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Properties of accessible subrings of topological rings when taking quotient rings

V.I. Arnautov

Abstract. A continuous isomorphism of topological rings is a superposition of a finite number of semi-topological isomorphisms if and only if it is a narrowing on an accessible subring of some topological homomorphism.

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In the general algebra, and in particular, in the theory of rings and groups the following isomorphism theorem is widely applied:

Theorem 1. *If A is a subring (subgroup) of a ring (group) R and I is an ideal (a normal divisor) in R, then the quotient rings (quotient groups) $A/(A\bigcap I)$ and $(A+I)/I$ are isomorphic. In particular, if $A\bigcap I=0$ then the ring (group) A is isomorphic to the ring (group) $(A+I)/I$, i.e. the rings (groups) A and $(A+I)/I$ possess the same algebraic properties.*

Therefore in discrete algebra algebraic systems (in particulars, rings and groups) are studied up to isomorphism and sometimes isomorphic rings (groups) are identified.

As topological properties together with algebraic properties are considered when studying topological rings and topological groups then instead of isomorphisms of rings and groups, accordingly, it is necessary to consider isomorphism which also are homeomorphisms of topological spaces (in topological algebra such isomorphisms are called topological isomorphisms).

In view of this the specified isomorphism theorem for topological rings is not always true. This fact was studied in the works [1–3].

As is shown in Theorem 1 from [2] for topological rings and topological groups the isomorphism specified in Theorem 1 is continuous and, in general case, it is impossible to tell anything more about this isomorphism.

However, if we demand that the subring A should be an ideal (one-sided ideal) in the ring R or that the subgroup A sould be a normal divisor in the group R then for topological rings and topological groups the corresponding isomorphism possesses additional properties.

[*]By a topological ring we is understan a not necessarily associative ring equipped with topology, in which the rings operations are continuous.

The case when A is an ideal of a topological ring (R,τ) was studied in [1], the case when A is a normal divisor of a topological group (R,τ) was studied in [2], and the case when A is one-sided ideal of a topological ring (R,τ) was studied in [3].

In this connection in work [1] the following definition has been introduced:

Definition 1. *Let $\Re$ be some class of topological rings, (R,τ) and $(\widehat{R},\widehat{\tau})$ be topological rings from the class $\Re$. A continuous isomorphism $f:(R,\tau)\to(\widehat{R},\widehat{\tau})$ is called semitopological if there exists a topological ring $(\widetilde{R},\widetilde{\tau})$ in the class $\Re$ such that the following conditions are satisfied:*

– *R is an ideal in the ring $\widetilde{R}$;*

– *(R,τ) is a subring in the topological ring $(\widetilde{R},\widetilde{\tau})$, i.e. $\widetilde{\tau}|_R=\tau$;*

– *the isomorphism f can be extended up to a topological (i.e. continuous and open) homomorphism $\widetilde{f}:(\widetilde{R},\widetilde{\tau})\to(\bar{R},\bar{\tau})$.*

In work [1] it is proved the following:

Theorem 2. *If (R,τ) and $(\bar{R},\bar{\tau})$ are topological rings then a continuous isomorphism $\varphi:(R,\tau)\to(\bar{R},\bar{\tau})$ is a semitopological isomorphism in the class of all topological rings iff for any element $r\in R$ and any neighbourhood V of zero in (R,τ) there are neighbourhoods U and $\bar{U}$ of zero in (R,τ) and $(\bar{R},\bar{\tau})$ accordingly such that*

$$r\cdot\varphi{-}1(\bar{U})+\varphi{-}1(\bar{U})\cdot r+\varphi{-}1(\bar{U})\cdot U+U\cdot\varphi{-}1(\bar{U})\subseteq V.$$

Remark 1. In work [1] it has been shown that in the class of all topological rings a superposition of semitopological isomorphisms can not be a semitopological isomorphism. In connection with this there is a question:

What is the superposition of finite number of semitopological isomorphisms?

The present work is a continuation of works [1–3 and in it the case when A is an accessible subring (see Definition 2) of the topological ring (R,τ) is studied and it is shown that a ring isomorphism is a superposition of semitopological isomorphisms iff it is a narrowing on the accessible subring A of some topological homomorphism.

Definition 2. *As usual, a subring A of a rings R is called an accessible subring of the stage no more than n of the ring R if there exist a chain $A=R_0\subseteq R_1\subseteq R_2\subseteq\ldots\subseteq R_k=R$ of subrings of the ring R such that R_i is an ideal in R_{i+1} for $i=0,1,\ldots k-1$. Further we shall designate it as $A=R_0\lhd R_1\lhd R_2\lhd\ldots\lhd R_k=R$.*

Proposition 1. *Let:*

– *$(\widehat{R},\widehat{\tau})$ be a separable topological ring;*

– *R be an ideal in $\widehat{R}$;*

– *$\widehat{I}$ be a closed ideal in $(\widehat{R},\widehat{\tau})$ and $I=\widehat{I}\bigcap R$;*

– *$\widetilde{I}=[I]_{(\widehat{R},\widehat{\tau})}$ and $\widetilde{R}=R+\widetilde{I}$;*

– *$\bar{\varepsilon}:R/I\to(R+\widehat{I})/I$ be the natural embedding;*

– *$\widehat{\omega}:\widehat{R}\to\widehat{R}/I$ and $\widetilde{\omega}:\widehat{R}/I\to\widehat{R}/\widetilde{I}$ be canonical homomorphisms.*

Then $\widetilde{\omega}|_{R/I} : (R, \widehat{\tau}|_R)/I \to (\widetilde{R}, \widehat{\tau}|_{\widetilde{R}})/\widetilde{I}$ *is a topological isomorphism.*

Proof. Consider the diagram 1:

$$\begin{array}{ccccc}
R \subseteq & & \widetilde{R} = R + \widetilde{I} \subseteq & \widehat{R} \\
\widehat{\omega}|_R \downarrow & & \widehat{\omega}|_{\widetilde{R}} \downarrow & \widehat{\omega} \downarrow \\
R/I & \xrightarrow{\bar{\varepsilon}} & \widetilde{R}/I & \subseteq \widehat{R}/I \\
\| & & \widetilde{\omega}|_{\widetilde{R}/I} \downarrow & \widetilde{\omega} \downarrow \\
R/I & \xrightarrow{\widetilde{\omega}|_{R/I}} & \widetilde{R}/\widetilde{I} & \subseteq \widehat{R}/\widetilde{I}
\end{array}$$

As $\widetilde{\omega} : (\widehat{R}, \widehat{\tau})/I \to (\widehat{R}, \widehat{\tau})/\widetilde{I}$ is a continuous homomorphism then $\widetilde{\omega}|_{(R/I)}$: $(R, \widehat{\tau}|_R)/I \to (\widetilde{R}, \widehat{\tau}_{\widetilde{R}}/\widetilde{I}$ is a continuous isomorphism.

We show that it is an open isomorphism.

Let $\bar{V}$ be any neighbourhood of the zero in $(R, \widehat{\tau}|_R)/I$. Then $V = (\widehat{\omega}|_R){-}1(\bar{V})$ is a neighbourhood of the zero in $(R, \widehat{\tau}|_R)$ and then there exist neighbourhoods $\widehat{V}$ and $\widehat{V}_1$ of the zero in $(\widehat{R}, \widehat{\tau})$ such that $V = R \bigcap \widehat{V}$ and $\widehat{V}_1 + \widehat{V}_1 \subseteq \widehat{V}$. As $\widetilde{\omega} \circ \widehat{\omega} : (\widehat{R}, \widehat{\tau}) \to (\widehat{R}, \widehat{\tau})/\widetilde{I}$ is an open mapping then $\widetilde{V}_1 = \widetilde{\omega}(\widehat{\omega}(\widehat{V}_1))$ is a neighbourhood of the zero in $(\widehat{R}, \widehat{\tau})/\widetilde{I}$, and hence $\widetilde{V}_1 \bigcap \widetilde{R}/\widetilde{I}$ is a neighbourhood of the zero in $(\widetilde{R}, \widehat{\tau}|_{\widetilde{R}})/\widetilde{I}$.

