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Locicaly separable algebras in varieties of algebras

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Abstract. Let Θ be an arbitrary variety of algebras and H be an algebra in Θ. Along with algebraic geometry in Θ over the distinguished algebra H we consider logical geometry in Θ over H. This insight leads to a system of notions and stimulates a number of new problems. We introduce a notion of logically separable in Θ algebras and consider it in the frames of logically-geometrical relations between different H_1 and H_2 in Θ. The paper is aimed to give a flavor of a rather new subject in a short and concentrated manner.

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1 Introduction

In the paper we give a list of main notions, formulate some results and specify problems. The paper is self-contained from the viewpoint of ideas of universal algebraic geometry and logical geometry.

The described theory has deep ties with model theory and some problems are of model theoretic nature [1, 2].

We fix an arbitrary variety of algebras Θ. Throughout the paper we consider algebras H in Θ. To each algebra $H \in \Theta$ one can attach an algebraic geometry (AG) in Θ over H and a logical geometry (LG) in Θ over H.

In algebraic geometry we consider algebraic sets over H, while in logical geometry we consider logical (elementary) sets over H. These latter sets are connected with the elementary logic, i.e. with the first order logic (FOL).

Consideration of these sets gives grounds to geometries in an arbitrary variety of algebras. We distinguish algebraic and logical geometries in Θ. However, there is a very little trace of the usual geometry in the geometries of such kind. It should be remembered that we consider an *arbitrary variety* Θ. Only some "good" varieties Θ and "good" algebras $H \in \Theta$ lead to a geometry which is, in some sense, close to a traditional one.

Algebraic sets are defined by systems of equations T of the type $w \equiv w'$. Here we assume that there is a fixed finitely generated free algebra $W = W(X)$ in Θ, and w, w' are the elements from W.

Elementary sets are defined by systems T of first order formulas. In this setting arbitrary FOL formulas play a role occupied by ordinary equations in the setting above.

Suppose that along with a free algebra $W(X)$ we consider a special algebra of formulas $\Phi = \Phi(X)$. This Φ is also associated with the variety Θ. Then a system T of "equations" i.e., a system of first order formulas turns to be a set of elements from Φ. In particular, an equation $w = w'$ is viewed as a formula of equality of the form $w \equiv w'$, and this is an element in the algebra $\Phi(X)$.

Algebraic sets and elementary sets lie in the affine space H^X which we identify in a standard way with the set of homomorphisms $Hom(W(X), H)$. Hence, any point of an affine space is a homomorphism $\mu : W \longrightarrow H$.

Let us consider the algebra $\Phi = \Phi(X)$ in more detail. We are not ready to give the precise definition of this algebra yet. For our immediate needs we shall say that $\Phi(X)$ is the algebra of the compressed first order formulas, i.e., the quotient algebra of the ordinary first order formulas modulo their semantic equivalence. It is important that:

1. $\Phi(X)$ is a Boolean algebra equipped with unary quantifier operations with respect to variables $x \in X$ and with equations $w \equiv w'$, $w \in W(X)$ which are viewed as nullary operations.

2. To each formula $u \in \Phi$ corresponds its value $Val_H^X(u) = A$, which is a subset in $Hom(W(X), H)$. In particular

$$Val_H^X(w \equiv w') = \{\mu : W \longrightarrow H | (w, w') \in Ker(\mu)\}.$$

3. Let us define the logical kernel $LKer(\mu)$ of a point $\mu : W \longrightarrow H$. A formula $u \in \Phi(X)$ belongs to $LKer(\mu)$ if and only if $\mu \in Val_H(u)$. The usual kernel $Ker(\mu)$ is the set of all (w, w') with $w \equiv w' \in LKer(\mu)$. We say that a point μ is a solution of the "equation" u if $u \in LKer(\mu)$.

The logical kernel $LKer(\mu)$ is an ultrafilter in the algebra Φ. Denote by M_X the set of all equalities $w \equiv w'$ over the algebra $W = W(X)$. Then

$$Ker(\mu) = LKer(\mu) \cap M_X.$$

Together with the usual FOL formulas we consider some weak variant of infinitary formulas. These are formulas of the type $\bigwedge_{u \in T} u \to v$, where T can be infinite and all u and v are FOL formulas, written in a given finite set X. We write shortly $T \to v$ and say that the $T \to v$ is an almost finite formula. We denote this new logic by LG keeping in mind the relation to logic geometry. We treat the logic LG as geometrical logic.

