \backslash\backslash)$ identities (6) and (8) are fulfilled.*

Proof. Case 1. Let $(Q, \cdot)$ be a division groupoid. From Zermelo Theorem [10] it follows that every set can be done a well-ordered set.

On a well-ordered set Q any groupoid $(Q, \cdot)$ can be defined in a unique way with the help of its left translations $\{L_x^{(\cdot)} \mid x \in Q\}$. In any division groupoid every its left translation L_a, $a \in Q$, is a surjective map of the set Q.

By Lemma 3 every surjective map f has a right inverse map g (the map g is not defined in a unique way). By Lemma 4 the map g is an injective map. We can consider the map g as a left translation $L_a^{(\backslash)}$ of a partial groupoid $(Q, \backslash)$.

The set of all injective maps $\{L_a^{(\backslash)} \mid a \in Q\}$ with the property $L_a^{(\cdot)} L_a^{(\backslash)} = 1_Q$ defines (in a unique way) a cancellation groupoid $(Q, \backslash)$.

Converse. If in the algebra $(Q, \cdot, \backslash)$ identity (1) is fulfilled, then in $(Q, \cdot, \backslash)$ identity (9) also is true. From Lemma 4 it follows that any left translation $L_a^{(\cdot)}$ is a surjective map, any left translation $L_a^{(\backslash)}$ is an injective map. The set $\{L_a^{(\cdot)} \mid a \in Q\}$ defines in a unique way the division groupoid $(Q, \cdot)$, the set $\{L_a^{(\backslash)} \mid a \in Q\}$ defines in a unique way the cancellation groupoid $(Q, \backslash)$, if the set Q is a well-ordered set.

Cases 2–16 are proved in the similar way.

Corollary 1. *An algebra $(Q, \cdot, //, \backslash)$ with identities (1), (3), (5) and (6) is an e-quasigroup.*

An algebra $(Q,\cdot,/,\backslash\backslash)$ *with identities (2), (4), (7) and (8) is an e-quasigroup.*
An algebra $(Q,\cdot,//,\backslash\backslash)$ *with identities (5), (6), (7) and (8) is an e-quasigroup.*

Proof. The proof follows from Lemma 5.

In books of G. Birkhoff [5] and A.I. Mal'tsev [18] (see, also, J.D.H. Smith's article [25]) can be found the following e-quasigroup identities

$$(x/y)\backslash x = y, \tag{17}$$

$$x/(y\backslash x) = y. \tag{18}$$

Identities (5) and (6) are some analogs of identities (17) and (18). Indeed, if we change operation "\" in identities (17) and (18) by operation "·", then the operation "/" passes in the operation "//".

Identities (17) and (18) are easy obtained from conditions of Theorem 2. For example, if $z/y = x \Longleftrightarrow x\backslash z = y$, then $z/(x\backslash z) = x$ (18).

Lemma 6. *Identities (17) and (18) hold in any e-quasigroup* $(Q,\cdot,\backslash,/)$ *[5, 25].*

Proof. From Table 1 it follows that $L_x^{(/)} = P_x^{-1}$, $R_x^{(\backslash)} = P_x$. Therefore identity (17) is equivalent to the equality $P_x^{-1}P_x y = y$ and identity (18) can be written in the form $P_x P_x^{-1} y = y$. It is clear that the last two equalities hold in the quasigroup $(Q,\cdot,\backslash,/)$ for all $x, y \in Q$.

Acknowledgment: Author thanks V.I. Arnautov, J.D. Phillips, D.I. Pushkashu, E.A. Zamorzaeva and Referee for their helpful comments.

References

[1] Belousov V.D. *Foundations of the theory of quasigroups and loops.* Moscow, Nauka, 1967 (in Russian).

[2] Belousov V.D. *On group associated with a quasigroup.* Mat. Issled., 1969, **4**, N 3, p. 21–39 (in Russian).

[3] Belousov V.D. *n-Ary quasigroups.* Kishinev, Shtiintsa, 1971 (in Russian).

[4] Belousov V.D. *Elements of quasigroup theory: a special course.* Kishinev State University Printing House, 1981 (in Russian).

[5] Birkhoff G. *Lattice theory.* Moscow, Nauka, 1984 (in Russian).

[6] Burris S., Sankappanavar H.P. *A course in universal algebra.* Berlin, Springer-Verlag, 1981.

[7] Dörnte W. *Untersuchungen über einen veralgemeinerten Gruppenbegriff.* Math. Z., 1928, **29**, p. 1–19.

[8] Dudek W.A., Trokhimenko V.S. *Menger algebras of multy-place functions.* Chisinau, USM, 2006.

[9] Duplak J. *A parastrophic equivalence in quasigroups.* Quasigroups Relat. Syst., 2000, **7**, p. 7–14.

[10] Engelking R. *General topology.* Moscow, Mir, 1986 (in Russian).

[11] Evans T. *Abstract mean values.* Duke Math. J., 1963, **30**, p. 331–347.

[12] Faith C. *Algebra: rings, modules and categories, Vol. 1.* Berlin, Springer-Verlag, 1973.

[13] Ježek J., Kepka T. *Medial groupoids.* Rozpravy Československe Academie VĚD, Vol. 93, sešit 2, Academia, Praha, 1983.

[14] Ježek J., Kepka T., Nemec P. *Distributive groupoids.* Rozpravy Československe Academie VĚD, Vol. 91, sešit 3, Academia, Praha, 1981.

[15] Keedwell A.D., Shcherbacov V.A. *Quasigroups with an inverse property and generalized parastrophic identities.* Quasigroups Relat. Syst., 2005, **13**, p. 109–124.

[16] Lindner C.C. *Quasigroup identities and orthogonal arrays.* London Math. Soc., Lect. Note Ser., 1983, **82**, p. 77–105.

[17] Mal'tsev A.I. *Identical relations on varieties of quasigroups.* Mat. Sb., 1966, **69**, N1, p. 3–12 (in Russian).

[18] Mal'tsev A.I. *Algebraic systems.* Moscow, Nauka, 1976 (in Russian).

[19] Moufang R. *Zur Structur von alternativ Körpern.* Math. Ann., 1935, **110**, p. 416–430.

[20] Mullen G.L., Shcherbacov V.A. *On orthogonality of binary operations and squares.* Bul. Acad. Stiinte Repub. Mold., Mat., 2005, N 2(48), p. 3–42.

[21] Pflugfelder H.O. *Quasigroups and loops: Introduction.* Berlin, Heldermann Verlag, 1990.

[22] Shcherbacov V.A. *On congruences of quasigroups.* 1990, Reg. in VINITI 01.08.90, No 4413-B90, Moscow, 16 pages (in Russian).

[23] Shcherbacov V.A. *On automorphism groups and congruences of quasigroups.* Ph. D. thesis, IM AN MSSR, 1991 (in Russian).

[24] Shcherbacov V.A. *On Bruck-Belousov problem.* Bul. Acad. Stiinte Repub. Mold., Mat., 2005, N 3, p. 123–140.

[25] Smith J.D.H. *Homotopy and semisymmetry of quasigroups.* Algebra Universalis, 1997, **38**, p. 175–184.

Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
5 Academiei str., MD-2028 Chişinău
Moldova

E-mail: *scerb@math.md*

Received June 26, 2007