Let's check that $\widetilde{R}/\widetilde{I} \bigcap \widetilde{V}_1 \subseteq \widetilde{\omega}(\bar{V})$. Really, if $\bar{r} \in \widetilde{V}_1 \bigcap \widetilde{R}/\widetilde{I}$ then there exists an element $\widetilde{r} \in \widehat{V}_1 \bigcap \widetilde{R}$ such that $\widetilde{\omega} \circ \widehat{\omega}(\widetilde{r}) = \bar{r}$. Then there are such elements $r \in R$ and $\widetilde{a} \in \widetilde{I}$ that $\widetilde{r} = r + \widetilde{a}$. As $\widetilde{I} = [I]_{(\widehat{R}, \widehat{\tau})} \subseteq I + \widehat{V}_1$ then

$$r = \widetilde{r} - \widetilde{a} \in \widehat{V}_1 + \widetilde{I} \subseteq I + \widehat{V}_1 + \widehat{V}_1 \subseteq I + \widehat{V},$$

and as $I \subseteq R$ then $r \in R \bigcap (I + \widehat{V}) = R \bigcap \widehat{V} = V$. Then $\bar{r} = \widetilde{\omega}(\widehat{\omega}(r)) \in \widetilde{\omega}(\widehat{\omega}(V)) = \widetilde{\omega}(\bar{V})$. Hence $(\widetilde{R}/\widetilde{I}) \bigcap \widetilde{V}_1 \subseteq \widetilde{\omega}(\bar{V})$, and $\widetilde{\omega}(\bar{V})$ is a neighborhood of the zero in $(\widetilde{R}, \widehat{\tau}|_{\widetilde{R}}$. Then from (see [4, Proposition 1.5.5]) it follows that the isomorphism $\widetilde{\omega}|_{R/I} : (R, \widehat{\tau}|_R)/I \to (\widetilde{R}, \widehat{\tau}|_{\widetilde{R}})/\widetilde{I}$ is open, and hence it is a topological isomorphism.

Theorem 3. *Let* $\Re$ *be one of following classes:*

– the class of all (separable) topological rings;

– the class of all (separable) topological commutative rings;

– the class of all (separable) topological rings possessing basis of neighborhood of the zero which consists of subgroups;

– the class of all (separable) topological commutative rings possessing basis neighborhood of the zero which consists of subgroups.

If (R, τ) , $(\bar{R}, \bar{\tau}) \in \Re$ *and* $\varphi : R \to \bar{R}$ *is a ring isomorphism then the following statements are equivalent:*

1. There exists a topological ring $(\widehat{R}, \widehat{\tau}) \in \Re$ *such that* (R, τ) *is an accessible subring of the stage not more than* n *of the topological ring* $(\widehat{R}, \widehat{\tau})$ *and the isomorphism* φ *can be extended up to a topological homomorphism* $\widehat{\varphi} : (\widehat{R}, \widehat{\tau}) \to (\bar{R}, \bar{\tau})$*;*

2. φ is a superposition of n semitopological isomorphisms, i.e. there exist topological rings

$$(R,\tau)=(R_0,\tau_0),(R_1,\tau_1),\ldots,(R_n,\tau_n)=(\bar{R},\bar{\tau})\in\Re$$

and semitopological isomorphisms $\varphi_i:(R_i,\tau_i)\to(R_{i+1},\tau_{i+1})$ for $i=0,1,\ldots,n-1$, such that $\varphi=\varphi_n\circ\varphi_{n-1}\circ\ldots\circ\varphi_0$.

Proof 1 $\Rightarrow$ 2.

Let $A=R_0\lhd R_1\lhd R_2\lhd\ldots\lhd R_k=R$ be a chain of subrings such that $\widehat{R}_i$ is an ideal in $\widehat{R}_{i+1}$ for $i=0,1,\ldots n-1$ and the isomorphism $\varphi:R\to\bar{R}$ can be extended to a topological homomorphism $\widehat{\varphi}:(\widehat{R},\widehat{\tau})\to(\bar{R},\bar{\tau})$.

If $\widehat{I}=\ker\widehat{\varphi}$ and $\widetilde{\omega}:R_{k+1}\to R_{k+1}/\widehat{I}$ is the canonical homomorphism (i.e. $\widetilde{\omega}(r)=r+\widehat{I}$), then there exists a topological isomorphism $\eta:(\widehat{R}_n,\widehat{\tau}_n)/\widehat{I}\to(\bar{R},\bar{\tau})$ such that $\widehat{\varphi}=\eta\circ\widetilde{\omega}$.

Let's consider the following diagram 2 (mappings entering into the diagram are defined below).

Diagram 2

$$\begin{array}{ccccccccccc}
R=\widehat{R}_0 & \lhd & \ldots & \lhd & \widehat{R}_k & \lhd & \widehat{R}_{k+1} & = & \widehat{R}_{k+1} & = & \widehat{R}\\
\Vert & & & & \omega|_{R_k}\downarrow & & \omega\downarrow & & & & \\
R & \xrightarrow{\varphi_0} & \ldots & \xrightarrow{\varphi_{k-1}} & \widehat{R}_k/I & \lhd & \widehat{R}_{k+1}/I & & \downarrow\widetilde{\omega} & & \downarrow\widehat{\varphi}\\
\varphi\downarrow & & & & \varphi_k\downarrow & & \widehat{\omega}\downarrow & & & & \\
\bar{R} & & = & & \widehat{R}_{k+1}/\widehat{I} & = & \widehat{R}_{k+1}/\widehat{I} & = & \widehat{R}_{k+1}/\widehat{I} & \xrightarrow{\eta} & \bar{R}
\end{array}$$

The further proof will be done by induction on number n.

If $n=1$ then (R,τ) is an accessible subring of the stage 1 (i.e. it is an ideal) of the topological ring $(\widehat{R},\widehat{\tau})$ and the isomorphism φ can be extended to a topological homomorphism $\widehat{\varphi}:(\widehat{R},\widehat{\tau})\to(\bar{R},\bar{\tau})$, and hence $\varphi:(R,\tau)\to(\bar{R},\bar{\tau})$ is a semitopological isomorphism.

Let's assume that the theorem is true for $n=k$, and let $n=k+1$.

In the beginning let's consider the case when $\Re$ is the class of all topological rings, i.e. the class $\Re$ contains topological rings which may be non-separable. Then

$$(\widehat{R}_k,\widehat{\tau}|_{R_k})/I\in\Re \text{ and } (\widehat{R}_{k+1},\widehat{\tau})/I\in\Re.$$

If $\omega:\widehat{R}_{k+1}\to\widehat{R}_{k+1}/I$ is the canonical homomorphism, then $\omega|_{\widehat{R}_k}:(\widehat{R}_k,\widehat{\tau}|_{\widehat{R}_k})\to(\widehat{R}_k,\widehat{\tau}|_{\widehat{R}_k})/I$ is a topological homomorphism. As

$$\widehat{R}_0\bigcap\ker\omega|_{\widehat{R}_k}=\widehat{R}_0\bigcap I=\widehat{R}_0\bigcap\widehat{I}=\widehat{R}_0\bigcap\ker\widehat{\varphi}=\ker\varphi=\{0\}$$

and

$$\widehat{R}_k = \widehat{R}_k \bigcap \widehat{R} = \widehat{R}_k \bigcap (R + \widehat{I}) = R + (\widehat{R}_k \bigcap \widehat{I}) = R + I$$

then $\omega|_{\widehat{R}_0} : \widehat{R}_0 \to \widehat{R}_k/I$ is an isomorphism and by the assumption $\omega|_{\widehat{R}_0}$ is a superposition of k semitopological isomorphisms, i.e. there are topological rings

$$(R, \tau) = (R_0, \tau_0), (R_1, \tau_1), \ldots, (R_k, \tau_k) = (R_k, \widehat{\tau}|_{R_k})/I$$

and topological isomorphisms $\varphi_i : (R_i, \tau_i) \to (R_{i+1}, \tau_{i+1})$ for $i = 0, 1, \ldots, k-1$, such that $\omega|_{\widehat{R}_0} = \varphi_{k-1} \circ \varphi_{k-2} \circ \ldots \circ \varphi_0$.

As $I = \widehat{I} \bigcap R_k = (\ker \widehat{\varphi}) \bigcap R_k = \ker(\widehat{\varphi}|_{R_k})$ then $\varphi'_k = \bar{\omega}|_{(\widehat{R}_k + \widetilde{I})/\widetilde{I}} : (\widehat{R}_k + \widetilde{I}, \widehat{\xi}_{\widehat{R}_k + \widetilde{I}})\widetilde{I} \to (\widehat{R}_{k+1}, \widehat{\xi})/\widehat{I}$ is a semitopological isomorphism. Then there exists a semitopological isomorphism $\varphi_k : (\widehat{R}_k, \widehat{\tau}|_{R_k})/I \to (\widehat{R}_{k+1}, \widehat{\tau})/\widehat{I}$ such that $\varphi_k = \widehat{\omega}$, and hence $\eta \circ \varphi_k : (\widehat{R}_k, \widehat{\tau}|_{R_k})/I \to (\bar{R}_{k+1}, \bar{\tau})$ is a semitopological isomorphism, and

$$\eta \circ \varphi_k \circ \varphi_{k-1} \circ \ldots \circ \varphi_0 = \eta \circ \varphi_k \circ \omega|_{\widehat{R}_0} = \eta \circ \widetilde{\omega}|_{R_0} = \widehat{\varphi}|_{R_0} = \varphi,$$

i.e. in this case (in the case when $\Re$ is a class of all topological rings) the isomorphism φ is a superposition of $k+1$ semitopological isomorphisms.

