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On natural and conatural sets of left ideals of a ring

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Dedicated to the memory of prof. V.A. Andrunachievici (1917–1997)

Abstract. The natural and conatural sets of left ideals of a ring R are defined by analogy with natural and conatural classes of left R-modules [4, 5]. Some characterizations and properties of such sets are indicated. The lattices of natural sets R-Nat and of conatural sets R-Conat of R are treated as skeletons of more wide lattices of closed and coclosed sets. In particular, R-Nat and R-Conat are boolean lattices.

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Introduction

The *natural classes* of modules are studied in a series of works (see, for example, [1–3]), the main results of which were summarized in the book [4]. In the paper [5] the notion of natural class was dualized, studying the *conatural classes* of modules. It is well known that some classes of modules $\mathcal{K} \subseteq R$-Mod can be completely described by the corresponding sets of left ideals of R: $\mathcal{E} = \{I \in \mathbb{L}(_RR) \mid R/I \in \mathcal{K}\}$. For example, this is the case of pretorsion classes, torsion classes, torsion-free classes, natural classes [6]. In such cases the study of the classes of R-Mod is reduced to the investigation of the corresponding sets of left ideals (linear topologies, radical filters, cofilters, etc.) of ring R.

In the present paper the transition from natural and conatural classes of R-Mod to the suitable sets of left ideals of R is realized. This transition is made substituting the closeness of the class $\mathcal{K} \subseteq R$-Mod under submodules (factor-modules) for the closeness of $\mathcal{E} \subseteq \mathbb{L}(_RR)$ under quotients (majorants). We define the natural and conatural sets of left ideals and study the lattices of such sets. These lattices are presented as skeletons of more wide lattices.

1 The natural sets of left ideals

An abstract class of modules $\mathcal{K} \subseteq R$-Mod is called *natural* if it is closed under submodules, direct sums and essential extensions (or injective envelopes). Every natural class $\mathcal{K}$ is completely described by the corresponding set of left ideals of R:

$$\mathcal{E} = \mathbf{\Gamma}(\mathcal{K}) = \{I \in \mathbb{L}(_RR) \mid R/I \in \mathcal{K}\}.$$

In the papers [3] and [6] some characterizations of sets of the form $\mathbf{\Gamma}(\mathcal{K})$, where $\mathcal{K}$ is a natural class, are indicated. These sets $\mathcal{E} = \mathbf{\Gamma}(\mathcal{K})$ are called *natural* and are described, in particular, by the following properties (in the notations of [6]):

(a_1) $I \in \mathcal{E} \Rightarrow (I : a) \in \mathcal{E}\ \ \forall a \in R$ (where $(I : a) = \{b \in R \,|\, ba \in I\}$);

(a_3) $I, J \in \mathcal{E} \Rightarrow I \cap J \in \mathcal{E}$;

(a_6) $J \subseteq I,\ (J : i) \in \mathcal{E}\ \ \forall i \in I,\ I/J \subseteq^{*} R/J \Rightarrow J \in \mathcal{E}$ (where $\subseteq^{*}$ means the essential inclusion).

Now we specify the terminology be used further. Let R be a ring with unity and R-Mod is the category of unitary left R-modules. We denote by $\mathbb{L}({}_RR)$ the lattice of left ideals of R. The left ideal $I \in \mathbb{L}({}_RR)$ is *proper* if $I \neq R$. The left ideal $(I : a) = \{b \in R \,|\, ba \in I\}$ is called the *quotient* of I by element $a \in R$. The left ideal $(I : a)$ is called *proper quotient* of I if $(I : a) \neq R$, i.e. $a \notin I$. The *majorant* of $I \in \mathbb{L}({}_RR)$ is a left ideal $J \in \mathbb{L}({}_RR)$ such that $I \subseteq J$. The left ideal J is a *proper majorant* of I if $I \subseteq J \neq R$.

The operators which realize the connection between classes of R-modules and sets of left ideals of R are defined as follows [7]:

$\mathcal{K} \subseteq R\text{-Mod}, \quad \mathbf{\Gamma}(\mathcal{K}) = \{(0 : m) \mid m \in M,\ M \in \mathcal{K}\}$;

$\mathcal{E} \subseteq \mathbb{L}({}_RR), \quad \mathbf{\Delta}(\mathcal{E}) = \{M \in R\text{-Mod} \mid (0 : m) \in \mathcal{E}\ \ \forall\, m \in M\}$,

where $(0 : m) = \{a \in R \mid am = 0\}$. The class of modules $\mathcal{K} \subseteq R$-Mod is called *closed* if $\mathcal{K} = \mathbf{\Delta\Gamma}(\mathcal{K})$. Similarly, the set of left ideals $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ is called *closed* if $\mathcal{E} = \mathbf{\Gamma\Delta}(\mathcal{K})$. It is known (see [7]) that the set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ is closed if and only if it satisfies the condition (a_1), i.e. it is closed under quotients. The class $\mathcal{K} \subseteq R$-Mod is closed if and only if it satisfies the condition:

$$(A_1) \quad M \in \mathcal{K} \Leftrightarrow Rm \in \mathcal{K}\ \ \forall\, m \in M.$$

The operators $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ determine a bijection between closed classes of modules and closed sets of left ideals [7]. This fact gives us the possibility to realize the transition from the classes of modules to the sets of left ideals of R. In continuation we formulate some results from the paper [6].