Denote by $Th(H)$ an elementary theory for the given algebra $H \in \Theta$ and by $LGTh(H)$ all $T \to v$ which hold in H.

Algebras H_1 and H_2 are elementary equivalent if $Th(H_1) = Th(H_2)$ and LG-equivalent if $LGTh(H_1) = LGTh(H_2)$.

If H_1 and H_2 are LG-equivalent, then they are elementary equivalent. The opposite is not true.

Now we able to manifest the main idea hidden in the title of the paper:

We call an algebra $H \in \Theta$ *logically ($LG-$)separable* in Θ if for every $H' \in \Theta$ not isomorphic to H the algebras H and H' are not $LG-$equivalent.

We are interested in the situation when every free in Θ algebra $W = W(X)$ with the finite X is logically separable in Θ.

The main goal of the paper is to suggest a general view on this problem and on related problems around logical geometry. In order to make the paper self-contained and clear we include some necessary material from universal algebra and algebraic logic. The paper does not contain proofs. It is aimed to give a flavor of a rather new subject in a short and concentrated manner.

Proceed now to the geometric approach.

2 Galois correspondence

Recall, first, how this correspondence looks like in universal algebraic geometry [6–9]. About universal AG see [4–9].

Let us fix a variety Θ and a finitely generated free algebra $W(X) \in \Theta$ and let T be a binary relation on $W(X)$. We view T as a system of equations $w \equiv w'$, $(w, w') \in T$ and as a system of formulas of the form $w \equiv w'$. Define:

$$T'_H = A = \{\mu : W \longrightarrow H | T \subset Ker(\mu)\}$$

for every $H \in \Theta$.

We can write the same as

$$T'_H = A = \bigcap_{(w,w')\in T} Val_H(w \equiv w').$$

Here A is the set of points, satisfying every equation in T, i.e. the set of solutions of all equations from T. Such sets A are called *algebraic sets*.

Define the correspondence in the opposite direction:

$$A'_H = T = \bigcap_{\mu\in A} Ker(\mu) = \{(w, w') | A \subset Val^X_H(w \equiv w')\}.$$

Here T is a congruence in W called an H-closed congruence.

The obtained correspondence between algebraic sets and H-closed congruence is a Galois correspondence. This means that

1. $A_1 \subset A_2$ implies $A'_{2H} \subset A'_{1H}$,
2. $T_1 \subset T_2$ implies $T'_{2H} \subset T'_{1H}$,
3. $A \subset A''_H$, $T \subset T''_H$.

Here, the algebraic set A''_H and the H-closed congruence T''_H are the closures of a set A and a system T, respectively.

Proposition 2.1. *A congruence T is H-closed if and only if $W/T \in SC(H)$.*

Here S is the operator of taking subgroups while C takes Cartesian products.

For each set of formulas T consider quasiidentities of the form

$$\bigwedge_{(w,w')\in T} w \equiv w' \longrightarrow w_0 \equiv w_0',$$

or, shortly, $T \longrightarrow w_0 \equiv w_0'$.

The set T is not necessarily finite, and the formulas above are considered in the infinitary logic.

Proposition 2.2. *The inclusion $(w_0, w_0') \in T_H^{''}$ holds if and only if the formula $T \longrightarrow w_0 \equiv w_0'$ holds in H.*

Now we define a Galois correspondence in logical geometry. We start with the algebra of formulas $\Phi(X)$ and consider an arbitrary subset T in Φ. In order to establish in this case the correspondence similar to the previous one we shall replace the kernel $Ker(\mu)$ by the logical kernel $LKer(\mu)$. Let us define

$$T_H^L = A = \{\mu : W \longrightarrow H | T \subset LKer(\mu)\} = \bigcap_{u\in T} Val_H(u).$$

Here A is an elementary set in $Hom(W,H)$ which consists of all points μ satisfying every "equation" $u \in T$. In the opposite direction:

$$A_H^L = T = \bigcap_{\mu\in A} LKer(\mu) = \{u \in \Phi(X) | A \subset Val_H^X(u)\}.$$

We defined the Galois correspondence in the case of logical geometry. The Galois closures are A_H^{LL} and T_H^{LL}, respectively. Here $T = A_H^L$ is always an H-closed Boolean filter in Φ.