Let's consider now the case when all the topological rings entering in the class $\Re$ are separable. For this case we shall add the diagram 2 by one line (definitions of unknown by now rings and mappings see below).

$$\begin{array}{ccccccccc}
R = \widehat{R}_0 & \lhd & \ldots & \lhd & \widehat{R}_k & \lhd & \widehat{R}_{k+1} = \widehat{R}_{k+1} = & & \widehat{R} \\
\| & & & & \omega|_{\widehat{R}_k}\downarrow & & \omega\downarrow & & \\
R & \xrightarrow{\varphi_0} & \ldots & \xrightarrow{\varphi_{k-1}} & \widehat{R}_k/I & \lhd & \widehat{R}_{k+1}/I & & \\
 & & & & \bar{\eta}\downarrow & & \omega'\downarrow & & \\
\varphi\downarrow & & & & (\widehat{R}_k + \widetilde{I})/\widetilde{I} & \lhd & \widehat{R}_{k+1}/\widetilde{I} & \downarrow\widetilde{\omega} & \downarrow\widehat{\varphi} \\
 & & & & \varphi'_k\downarrow & & \bar{\omega}\downarrow & & \\
\bar{R} & & = & & \widehat{R}_{k+1}/\widehat{I} & & = \widehat{R}_{k+1}/\widehat{I} = \widehat{R}_{k+1}/\widehat{I} & \xrightarrow{\eta} & \bar{R}
\end{array}$$

As $\widehat{R}_k$ is an ideal in $\widehat{R}_{k+1}$, then $I = \widehat{I} \bigcap \widehat{R}_k$ is an ideal in $\widehat{R}$, and hence $\widetilde{I} = [\widehat{I} \bigcap \widehat{R}_k]_{(\widehat{R}, \widehat{\tau})}$ is a closed ideal in $(\widehat{R}, \widehat{\tau}) = (\widehat{R}_{k+1}, \widehat{\tau})$. Then $\widehat{R}_{k+1}/\widetilde{I}$ and $(\widehat{R}_k + \widetilde{I}, \widehat{\tau}|_{\widehat{R}_k + \widetilde{I}})/\widetilde{I} \in \Re$. If $\omega' : \widehat{R}/I \to \widehat{R}/\widetilde{I}$ and $\bar{\omega} : \widehat{R}/\widetilde{I} \to \widehat{R}/\widehat{I}$ are the canonical homomorphisms then $\widetilde{\omega} = \widehat{\omega} \circ \omega' \circ \omega$. As $(\widehat{R}_k + \widetilde{I})/\widetilde{I}$ is an ideal in $\widehat{R}_{k+1}/\widetilde{I}$ then

$$\varphi'_k = \bar{\omega}|_{(\widehat{R}_k + \widetilde{I})/\widetilde{I}} : (\widehat{R}_k + \widetilde{I}, \widehat{\tau}|_{\widehat{R}_k + \widetilde{I}})/\widetilde{I} \to (\widehat{R}_{k+1}, \widehat{\tau})/\widehat{I}$$

is a semitopological isomorphism.

According to Proposition 1 $\bar{\eta} = \omega'|_{(\widehat{R}_k/I)} : (\widehat{R}_k, \widehat{\tau}|_{\widehat{R}_k})/I \to (\widehat{R}_k + \widetilde{I}, \widehat{\tau}|_{\widehat{R}_k+\widetilde{I}})/\widetilde{I}$ is a topological isomorphism and hence

$$\varphi"_k = \eta \circ \varphi'_k \circ \bar{\eta} : (\widehat{R}_k, \widehat{\tau}|_{\widehat{R}_k})/I \to (\bar{R}, \bar{\tau})$$

is a semitopological isomorphism, and

$$\varphi = \widehat{\varphi}|_R = \eta \circ \widetilde{\omega}|_R = \eta \circ \bar{\omega} \circ \omega' \circ \omega|_R = \eta \circ \varphi'_k \circ \bar{\eta} \circ \omega|_R = \varphi"_k \circ \varphi_{k-1} \circ \ldots \circ \varphi_0,$$

i.e. the isomorphism φ is the superposition of $k+1$ semitopological isomorphisms.

Thus we have proved that 2 follows from 1 for any natural number n.

Proof 2 $\Rightarrow$ 1. Assume there are topological rings

$$(R, \tau) = (R_0, \tau_0), (R_1, \tau_1), \ldots, (R_n, \tau_n) = (\bar{R}, \bar{\tau})$$

and semitopological isomorphisms $\varphi_i : (R_i, \tau_i) \to (R_{i+1}, \tau_{i+1})$ for $i = 0, 1, \ldots, n-1$ such that φ is the superposition of these semitopological isomorphisms, i.e. $\varphi = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_0$. For any $0 \leq i \leq j \leq n$ we consider the isomorphism $f_{i,j} = \varphi_{j-1} \circ \ldots \circ \varphi_i : R_i \to R_j$. and $f_{i,i} : R_i \to R_i$ is the identical mapping.

The further proof will be done in some stages.

I. The construction of the ring $\widehat{R}$ and checking of some its algebraic properties.

Let's define on the set $\widehat{R} = \{(r_0, r_0, \ldots, r_n \mid r_i \in R_i, i = 0, 1, \ldots, n)\}$ the operations of addition and multiplication as follows:

$$(a_0, a_1, \ldots, a_n) + (b_0, b_1, \ldots, b_n) = (a_0 + b_0, a_1 + b_1, \ldots, a_n + b_n)$$

and

$$(a_0, a_1, \ldots, a_n) \cdot (b_0, b_1, \ldots, b_n) = (r_0, r_1, \ldots, r_n),$$

where $r_i = a_i \cdot b_i$ for $i \in \{0, n\}$ and $r_i = a_i \cdot b_i + (f_{0,i}(a_0) - a_i) \cdot \varphi_i{-}1(b_{i+1}) + \varphi_i{-}1(a_{i+1} \cdot (f_{0,i}(b_0) - b_i)$ for $0 < i < n$.

As the mappings $\varphi_i : R_i \to R_{i+1}$ and $f_{0,i} : R_0 \to R_i$ are isomorphism then it is easily checked, that:

I.1. $\widehat{R}$ is a non-associative ring with respect to these operations (even if the initial rings were associative);

I.2. If all rings $R_1, \ldots, R_n$ are commutative then the ring $\widehat{R}$ is commutative.

I.3. For any $0 \leq k < n$ the set $\widehat{R}_k = \{(r_0, \ldots, r_n) \in \widehat{R} \mid r_i = 0 \text{ if } i > k\}$ is an ideal in the ring $\widehat{R}_{k+1} = \{(r_0, \ldots, r_n) \in \widehat{R} \mid r_i = 0 \text{ if } i > k+1\}$;

I.4. $\widehat{R}_0 = \{(r_0, \ldots, r_n) \in \widehat{R} \mid r_i = 0 \text{ if } i \leq 1\}$ is an accessible ring of the stage no more than n in the ring $\widehat{R_n} = \widehat{R}$;

I.5. The mapping $\psi : \widehat{R_0} \to R_0 = R$ which transfers the element $(a, 0, \ldots, 0) \in \widehat{R_0}$ into the element $a \in R_0$ is isomorphic.

I.6. For the definition of the operations of addition and multiplication in $\widehat{R}$ it follow that $\widehat{I} = \{(0, r_1, \ldots r_n) \mid r_i \in R_i, i = 1, \ldots, n\}$ is an ideal in the ring $\widehat{R}$ and $\widehat{R}_0 \bigcap \widehat{I} = \{0\}$ and $\widehat{R}_0 + \widehat{I} = \widehat{R}$.

I.7. If $\widehat{\varphi} : \widehat{R} \to \bar{R}$ is a mapping such that $\widehat{\varphi}(r_0, \ldots, r_n) = \varphi(r_0)$ for any $(r_0, \ldots, r_n) \in \widehat{R}$, then $\widehat{\varphi} : \widehat{R} \to \bar{R}$ is a homomorphism of rings and $\ker \widehat{\varphi} = \widehat{I}$ and $\widehat{\varphi}|_R = \varphi$.

Identifying any elements $(a, 0, \ldots, 0) \in \widehat{R_0}$ with the element $a \in R_0$, we shall identify the ring $\widehat{R}_0$ with the ring R_0, therefore we can consider that $R = R_0$ is an accessible subring the stage no more than n of the ring $\widehat{R_n} = \widehat{R}$.

II. The definition of a ring topology $\widehat{\tau}$ in the ring $\widehat{R}$ and checking some properties of the topological ring $(\widehat{R}, \widehat{\tau})$.

For every $0 \leq i \leq n$ we shall designate by $\mathbf{B_i}$ the set of all neighborhoods of zero of the topological ring (R_i, τ_i).

Consider the set Ω of all sequences $\mathcal{V} = (V_0, V_1, \ldots, V_n)$ where $V_i \in \mathbf{B_i}$ for $0 \leq i \leq n$ and for every $\mathcal{V} = (V_0, V_1, \ldots, V_n) \in \Omega$ define $W(\mathcal{V}) = \{(r_0, \ldots, r_n) \mid r_n \in V_n$ and $r_i - \varphi{-}1_i(r_{i+1}) \in V_i$ for $0 \leq i < n\}$.

II.1. Let's check that the set $\{W(\mathcal{V}) \mid \mathcal{V} \in \Omega\}$ satisfies the conditions BN1–BN6 of Theorem 1.2.2 from [4], and hence by Theorem 1.2.5 from [4] it it is possible to take it for basis of neighbourhoods of zero in order to set some ring topology in the ring $\widehat{R}$.

As $0 \in V_i$ for any $V_i \in \mathbf{B_i}$, $0 \leq i \leq n$, then $(0, \ldots 0) \in W(\mathcal{V})$ for any $\mathcal{V} \in \Omega$, i.e. the condition BN1 is executed.

From the definition of sets $W(\mathcal{V})$ it follows that $W(\mathcal{V}) \subseteq W(\mathcal{V}')$ for any $\mathcal{V} = (V_0, V_1, \ldots, V_n)$ and $\mathcal{V}' = (V_0', V_1', \ldots, V_n')$ such that $V_i \subseteq V_i'$, $0 \leq i \leq n$. From this inclusions it follows that the condition BN2 is satisfied.

Let $\mathcal{V} = (V_0, V_1, \ldots, V_n) \in \Omega$. For every $0 \leq i \leq n$ there exists $V_i' \in \mathbf{B_i}$ such that $V_i' + V_i' \subseteq V_i$ and $-V_i' \subseteq V_i$. Then $\mathcal{V}' = (V_0', V_1', \ldots, V_n') \in \Omega$ and $W(\mathcal{V}') + W(\mathcal{V}') \subseteq W(\mathcal{V})$ and $-W(\mathcal{V}') \subseteq W(\mathcal{V})$, i.e. the conditions BN3 and BN4 are executed.

Let's check now the fulfilment of the condition BN5.

Let $\mathcal{V} = (V_0, V_1, \ldots, V_n) \in \Omega$. For every $0 \leq i \leq n$ there exists $V_i' \in \mathbf{B_i}$ such that $\underbrace{V_i' + \ldots + V_i'}_{n\text{items}} \subseteq V_i$. As for every $0 \leq i < n$ the isomorphism $\varphi_i : (R_i, \tau_i) \to (R_{i+1}, \tau_{i+1})$ is topological then according to Theorem 2 there exist neighborhoods of the zero $U_0, U_1, \ldots, U_n$ in $(R_0, \tau_0), (R_1, \tau_1), \ldots, (R_n, \tau_n)$ correspondingly such that $\varphi_i{-}1(U_{i+1}) \cdot U_i + U_i \cdot \varphi_i{-}1(U_{i+1}) + U_i \cdot U_i \subseteq V_i'$ and $U_n \cdot U_n \subseteq V_n'$. As the isomorphism $\varphi_i : (R_i, \tau_i) \to (R_{i+1}, \tau_{i+1})$ is continuous, then without loss of generality we can consider that $\varphi_i(U_i) \subseteq U_{i+1}$. Then from the definition of mappings $f_{i,j}$ it follows that $f_{i,j}(U_i) \subseteq U_j$ for any $0 \leq i < j \leq n$. Then $\mathcal{U} = (U_0, \ldots, U_n) \in \Omega$.

If $\widehat{a} = (a_0, \ldots, a_n) \in W(\mathcal{U})$ and $\widehat{b} = (b_0, \ldots, b_n) \in W(\mathcal{U})$ then $h_i = a_i - \varphi{-}1_i(a_{i+1}) \in U_i$ and $h_i' = b_i - \varphi{-}1_i(b_{i+1}) \in U_i$ for $0 \leq i < n$ and $h_n = a_n \in U_n$

and $h'_n = b_n \in U_n$. Taking in consideration the definitions of mappings $f_{i,j}$, by induction on of the number $j-i$ it is easily proved that

$$f_{i,j}(a_i) - a_j = f_{i,j}(a_i) - \varphi_j(\varphi{-}1_j(a_j)) =$$

$$= f_{i,j}(a_i - \varphi{-}1_i(a_{i+1})) + f_{i,j}(\varphi{-}1_i(a_{i+1})) - f_{j-1,j}(\varphi{-}1_{j-1}(a_j)) =$$

$$= f_{i,j}(h_i) + f_{i+1,j}(a_{i+1}) - f_{j-1,j}(\varphi{-}1_{j-1}(a_j)) = f_{i,j}(h_i) + f_{i+1,j}(h_{i+1}) + \ldots$$

$$\ldots + f_{j-1,j}(h_{j-1}) \in f_{i,j}(U_i) + f_{i+1,j}(U_{i+1}) + \ldots$$

$$\ldots + f_{j-1,j}(U_{j-1}) \subseteq \underbrace{U_j + U_j + \ldots + U_j}_{j-i \text{ items}} \quad \text{for any} \quad 0 \le i < j \le n.$$

If $r = (r_0, \ldots, r_n) = a \cdot b = (a_0, \ldots a_n) \cdot (b_0, \ldots, b_n)$, then:

$$r_n = a_n \cdot b_n \in U_n \cdot U_n \subseteq V'_n \subseteq V_n;$$

$$r_{n-1} - \varphi_{n-1}^{-1}(r_n) = a_{n-1} \cdot b_{n-1} + \big(f_{0,n-1}(a_0) - a_{n-1}\big) \cdot \varphi_{n-1}^{-1}(b_n) +$$

$$+ \varphi_{n-1}^{-1}(a_n) \cdot \big(f_{0,n-1}(b_0) - b_{n-1}\big) - \varphi_{n-1}^{-1}(a_n \cdot b_n) =$$

$$= (\varphi_{n-1}^{-1}(a_n) + h_{n-1}) \cdot (\varphi_{n-1}^{-1}(b_n) + h'_{n-1}) + \big(f_{0,n-1}(a_0) - a_{n-1}\big) \cdot \varphi_{n-1}^{-1}(b_n) +$$

$$+ \varphi_{n-1}^{-1}(a_n) \cdot \big(f_{0,n-1}(b_0) - b_{n-1}\big) - \varphi_{n-1}^{-1}(a_n \cdot b_n) \in \varphi_{n-1}^{-1}(a_n) \cdot h'_{n-1} +$$

$$+ h_{n-1} \cdot \varphi_{n-1}^{-1}(b_n) + h_{n-1} \cdot h'_{n-1} + \big(\underbrace{U_{n-1} + U_{n-1} + \ldots + U_{n-1}}_{n-1 \text{ items}}\big) \cdot \varphi_{n-1}{-}1(h'_n) +$$

$$+ \varphi_{n-1}^{-1}(h_n) \cdot \big(\underbrace{U_{n-1} + U_{n-1} + \ldots + U_{n-1}}_{n-1 \text{ items}}\big) \subseteq$$

$$\subseteq \varphi_{n-1}^{-1}(U_n) \cdot U_{n-1} + U_{n-1} \cdot \varphi_{n-1}^{-1}(U_n) + U_{n-1} \cdot U_{n-1} +$$

$$\underbrace{\big(U_{n-1} \cdot \varphi_{n-1}^{-1}(h'_n) + U_{n-1} \cdot \varphi_{n-1}^{-1}(h'_n) + \ldots + U_{n-1} \cdot \varphi_{n-1}^{-1}(h'_n)\big)}_{n-1 \text{ items}} +$$

$$+ \underbrace{\varphi_{n-1}^{-1}(U_n) \cdot U_{n-1} + \varphi_{n-1}^{-1}(U_n) \cdot U_{n-1} + \ldots + \varphi_{n-1}^{-1}(U_n) \cdot U_{n-1}}_{n-1 \text{ items}} \subseteq$$

$$\subseteq \underbrace{V'_{n-1} + V'_{n-1} + \ldots + V'_{n-1}}_{n \text{ items}} \subseteq V_{n-1};$$

$$r_i - \varphi_i^{-1}(r_{i+1}) = a_i \cdot b_i + \big(f_{0,i}(a_0) - a_i\big) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}(a_{i+1}) \cdot \big(f_{0,i}(b_0) - b_i\big) -$$

$$- \varphi_i^{-1}\Big(a_{i+1} \cdot b_{i+1} + \big(f_{0,i+1}(a_0) - a_{i+1}\big) \cdot \varphi_{i+1}^{-1}(b_{i+2}) + \varphi_{i+1}^{-1}(a_{i+2}) \cdot \big(f_{0,i+1}(b_0) - b_{i+1}\big)\Big) =$$

$$= \big(\varphi_i^{-1}(a_{i+1}) + h_i\big) \cdot \big(\varphi_i^{-1}(b_{i+1}) + h'_i\big) + \big(f_{0,i}(h_0) + \ldots + f_{i-1,i}(h_{i-1})\big) \cdot \varphi_i^{-1}(b_{i+1}) +$$

$$+ \varphi_i^{-1}(a_{i+1}) \cdot \big(f_{0,i}(h'_0) + \ldots + f_{i-1,i}(h'_{i-1})\big) - \varphi_i^{-1}(a_{i+1}) \cdot \varphi_i^{-1}(b_{i+1}) -$$