For a given set of formulas $T \subset \Phi(X)$ and a given $v \in \Phi(X)$ consider the formula

$$\bigwedge_{u\in T} u \longrightarrow v,$$

or equally $T \longrightarrow v$, where T is not necessarily finite.

Proposition 2.3. *The inclusion $u \in T_H^{LL}$ holds if and only if the formula $T \longrightarrow v$ holds in H.*

3 Geometrical equivalence and logical equivalence of algebras

Recall that [6, 7] algebras H_1 and H_2 are *geometrically equivalent* (AG-equivalent for short) if for every finite X and T in $W = W(X)$ we have

$$T''_{H_1} = T''_{H_2}.$$

See the survey [8] and [7] for details.

Now we are able to define the notion of logically equivalent algebras.

Definition 3.1. *Algebras H_1 and H_2 are logically equivalent (LG-equivalent for short) if for every finite X and T in $\Phi = \Phi(X)$ we have*

$$T^{LL}_{H_1} = T^{LL}_{H_2}.$$

Let us look at the idea of LG-equivalence from the yet another point of view. Consider formulas of the form $u_1 \wedge u_2 \wedge \ldots \wedge u_n \longrightarrow v$, i.e., formulas $T_0 \longrightarrow v$, where T_0 is a finite set. Denote by $LG^0Th(H)$ the set of all formulas of such kind which hold in H. It is easy to see that H_1 and H_2 are elementary equivalent if and only if $LG^0Th(H_1) = LG^0Th(H_2)$.

Along with the invariants $Th(H)$ and $LG^0Th(H)$ of the algebra H consider an invariant $LGTh(H)$. We call $LGTh(H)$ the *implicative theory* of the algebra H. It consists of all formulas of the form $T \longrightarrow v$ which hold in H. Here, T is a set of formulas in Φ (possibly inifinite) and v is a formula in Φ.

Proposition 3.2. *Algebras H_1 and H_2 are logically equivalent if and only if their implicative theories coincide, i.e.,*

$$LGTh(H_1) = LGTh(H_2).$$

This proposition motivates the main idea formulated in the introduction. Hence,

Proposition 3.3. *If the algebras H_1 and H_2 are LG-equivalent then they are elementary equivalent.*

The opposite implication is not true and thus the relation of LG-equivalence is more strong than the relation of elementary equivalence.

It is also clear that

Proposition 3.4. *If the algebras H_1 and H_2 are LG-equivalent then they are geometrically equivalent.*

Definition 3.5. *Algebras H_1 and H_2 are called weakly LG-equivalent if the equality*

$$T^{LL}_{H_1} = T^{LL}_{H_2}.$$

holds for all finite T.

Proposition 3.6. *Algebras H_1 and H_2 are weakly LG-equivalent if and only if they are elementary equivalent.*

4 Some categories and lattices

Let us fix an infinite set X^0 and let Γ^0 be the set of all finite subsets of X^0.

For a given variety Θ denote by Θ^0 the category whose objects are the free algebras $W = W(X)$ in Θ with finite $X \in \Gamma^0$. The category Θ^0 is a full subcategory in the category Θ. Its morphisms are homomorphisms of free algebras.

Observe that we can consider also the special Halmos category Hal^0_Θ of all algebras of formulas $\Phi(X)$, where X runs all finite subsets of X^0. In logical geometry this category plays the role similar to that of the category of free algebras Θ^0 in the case of algebraic geometry.

Given algebra $H \in \Theta$, define the category of affine spaces $K^0_\Theta(H)$ over H. Its objects are the sets of homomorphisms $Hom(W,H)$, $W \in \Theta^0$, and morphisms are of the form

$$\tilde{s} : Hom(W_1, H) \longrightarrow Hom(W_2, H),$$

where $s : W_2 \longrightarrow W_1$ is a morphism in Θ^0 and the mapping $\tilde{s}$ is given by the rule $\tilde{s}(\nu) = \nu s : W_2 \longrightarrow H$ for $\nu : W_1 \longrightarrow H$. We have a contravariant functor $\Theta^0 \longrightarrow K^0_\Theta(H)$ which implies a duality of the categories if and only if the identities of the algebra H determine the whole variety Θ, i.e., $Var(H) = \Theta$ [9].