$$
\begin{aligned}
&-\varphi_i^{-1}\Big(\big(f_{0,i+1}(h_0)+\ldots+f_{i,i+1}(h_i\big)\cdot\varphi_{i+1}^{-1}(b_{i+2})-\varphi_{i+1}^{-1}(a_{i+2})\cdot\big(f_{0,i+1}(h'_0)+\ldots\\
&\ldots+f_{i,i+1}(h'_i))\Big)=\varphi_i^{-1}(a_{i+1})\cdot\varphi_i^{-1}(b_{i+1})+h_i\cdot\varphi_i^{-1}(b_{i+1})+\varphi_i^{-1}(a_{i+1})\cdot h'_i+h_i\cdot h'_i+\\
&+\big(f_{0,i}(h_0)+\ldots+f_{i-1,i}(h_{i-1})\big)\cdot\varphi_i-1(b_{i+1})+\varphi_i^{-1}(a_{i+1})\cdot\big(f_{0,i}(h'_0)+\ldots+f_{i-1,i}(h'_{i-1})\big)-\\
&-\varphi_i^{-1}(a_{i+1}))\cdot\varphi_i^{-1}(b_{i+1}))-\varphi_i^{-1}\Big(\big(f_{0,i+1}(h_0)+\ldots+f_{i,i+1}(h_i)\Big)\cdot\varphi_{i+1}^{-1}(b_{i+2})-\\
&-\varphi_{i+1}^{-1}(a_{i+2})\cdot\big(f_{0,i+1}(h'_0)+\ldots+f_{i,i+1}(h'_i))\Big)=\\
&=\varphi_i^{-1}(a_{i+1})\cdot\varphi_i-1(b_{i+1})+h_i\cdot\varphi_i^{-1}(b_{i+1})+\varphi_i^{-1}(a_{i+1})\cdot h'_i+h_i\cdot h'_i+\\
&+\big(f_{0,i}(h_0)+\ldots+f_{i-1,i}(h_{i-1})\big)\cdot\varphi_i^{-1}(b_{i+1})+\varphi_i^{-1}(a_{i+1})\cdot\big(f_{0,i}(h'_0)+\ldots+f_{i-1,i}(h'_{i-1})\big)-\\
&-\varphi_i^{-1}(a_{i+1})\cdot\varphi_i^{-1}(b_{i+1})-\big(f_{0,i}(h_0)+\ldots+f_{i-1,i}(h_{i-1})+h_i\big)\cdot\varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2}))-\\
&-\varphi_i^{-1}(\varphi_{i+1}^{-1}(a_{i+2}))\cdot\big(f_{0,i}(h'_0)+\ldots+f_{i-1,i}(h'_{i-1})+h'_i\big)=\\
&=h_i\cdot\varphi_i-1\big(b_{i+1}-\varphi_{i+1}^{-1}(b_{i+2})\big)+\varphi_i^{-1}\big(a_{i+1}-\varphi_{i+1}^{-1}(a_{i+2})\big)\cdot h'_i+h_i\cdot h'_i+\\
&+\big(f_{0,i}(h_0)+\ldots+f_{i-1,i}(h_{i-1})\big)\cdot\varphi_i^{-1}\big(b_{i+1}-\varphi_{i+1}^{-1}(b_{i+2})\big)+\\
&+\varphi_i^{-1}\big(a_{i+1}-\varphi_{i+1}^{-1}(a_{i+2})\big)\cdot\big(f_{0,i}(h'_0)+\ldots+f_{i-1,i}(h'_{i-1})\big)\in\\
&\in U_i\cdot\varphi_i^{-1}(U_{i+1})+\varphi_i^{-1}(U_{i+1})\cdot U_i+U_i\cdot U_i+\big(f_{0,i}(U_0)+\ldots+f_{i-1,i}(U_{i-1})\big)\cdot\varphi_i^{-1}(U_{i+1})+\\
&+\varphi_i^{-1}(U_{i+1})\cdot\big(f_{0,i}(U_0)+\ldots+f_{i-1,i}(U_{i-1})\big)\subseteq\\
&\subseteq U_i\cdot\varphi_i^{-1}(U_{i+1})+\varphi_i^{-1}(U_{i+1})\cdot U_i+U_i\cdot U_i+\\
&+\underbrace{U_i\cdot\varphi_i-1(U_{i+1})+\ldots+U_i\cdot\varphi_i^{-1}(U_{i+1})}_{i\text{items}}+\underbrace{\varphi_i^{-1}(U_{i+1})\cdot U_i+\ldots+\varphi_i^{-1}(U_{i+1})\cdot U_i}_{i\text{ items}}\subseteq\\
&\subseteq\underbrace{V'_i+\ldots+V'_i}_{n\text{items}}\subseteq V_i\quad\text{for } 0<i<n;
\end{aligned}
$$

$$
\begin{aligned}
&r_0-\varphi-1_0(r_1)=a_0\cdot b_0-\varphi-1_0\big(a_1\cdot b_1+(f_{0,1}(a_0)-a_1)\cdot\varphi_1^{-1}(b_2)+\\
&+\varphi_1^{-1}(a_2)\cdot(f_{0,1}(b_0)-b_1)\big)=(\varphi_0^{-1}(a_1)+h_0)\cdot(\varphi_0^{-1}(b_1)+h'_0)-\varphi_0^{-1}(a_1\cdot b_1)-\\
&-\big(a_0-\varphi-1_0(a_1)\big)\cdot\varphi_0^{-1}(\varphi_1^{-1}(b_2))-\varphi_0^{-1}(\varphi-1_1(a_2))\cdot\big(b_0-\varphi_0^{-1}(b_1)\big)=\\
&=\varphi_0^{-1}(a_1\cdot b_1)+h_0\cdot\varphi_0^{-1}(b_1)+\varphi_0^{-1}(a_1)\cdot h'_0+h_0\cdot h'_0-\\
&-\varphi_0^{-1}(a_1\cdot b_1)-h_0\cdot\varphi_0^{-1}(\varphi_1^{-1}(b_2))-\varphi_0^{-1}(\varphi_1^{-1}(a_2))\cdot h'_0=h_0\cdot\varphi_0^{-1}\big(b_1-\varphi_1^{-1}(b_2)\big)+\\
&+\varphi_0^{-1}\big(a_1-\varphi_1^{-1}(a_2)\big)\cdot h'_0+h_0\cdot h'_0\in U_0\cdot\varphi-1_0(U_1)+\varphi-1_0(U_1)\cdot U_0+U_0\cdot U_0\subseteq V'_0\subseteq V_0.
\end{aligned}
$$

Thus, we have obtained that $r_i-\varphi-1_i(r_{i+1}\in V_i)$ for $0\le i<n$ and $r_n\in V_n$, i.e. $\widehat{r}=\widehat{a}\cdot\widehat{b}=(r_0,\ldots,r_n)\in W(\mathcal{V})$. As the elements $\widehat{a}$ and $\widehat{b}$ are arbitrary elements then $\mathcal{U}\cdot\mathcal{U}\subseteq\mathcal{V}$, i.e. the condition BN5 is executed.

Let's check now that the condition BN6 is satisfied.

Let $\mathcal{V} = (V_0, V_1, \dots, V_n) \in \Omega$ and $\widehat{b} = (b_0, \dots, b_n) \in \widehat{R}$. For everyone $0 \le i \le n$ there exists $V_i' \in \mathbf{B_i}$ such that $-V_i' = V_i'$ and $\underbrace{V_i' + \ldots + V_i'}_{2n+4 \text{ items}} \subseteq V_i$.

As for every $0 \le i < n$ the isomorphism $\varphi_i : (R_i, \tau_i) \to (R_{i+1}, \tau_{i+1})$ is semi-topological, then according to Theorem 2 there exist neighbourhoods of the zero $U_0, U_1, \dots, U_n$ in $(R_0, \tau_0), (R_1, \tau_1), \dots, (R_n, \tau_n)$ correspondingly such that

$$\varphi_i^{-1}(U_{i+1}) \cdot \{f_{0,i}(b_0), b_i, \varphi_i^{-1}(b_{i+1}), \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2}))\} +$$

$$+ \{f_{0,i}(b_0), b_i, \varphi_i^{-1}(b_{i+1}), \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2}))\} \cdot \varphi_i^{-1}(U_{i+1}) \subseteq V_i'$$

for $0 < i < n$ and $U_n \cdot b_n + b_n \cdot U_n \subseteq V_n'$.

As the isomorphism $\varphi_i : (R_i, \tau_i) \to (R_{i+1}, \tau_{i+1})$ is continuous then (without loss of generality) we can consider that $\varphi_i(U_i) \subseteq U_{i+1}$. From the definition of mappings $f_{i,j}$ it follows that $f_{i,j}(U_i) \subseteq U_j$ for any $0 \le i < j \le n$.

Then $\mathcal{U} = (U_0, U_1, \dots, U_n) \in \Omega$. If $\widehat{a} = (a_0, \dots, a_n) \in W(\mathcal{U})$ then $h_i = a_i - \varphi_i^{-1}(a_{i+1}) \in U_i$ for $0 \le i < n$ and $h_n = a_n \in U_n$. As it was done by checking the condition BN5 it is proved that

$$f_{i,j}(a_i) - a_j = f_{i,j}(h_i) + f_{i+1,j}(h_{i+1}) + \ldots + f_{j-1,j}(h_{j-1}) \in$$

$$\in f_{i,j}(U_i) + f_{i+1,j}(U_{i+1}) + \ldots + f_{j-1,j}(U_{j-1}) \subseteq \underbrace{U_j + U_j + \ldots + U_j}_{j-i \text{ items}}$$

for any $0 \le i < j \le n$.

If $\widehat{a} \cdot \widehat{b} = (r_0, \dots, r_n)$, then:

$$r_n = a_n \cdot b_n \in U_n \cdot b_n \subseteq V_n' \subseteq V_n;$$

$$r_i - r_{i+1} = a_i \cdot b_i + (f_{0,i}(a_0) - a_i)) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi^{-1}(a_{i+1}) \cdot (f_{0,i}(b_0) - b_i)) -$$

$$-\varphi_i^{-1}\big(a_{i+1} \cdot b_{i+1} + (f_{0,i+1}(a_0) - a_{i+1}) \cdot \varphi_{i+1}^{-1}(b_{i+2}) + \varphi_{i+1}^{-1}(a_{i+2}) \cdot (f_{0,i+1}(b_0) - b_{i+1})\big) =$$

$$= a_i \cdot b_i + \big(f_{0,i}(a_0) - a_i\big) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}(a_{i+1}) \cdot f_{0,i}(b_0) - \varphi_i^{-1}(a_{i+1}) \cdot b_i -$$

$$-\varphi_i^{-1}(a_{i+1}) \cdot \varphi_i^{-1}(b_{i+1}) - \varphi_i^{-1}(f_{0,i+1}(a_0)) \cdot \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2})) +$$

$$+\varphi_i^{-1}(a_{i+1}) \cdot \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2})) - \varphi_i^{-1}(\varphi_{i+1}^{-1}(a_{i+2})) \cdot \varphi_i^{-1}(f_{0,i+1}(b_0)) +$$

$$+\varphi_i^{-1}(\varphi_{i+1}^{-1}(a_{i+2})) \cdot \varphi_i^{-1}(b_{i+1}) = a_i \cdot b_i + \big(f_{0,i}(a_0) - a_i\big) \cdot \varphi_i^{-1}(b_{i+1}) +$$

$$+\varphi_i^{-1}(a_{i+1}) \cdot f_{0,i}(b_0) - \varphi_i^{-1}(a_{i+1}) \cdot b_i - \varphi_i^{-1}(a_{i+1}) \cdot \varphi_i^{-1}(b_{i+1}) -$$

$$-f_{0,i}(a_0) \cdot \varphi_i^{-1}(\varphi_{i+1} - 1(b_{i+2})) + \varphi_i^{-1}(a_{i+1}) \cdot \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2})) -$$

$$-\varphi_i^{-1}(\varphi_{i+1}^{-1}(a_{i+2})) \cdot f_{0,i}(b_0) + \varphi_i^{-1}(\varphi_{i+1}^{-1}(a_{i+2})) \cdot \varphi_i^{-1}(b_{i+1}) =$$

$$= \big(a_i - \varphi_i^{-1}(a_{i+1})\big) \cdot b_i + \big(f_{0,i}(a_0) - a_i\big) \cdot \varphi_i^{-1}(b_{i+1}) +$$

$$+\big(\varphi_i^{-1}(a_{i+1}) - \varphi_i^{-1}(\varphi_{i+1}^{-1}(a_{i+2}))\big) \cdot f_{0,i}(b_0) - \big(\varphi_i^{-1}(a_{i+1}) -$$

$$-\varphi_i^{-1}(\varphi_{i+1}^{-1}(a_{i+2}))\big) \cdot \varphi_i^{-1}(b_{i+1}) - f_{0,i}(a_0) \cdot \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2})) + a_i \cdot \varphi_i^{-1}(b_{i+2}) -$$
$$-a_i \cdot \varphi_i^{-1}(b_{i+2}) + \varphi_i^{-1}(a_{i+1}) \cdot \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2})) = \big(a_i - \varphi_i^{-1}(a_{i+1})\big) \cdot b_i +$$
$$+\big(f_{0,i}(a_0) - a_i\big) \cdot \varphi_i^{-1}(b_{i+1}) + \big(\varphi_i^{-1}(a_{i+1}) - \varphi_i^{-1}(\varphi_{i+1}^{-1}(a_{i+2}))\big) \cdot f_{0,i}(b_0) -$$
$$-\big(\varphi_i^{-1}(a_{i+1}) - \varphi_i^{-1}(\varphi_{i+1}^{-1}(a_{i+2}))\big) \cdot \varphi_i^{-1}(b_{i+1}) - \big(f_{0,i}(a_0) - a_i\big) \cdot \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2})) -$$
$$-\big(a_i - \varphi_i^{-1}(a_{i+1})\big) \cdot \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2})) = h_i \cdot b_i + \big(f_{0,i}(h_0) + \ldots$$
$$+f_{i-1,i}(h_{i-1}) + h_i\big) \cdot \varphi_i^{-1}(b_{i+1}) + \varphi_i^{-1}\big(a_{i+1}) - \varphi_{i+1}^{-1}(a_{i+2})\big) \cdot f_{0,i}(b_0) -$$
$$-\varphi_i^{-1}\big(a_{i+1}) - \varphi_{i+1}^{-1}(a_{i+2})\big) \cdot \varphi_i^{-1}(b_{i+1}) - \big(f_{0,i}(h_0) + \ldots$$
$$+f_{i-1,i}(h_{i-1}) + h_i\big) \cdot \varphi_i^{-1}\big(\varphi_{i+1}^{-1}(b_{i+2})\big) - h_i \cdot \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2})) \in$$
$$\in U_i \cdot b_i + \big(f_{0,i}(U_0) + \ldots + f_{i-1,i}(U_{i-1}) + U_i\big) \cdot \varphi^{-1}(b_{i+1}) + \varphi^{-1}(U_{i+1}) \cdot f_{0,i}(b_0) -$$
$$-\varphi_i^{-1}(U_{i+1}) \cdot \varphi_i^{-1}(b_{i+1}) - \big(f_{0,i}(U_0) + \ldots + f_{i-1,i}(U_{i-1}) + U_i\big) \cdot \varphi_i^{-1}\big(\varphi_{i+1}^{-1}(b_{i+2})\big) -$$
$$-U_i \cdot \varphi_i^{-1}(\varphi_{i+1}^{-1}(b_{i+2})) \subseteq \underbrace{V_i' + \ldots + V_i'}_{2n+4 \text{ items}} \subseteq V_i \quad \text{for} \quad 0 < i < n;$$

$$r_0 - r_1 = a_0 \cdot b_0 - \varphi_0^{-1}\big(a_1 \cdot b_1 + (f_{0,1}(a_0) - a_1) \cdot \varphi_1^{-1}(b_2) + \varphi_1^{-1}(a_2) \cdot (f_{0,1}(b_0) - b_1\big) =$$
$$= a_0 \cdot b_0 - \varphi_0^{-1}(a_1) \cdot \varphi_0^{-1}(b_1) - a_0 \cdot \varphi_0^{-1}(\varphi_1^{-1}(b_2)) +$$
$$+\varphi_0^{-1}(a_1) \cdot \varphi_0^{-1}(\varphi_1^{-1}(b_2)) - \varphi_0^{-1}(\varphi_1^{-1}(a_2)) \cdot b_0 + \varphi_0^{-1}(\varphi_1^{-1}(a_2)) \cdot \varphi_0^{-1}(b_1)\big) =$$
$$= a_0 \cdot b_0 - \varphi_0^{-1}(a_1) \cdot b_0 + \varphi_0^{-1}(a_1) \cdot b_0 - \varphi_0^{-1}(a_1) \cdot \varphi_0^{-1}(b_1) - a_0 \cdot \varphi_0^{-1}(\varphi_1^{-1}(b_2)) +$$
$$+\varphi_0^{-1}(a_1) \cdot \varphi_0^{-1}(\varphi_1^{-1}(b_2)) - \varphi_0^{-1}(\varphi_1^{-1}(a_2)) \cdot b_0 + \varphi_0^{-1}(\varphi_1^{-1}(a_2)) \cdot \varphi_0^{-1}(b_1)\big) =$$
$$= (a_0 - \varphi_0^{-1}(a_1)) \cdot b_0 + \varphi_0^{-1}(a_1 - \varphi_1^{-1}(a_2) \cdot b_0 - \varphi_0^{-1}(a_1 - \varphi_1^{-1}(a_2)) \cdot \varphi_0^{-1}(b_1) -$$
$$-(a_0 - \varphi_0^{-1}(a_1)) \cdot \varphi_0^{-1}(\varphi_1^{-1}(b_2)) \in U_0 \cdot b_0 + \varphi_0^{-1}(U_1) \cdot b_0 - \varphi_0^{-1}(U_1) \cdot \varphi_0^{-1}(b_1) -$$
$$-U_0 \cdot \varphi_0^{-1}(\varphi_1^{-1}(b_2)) \subseteq V_0' + V_0' + V_0' + V_0' \subseteq V_0.$$

Thus, we have obtained that $r_i - \varphi_i^{-1}(r_{i+1}) \in V_i$ for $0 \leq i < n$ and $r_n \in V_n$, i.e. $\widehat{a} \cdot \widehat{b} = (r_0, \ldots, r_n) \in W(\mathcal{V})$. As the element $\widehat{a}$ is arbitrary then $\mathcal{U} \cdot \widehat{b} \subseteq \mathcal{V}$.