The next category is the category $Set_\Theta(H)$ of affine sets over an algebra H. Its objects are of the form (X, A), where A is an arbitrary subset in the affine space $Hom(W(X), H)$. The morphisms are

$$[s] : (X, A) \longrightarrow (Y, B).$$

Here $s : W(Y) \longrightarrow W(X)$ is a morphism in Θ^0. The corresponding $\tilde{s} : Hom(W(X), H) \longrightarrow Hom(W(Y), H)$ should be coordinated with A and B by the condition: if $\nu \in A \subset Hom(W(X), H)$, then $\tilde{s}(\nu) \in B \subset Hom(W(Y), H)$. Then we consider the induced mapping $[s] : A \longrightarrow B$ as a morphism $(X, A) \longrightarrow (Y, B)$.

Now we define the category of algebraic sets $K_\Theta(H)$ and the category of elementary sets $LK_\Theta(H)$. Both these categories are full subcategories in $Set_\Theta(H)$ and are viewed as important invariants of the algebra H. We call them AG and LG-invariants of H.

The objects of $K_\Theta(H)$ are of the form (X, A), where A is an algebraic set in $Hom(W(X), H)$. If we take for A the elementary sets, then we are getting the category of elementary sets $LK_\Theta(H)$. The category $K_\Theta(H)$ is a full subcategory in $LK_\Theta(H)$.

Let us turn to the lattices. We will see that if A and B are the elementary sets in $Hom(W, H)$, then the union $A \cup B$ is also an elementary set. This means that elementary sets in $Hom(W, H)$ constitute a lattice which is a sublattice in the lattice of all subsets in the given affine space. The similar fact in AG is not true. Given two algebraic sets A and B, the set $A \cup B$ is not always an algebraic set. (Clearly, $A \cup B$ is an elementary set.)

5 Algebras with the same logic

Recall, first, some general facts from category theory [3]. Let φ_1, φ_2 be two functors $C_1 \longrightarrow C_2$. An *isomorphism of functors* $s : \varphi_1 \longrightarrow \varphi_2$ is defined by the following conditions:

1. To every object A of the category C_1 an isomorphism $s_A : \varphi_1(A) \longrightarrow \varphi_2(A)$ in C_2 is assigned.

2. If $\nu : A \longrightarrow B$ is a morphism in C_1 then there is a commutative diagram in C_2:

$$\begin{array}{ccc} \varphi_1(A) & \xrightarrow{\quad s_A \quad} & \varphi_2(A) \\ {\scriptstyle \varphi_1(\nu)}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_2(\nu)} \\ \varphi_1(B) & \xrightarrow{\quad s_B \quad} & \varphi_2(B) \end{array}$$

An isomorphism of functors φ_1 and φ_2 is denoted by $\varphi_1 \simeq \varphi_2$. Let now a pair (φ, ψ) of functors $\varphi : C_1 \longrightarrow C_2$ and $\psi : C_2 \longrightarrow C_1$ be given. We say that it defines a *category equivalence* C_1 and C_2 if $\psi\varphi \simeq 1_{C_1}$ and $\varphi\psi \simeq 1_{C_2}$. Here, 1_{C_1} and 1_{C_2} are identity functors. The conditions $\psi\varphi = 1_{C_1}$ and $\varphi\psi = 1_{C_2}$ define an isomorphism of categories. If $C_1 = C_2 = C$ then we get the notions of *automorphism* and *autoequivalence* of the category C.

An automorphism φ of the category C is called inner if it is isomorphic to the identity automorphism 1_C. The latter means that if $s : 1_C \longrightarrow \varphi$ is an isomorphism of functors, then for every object A of the category C there is an isomorphism $s_A : A \longrightarrow \varphi(A)$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad s_A \quad} & \varphi(A) \\ {\scriptstyle \nu}\big\downarrow & & \big\downarrow{\scriptstyle \varphi(\nu)} \\ B & \xrightarrow{\quad s_B \quad} & \varphi(B) \end{array}$$

is commutative for any morphism $\nu : A \longrightarrow B$ in C. So, φ is inner if and only if it can be represented in the form:

$$\varphi(\nu) = s_B \nu s_A^{-1} : \varphi(A) \longrightarrow \varphi(B).$$

This formula motivates the term "inner automorphism".