It is similarly proved that $\widehat{b} \cdot W(\mathcal{U}) \subseteq W(\mathcal{V})$ i.e. the condition BN6 is executed.

Hence we can take the set $\{W(\mathcal{V}) \mid \mathcal{V} \in \Omega\}$ for basis of neighbourhoods of the zero, sets which determines some, probably nonseparable, ring topology $\widehat{\tau}$ on the ring $\widehat{R}$.

II.2. From the definition of the set $W(\mathcal{V})$ it follows that if $\mathcal{V} = (V_0, \ldots V_n)$ and V_i is a subgroup of the additive group of the ring R_i, for $0 \leq i \leq n$, then $W(\mathcal{V})$ is a subgroup of the additive group of the ring $\widehat{R}$.

II.3. Let's show now that if for every $0 \leq i \leq n$ the topological ring (R_i, τ_i) is separable, then the topological ring $(\widehat{R}, \widehat{\tau})$ is separable, too.

Let's assume the contrary i.e. $0 \neq \widehat{r} = (r_0, \dots, r_n) \in \bigcap_{\mathcal{V}\in\Omega} W(\mathcal{V})$ and let $k = max\{i \mid r_i \neq 0\}$. As (R_k, τ_k) is a separable ring there exists a neighborhood $V_k \in \mathbf{B_k}$ of the zero such that $r_k \notin V_k$. From the definition of the number k it follows that either $k = n$ or $r_{k+1} = 0$. Then $r_n \in V_n$ if $k = n$ and $r_k = r_k - 0 = r_k - \varphi_k(r_{k+1}) \in V_k$ if $k < n$. We have obtained the contradiction with the choice of the neighborhood V_k. Hence the topological ring $(\widehat{R}, \widehat{\tau})$ is separable.

From subitems II.2 and II.3 it follows that if $\Re$ is any class from the classes specified in the formulation of this theorem and $(R_i, \tau_i) \in \Re$ for $0 \leq i \leq n$, then $(\widehat{R}, \widehat{\tau}) \in \Re$.

II.4. Let's check now that the isomorphism $\psi : (\widehat{R}_0, \widehat{\tau}|_{\widehat{R}_0}) \to (R, \tau)$ (the definition of ψ see in I.5.) is a topological isomorphism.

If $\mathcal{V} = (V_0, \dots, V_n) \in \Omega$ then $\psi(W(\mathcal{V}) \bigcap \widehat{R}_0) = \psi(\{(r_0, 0, \dots, 0) \mid r_0 \in V_0\}) = V_0$. Then (see [4, 1.5.5]) ψ is a topological isomorphism, and hence (see I.5), identifying any element $r \in R = R_0$ with the element $(r, 0, \dots, 0) \in \widehat{R}$, we can consider, that a topological ring (R, τ) is an accessible subring of the topological ring $(\widehat{R}, \widehat{\tau})$of the stage not more than n.

II.5. If we define mapping $\widehat{\varphi} : \widehat{R} \to \bar{R}$ as follows $\widehat{\varphi}(r_0, \dots, r_n) = \varphi(r_0)$, then it is easy to see that $\widehat{\varphi} : \widehat{R} \to \bar{R}$ is a ring homomorphism. As we have identified the element $r \in R_0 = R$ with the element $(r_0, 0, \dots, 0) \in \widehat{R}$, then the homomorphism $\widehat{\varphi}$ is an extension of the isomorphism φ.

To complete the proof of this theorem it is necessary to check that the homomorphism $\widehat{\varphi} : (\widehat{R}, \widehat{\tau}) \to (\bar{R}, \bar{\tau})$ is a topological homomorphism.

Let $\bar{V}$ be any neighborhood of the zero in the topological ring $(\bar{R}, \bar{\tau}) = (R_n, \tau_n)$. As the homomorphism $\varphi_i : (R_i, \tau_i) \to (R_{i+1}, \tau_{i+1})$ is continuous for every $0 \leq i < n$ then there are neighborhoods $V_0, \dots, V_n$ of the zero in the topological rings $(R_0, \tau_0), \dots, (R_n, \tau_n)$ correspondingly such that $\underbrace{V_n + \ldots + V_n}_{n+1 \text{ items}} \subseteq \bar{V}$ and $\varphi_i(V_i) \subseteq V_{i+1}$ for $0 \leq i < n$. Then $\mathcal{V} = V_0, \dots, V_n \in \Omega$ and $f_{i,j}(V_i) \subseteq V_j$ for any $0 \leq i < j \leq n$. If $(r_0, \dots r_n) \in W(\mathcal{V})$, then $r_n \in V_n$ and $r_i - \varphi_i^{-1}(r_{i+1}) \in V_i$ for $0 \leq i < n$. Then $\widehat{\varphi}(r_0, \dots, r_n) = \varphi(r_0) = f_{0,n}(r_0) = f_{0,n}(r_0) - f_{1,n}(r_1) + f_{1,n}(r_1) - f_{2,n}(r_2) + \ldots + f_{n-1,n}(r_n) - r_n + r_n = f_{0,n}(r_0 - \varphi_0^{-1}(r_1)) + \ldots + f_{n-1,n}(r_{n-1} - \varphi_{n-1}^{-1}(r_n)) + r_n \in f_{0,n}(V_0) + \ldots + f_{n-1,n}(V_{n-1}) + V_n \subseteq \underbrace{V_n + \ldots + V_n}_{n+1 \text{ items}} \subseteq \bar{V}$. As the element $(r_0, \dots r_n) \in W(\mathcal{V})$ is arbitrary then $\widehat{\varphi}(W(\mathcal{V})) \subseteq \bar{V}$, and according to (see [4, 1.5.5]) $\widehat{\varphi} : (\widehat{R}, \widehat{\tau}) \to (\bar{R}, \bar{\tau})$ is a continuous homomorphism.

Let now $\mathcal{V} = (V_0, \dots, V_n) \in \Omega$. Taking in consideration the condition of the theorem (see the statement 2 of formulation of the theorem) and the definition of the isomorphisms φ and $f_{i,j}$ it follows that $f_{0,n} = \varphi$. Then $\{(f_{0,n}^{-1}(r), \dots, f_{n-1,n}^{-1}(r), r) \mid r \in V_n\} \subseteq W(\mathcal{V})$, that is $V_n = \{\varphi(f_{0,n}^{-1}(r)) \mid r \in V_n\} = \widehat{\varphi}\big(\{(f_{0,n}^{-1}(r), \dots, f_{n-1,n}^{-1}(r), r) \mid r \in V_n\}\big) \subseteq W(\mathcal{V})$.

As $(\bar{R},\bar{\tau})=(R_n,\tau_n)$ then V_n is a neighborhood of the zero in $(\bar{R},\bar{\tau})$ and then according to (see [4, 1.5.5]) $\widehat{\varphi}:(\widehat{R},\widehat{\tau})\to(\bar{R},\bar{\tau})$ is an open homomorphism.

The theorem is completely proved.

Designation 1. Let R be any not necessarily associative ring. Put $R^1=R$ and for any natural number n define R^n as the subgroup generated by the set $\{a\cdot b \mid a\in R^s,\ b\in R^t,\ 0<s,t<n,\ s+t=n\}$.

It is easy to note that R^n is an ideal in the ring R.

Definition 3. *A ring R is called a nilpotent ring if $R^n=0$ for some natural number n. The minimal one from these natural numbers is called the index of nilpotence.*

Theorem 4. *Let $\Re$ be one of following classes:*

– the class of all (separable) topological rings;

– the class of all (separable) topological commutative rings;

– the class of all (separable) topological rings possessing basis of neighborhood of zero which consists of subgroups;

– the class of all (separable) topological commutative rings possessing basis of neighborhood of zero which consists of subgroups.

If (R,τ) and $(\bar{R},\bar{\tau})\in\Re$ and R is a nilpotent ring of index of nilpotence n, then every continuous isomorphism $\varphi:(R,\tau)\to(\bar{R},\bar{\tau})$ is a superposition of n semitopological isomorphism.

Proof. Let $\mathbf{B}$ and $\bar{\mathbf{B}}$ be the sets of all neighbourhoods of zero in the topological rings (R,τ) and $(\bar{R},\bar{\tau})$ correspondingly. For each natural number $0\le k<n-1$, we shall define the set $\mathbf{B_k}=\{W_k(V,\bar{V})=V+\big(R^{n-k}\bigcap\varphi^{-1}(\bar{V})\big)\mid V\in\mathbf{B},\ \bar{V}\in\bar{\mathbf{B}}\}$ in the ring R. Let's show that each of these sets $\mathbf{B_k}$ satisfies the conditions BN1–BN6 of Theorem 1.2.2 from [4]. Then by Theorem 1.2.5 from [4] it is possible to take $\mathbf{B_k}$ as a basis of neighborhoods of the zero for determining some ring topology τ_k in the ring R.

It is easy to notice that the set $\mathbf{B_k}$ satisfies the conditions BN1–BN4.

Let's check the condition BN5. If $W_k(V,(V)\in\mathbf{B_k}$, then there exist $V_1\in\mathbf{B}$ and $\bar{V}_1\in\bar{\mathbf{B}}$ such that $V_1\cdot V_1\subseteq V$ and $\bar{V}_1\cdot\bar{V}_1+\bar{V}_1\cdot\bar{V}_1+\bar{V}_1\cdot\bar{V}_1\subseteq\bar{V}$. As $\varphi:(R,\tau)\to(\bar{R},\bar{\tau})$ is a continuous mapping then we can consider that $V_1\subseteq\varphi^{-1}(\bar{V}_1)$. Then $W_k(V_1,\bar{V}_1)\cdot W_k(V_1,\bar{V}_1)=V_1\cdot V_1+(R^{n-k}\bigcap(V_1\cdot\varphi^{-1}(\bar{V}_1)+\varphi^{-1}(\bar{V}_1)\cdot V_1+\varphi^{-1}(\bar{V}_1)\cdot\varphi^{-1}(\bar{V}_1)))\subseteq V+(R^{n-k}\bigcap(\varphi^{-1}(\bar{V}_1)\cdot\varphi^{-1}(\bar{V}_1)+\varphi^{-1}(\bar{V}_1)\cdot\varphi^{-1}(\bar{V}_1)+\varphi^{-1}(\bar{V}_1)\cdot\varphi^{-1}(\bar{V}_1)))\subseteq V+(R^{n-k}\bigcap\varphi^{-1}(\bar{V}))=W_k(V,\bar{V})$, i.e. the condition BN5 is executed.

Let's check the condition BN6.

If $W_k(V,\bar{V})\in\mathbf{B_k}$ and $r\in R$, then there exist $V_1\in\mathbf{B}$ and $\bar{V}_1\in\bar{\mathbf{B}}$ such that $(r\cdot V_1)\bigcup(V_1\cdot r)\subseteq V$ and $(\varphi(r)\cdot\bar{V}_1)\bigcup(\bar{V}_1\cdot\varphi(r))\subseteq\bar{V}$. Then $r\cdot W_k(V_1,\bar{V}_1)=r\cdot\big(V_1+(R^{n-k}\bigcap\varphi^{-1}(\bar{V}_1))\big)\subseteq V+\big(R^{n-k}\bigcap\varphi^{-1}(\bar{V}_1)\big)=W_k(V,\bar{V})$.

It is similarly proved that $W_k(V_1,\bar{V}_1)\cdot r\subseteq W_k(V,\bar{V})$.

Thus, we have proved that each of these sets $\mathbf{B_k}$ determines some ring topology τ_k in the ring R.

Let's notice now that if topological rings (R,τ) and $(\bar{R},\bar{\tau}) \in \Re$ are separable then considering (see [4, Proposition 1.2.2]) we obtain that

$$\bigcap_{W_k(V,(V)\in \mathbf{B}_k} W_k(V,\bar{V}) = \bigcap_{\bar{V}\in\bar{\mathbf{B}}} \bigcap_{V\in\mathbf{B}} \big(V + (R^{n-k} \bigcap \varphi^{-1}(\bar{V}))\big) =$$

$$= \bigcap_{\bar{V}\in\bar{\mathbf{B}}} [(R^{n-k} \bigcap \varphi^{-1}(\bar{V}))]_{(R,\tau)} \subseteq \bigcap_{\bar{V}\in\bar{\mathbf{B}}} \varphi^{-1}([\bar{V}]_{(\bar{R},\bar{\tau})}) =$$

$$= \bigcap_{\bar{V}\in\bar{\mathbf{B}}} \varphi^{-1}(\bar{V}) = \varphi^{-1}(\{0\}) = \{0\}$$

for any $0 \le k < n-1$, and according to (see [4, Corollary 1.3.6]) the topological ring (R,τ_k) is separable.

As for any $0 \le k < n-1$ the set $W_k(V,\bar{V}) = V + (R^{n-k} \bigcap \varphi^{-1}(\bar{V}))$ is a group if V and $\bar{V}$ are subgroups then $(R,\tau_k) \in \mathcal{K}$ if $(R,\tau) \in \mathcal{K}$ and $(\bar{R},\bar{\tau}) \in \mathcal{K}$ for each of the classes which are specified in the formulation of this theorem.

As $R^n = \{0\}$ then $\tau = \tau_0$ and as $\varphi = \varphi \circ \varepsilon_{n-2} \circ \ldots \circ \varepsilon_0$, where $\epsilon_i : R \to R$ is the identical mapping for any $0 \le i \le n-2$, then to complete the proof of the theorem we need to check that for any $0 \le k < n-1$ the identical mapping $\varepsilon_k : (R,\tau_k) \to (R,\tau_{k+1})$ and the mapping $\varphi : (R,\tau_{n-1}) \to (\bar{R},\bar{\tau})$ are semitopological isomorphisms.

Let $0 \le k < n-1$ and $W_k(V,\bar{V}) = V + (R^{n-k} \bigcap \varphi^{-1}(\bar{V}))$ be any neighbourhood of the zero in (R,τ_k) and $r \in R$. There are neighbourhoods U and $\bar{U}$ of the zero in (R,τ) and $(\bar{R},\bar{\tau})$, correspondingly such that $r\cdot U + U\cdot r + U\cdot U + U\cdot U \subseteq V$ and $\bar{U} + \varphi(r)\cdot\bar{U} + \bar{U}\cdot\varphi(r) + \bar{U}\cdot\bar{U} + \bar{U}\cdot\bar{U} \subseteq \bar{V}$. As $\varphi : (R,\tau) \to (\bar{R},\bar{\tau})$ is a continuous isomorphism then, without loss of generality, we can consider that $\varphi(U) \subseteq \bar{U}$. Then

$$r\cdot\varepsilon_k^{-1}(W_{k+1}(U,\bar{U})) + \varepsilon_k^{-1}(W_{k+1}(U,\bar{U}))\cdot r +$$

$$+U\cdot\varepsilon_k^{-1}(W_{k+1}(U,\bar{U})) + \varepsilon_k^{-1}(W_{k+1}(U,\bar{U}))\cdot U = r\cdot\Big(U + \big(R^{n-k}\bigcap\varphi^{-1}(\bar{U})\big)\Big) +$$

$$+\Big(U\cdot r + \big(R^{n-k}\bigcap\varphi^{-1}(\bar{U})\big)\Big)\cdot r + U\cdot\Big(U + \big(R^{n-k}\bigcap\varphi^{-1}(\bar{U})\big)\Big) +$$

$$+\Big(U + \big(R^{n-k}\bigcap\varphi^{-1}(\bar{U})\big)\Big)\cdot U \subseteq r\cdot U + r\cdot\big(R^{n-k}\bigcap\varphi^{-1}(\bar{U})\big) + U\cdot r +$$

$$+\big(R^{n-k}\bigcap\varphi^{-1}(\bar{U})\big)\Big)\cdot r + U\cdot U + U\cdot\big(R^{n-k}\bigcap\varphi^{-1}(\bar{U})\big)\Big) +$$

$$+U\cdot U + \big(R^{n-k}\bigcap\varphi^{-1}(\bar{U})\big)\cdot U \subseteq V + R^{n-k+1}\bigcap\big(\varphi^{-1}(\varphi(r)\cdot\bar{U} + \bar{U}\cdot(\varphi(r) + \varphi(U)\cdot\bar{U} +$$

$$+\bar{U}\cdot\varphi(r))\big) \subseteq V + R^{n-k+1}\bigcap\varphi^{-1}(\bar{V}) = W_k(U,\bar{U})$$

if $k < n-2$ and $r\cdot\varphi^{-1}(\bar{U})) + \varphi^{-1}(\bar{U})\cdot r + U\cdot\varphi^{-1}(\bar{U})) + \varphi^{-1}(\bar{U}))\cdot U = R^2\bigcap\varphi^{-1}\big(\varphi(r)\cdot \bar{U} + \bar{U}\cdot\varphi(r) + \varphi(U)\cdot\bar{U} + \bar{U}\cdot\varphi(U)\big) \subseteq \varphi^{-1}(\bar{V}) \subseteq W_{n-2}(V,\bar{V})$.

Then, according to Theorem 2, for any $0 \leq k < n-1$ the identical mapping $\varepsilon_k : (R, \tau_k) \to (R, \tau_{k+1})$ and the mapping $\varphi : (R, \tau_{n-1}) \to (\bar{R}, \bar{\tau})$ are semitopological isomorphisms.

The theorem is completely proved.

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Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
str. Academiei 5, MD-2028 Chişinău
Moldova

E-mail: *arnautov@math.md*

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