Now we consider the main topic of this subsection. Let $Lat_H : \Theta^0 \longrightarrow Lat$ be the functor which assigns to each $W(X)$ the lattice of elementary sets in $Hom(W(X)), H)$. We say that algebras H_1 and H_2 have the same lattices if there exists an automorphism $\varphi : \Theta^0 \longrightarrow \Theta^0$ such that the functors Lat_{H_1} and $Lat_{H_2}\varphi$ are isomorphic. Coincidence of lattices can be represented by the commutative diagram

$$\begin{array}{ccc} \Theta^0 & \xrightarrow{\quad \varphi \quad} & \Theta^0 \\ & {\scriptstyle \mathrm{Lat}_{H_1}}\searrow \quad \swarrow{\scriptstyle \mathrm{Lat}_{H_2}} & \\ & \mathrm{Lat} & \end{array}$$

Here Lat is the category of lattices and commutativity means the existence of an isomorphism $Lat_{H_1} \longrightarrow Lat_{H_2}\varphi$.

Consider also isomorphisms of categories $LK_\Theta(H_1) \longrightarrow LK_\Theta(H_2)$. We require that there exists a special correct isomorphism of these categories. Informally speaking the correctness means coordination of the category isomorphism with lattices. We regard algebras H_1 and H_2 to have the same logic if the categories $LK_\Theta(H_1)$ and $LK_\Theta(H_2)$ are correctly isomorphic.

This approach repeats similar definitions in the case of AG. In AG we prove that if H_1 and H_2 are AG-equivalent, then the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are correctly isomorphic. Analogously:

Theorem 5.1. *If the algebras H_1 and H_2 are LG-equivalent, then the categories $LK_\Theta(H_1)$ and $LK_\Theta(H_2)$ are correctly isomorphic.*

Here, LG-equivalence means that the corresponding logics are the same. In the case of AG the proof of a theorem of such kind is trivial and based on the duality of $K_\Theta(H)$ and $C_\Theta(H)$. In the case of LG there is no such a duality and the proof is not trivial [9].

Note that the definition of the coincidence of logics can be grounded also on the correct equivalence of the categories of algebraic and elementary sets.

6 Some problems

We shall provide the reader with the list of problems related to the general scheme described above.

Problem 6.1. Consider various non-isomorphic LG-equivalent algebras.

It is hard to believe that LG-equivalence always implies isomorphism. Moreover, the general model-theoretic methods allow to construct non-isomorphic LG-equivalent algebras. We are mostly interested in the cases when the LG-equivalence of algebras implies their isomorphism. Grounding on this aim we introduce the following:

Definition 6.2. *Let H_1 and H_2 be two non-isomorphic algebras in Θ. We call them LG-separated if they are not LG-equivalent.*

Definition 6.3. *An algebra H is called LG-separable in Θ if H is not LG-equivalent to any other algebra H' in Θ.*

Problem 6.4. Consider varieties Θ such that every free in Θ algebra $W(X)$ is separable.

It can be proved that the varieties of semigroups and inverse semigroups possess this property. There are also other examples of such kind.

Problem 6.5. What is the situation in the case of the variety of all groups, i.e., $\Theta = Grp$?

It can be proved (Z.Sela, unpublished) that every free group F_n can be separated from other free groups F_m, $n \neq m$. Hence, the question is what can be said if the second group is not free.

Problem 6.6. What is the situation in the case of the variety of all commutative and associative algebras over a field P, i.e., $\Theta = Com - P$?

Problem 6.7. What is the situation in the case of the variety of all associative algebras over a field P, i.e., $\Theta = Ass - P$, or in the case of the variety of all Lie algebras over a field P, i.e., $\Theta = Lie - P$?

Problem 6.8. Let us fix a free algebra $W(X)$ in the variety Θ. Describe all LG-equivalent to $W(X)$ algebras H. In particular, consider $\Theta = Grp$.

Problem 6.9. Let H_1 and H_2 be two abelian groups. Suppose that they are LG-equivalent. Is it true that they are isomorphic?

It can happen that this question has a positive answer.

Problem 6.10. Let L_1 and L_2 be two extensions of the field P. Suppose that they are LG-equivalent. Is it true that they are isomorphic?

In fact, a negative answer to this question in the case of arbitrary L_1 and L_2 also can be deduced from model theory (see the survey [2] and references therein). The most interesting case is to consider extensions with some natural restrictions on L_1 and L_2.

Note now the following important question. Let H_1 and H_2 be elementary equivalent. When a single formula of the form $T \longrightarrow v$ makes H_1 and H_2 isomorphic? Or, vice versa, to separate non-isomorphic H_1 and H_2 with the help of a single formula of the form $T \longrightarrow v$.

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