We denote by R-Cl the family of all closed sets $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ of left ideals of R. It can be transformed in to a complete lattice using the inclusion and the following lattice operations:

$$\bigwedge_{\alpha \in \mathfrak{A}} \mathcal{E}_\alpha = \bigcap_{\alpha \in \mathfrak{A}} \mathcal{E}_\alpha, \quad \bigvee_{\alpha \in \mathfrak{A}} \mathcal{E}_\alpha = \bigcup_{\alpha \in \mathfrak{A}} \mathcal{E}_\alpha,$$

where $\cap$ and $\cup$ are the intersection and union in the sense of set theory. Then the lattice $(R\text{-Cl}, \subseteq, \wedge, \vee)$ satisfies the both infinite distributive laws, it has the smallest element $\{{}_RR\}$ and the greatest element $\mathbb{L}({}_RR)$. Therefore R-Cl is a frame and every element $\mathcal{E} \in R$-Cl has a unique *pseudocomplement* $\mathcal{E}^*$, which is the greatest of the elements $\mathcal{E}' \in R$-Cl with the property $\mathcal{E}' \wedge \mathcal{E} = \{{}_RR\}$. To determine the pseudocomplement $\mathcal{E}^*$ of the element $\mathcal{E} \in R$-Cl the following operator is used:

$$\mathcal{E} \subseteq \mathbb{L}({}_RR), \quad \mathcal{E}^{\perp} = \{I \in \mathbb{L}({}_RR) \,|\, (I : a) \notin \mathcal{E}\ \forall a \in R \setminus I\},$$

i.e. the set $\mathcal{E}^{\perp}$ consists of all left ideals of R which have no proper quotient from the set $\mathcal{E}$.

Proposition 1.1. *For every set $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ the set $\mathcal{E}^{\perp}$ is closed and if $\mathcal{E} \in R$-Cl, then $\mathcal{E}^{\perp}$ is the pseudocomplement of $\mathcal{E}$ in the lattice R-Cl, i.e. $\mathcal{E}^* = \mathcal{E}^{\perp}$.*

From this result and from the definition of $\mathcal{E}^{\perp}$ the characterization of the second pseudocomplement $\mathcal{E}^{\perp\perp}$ of $\mathcal{E}$ in R-Cl follows.

Proposition 1.2. *If $\mathcal{E} \in R$-Cl then:*
$\mathcal{E}^{\perp\perp} = \{I \in \mathbb{L}({}_R R) \mid \forall\, a \notin I,\ \exists\, b \notin (I:a)$ *such that* $\big((I:a):b\big) \in \mathcal{E}\}$.

As was mentioned above, the set $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ is *natural* (i.e. $\mathcal{E} = \mathbf{\Gamma}(\mathcal{K})$ for $\mathcal{K} \in R$-nat) if and only if it satisfies the conditions (a_1), (a_3) and (a_6), which represent the „translations” in the language of $\mathbb{L}({}_R R)$ of the conditions from the definition of natural classes (closeness under submodules, direct sums and essential extensions, respectively).

We denote by R-Nat the family of all natural sets of $\mathbb{L}({}_R R)$. It is obvious that R-Cl $\supseteq$ R-Nat. The relation between the closed sets and natural sets of left ideals of R is mentioned in the following statement [6, Prop. 3.7].

Proposition 1.3. *If $\mathcal{E} \in R$-Cl, then $\mathcal{E}^{\perp}$ is a natural set.*

Therefore, for every set $\mathcal{E} \in R$-Cl the set $\mathcal{E}^{\perp\perp}$ is natural. Moreover, the set $\mathcal{E} \in R$-Cl is natural if and only if $\mathcal{E} = \mathcal{E}^{\perp\perp}$.

Corollary 1.4. *The lattice R-Nat is the skeleton of the lattice R-Cl* (*in particular, R-Nat is a boolean lattice*).

2 The conatural sets of left ideals

In this section we will dualize the results of §1, using the method of dualization of natural classes from the paper [5]. In contrast to the previous case, now we will not resort to the relation between classes of modules and sets of left ideals, since now we will not deal with the closed classes and closed sets, for which such relation is valid.

Definition 1. *The set of left ideals $\mathcal{E} \subseteq \mathbb{L}({}_R R)$ will be called* **coclosed** *if it satisfies the condition:*

$$(a_2)\quad I \in \mathcal{E},\ I \subseteq J \in \mathbb{L}({}_R R) \Rightarrow J \in \mathcal{E},$$

i.e. $\mathcal{E}$ is closed under majorants.

We denote by R-Cocl the family of all coclosed sets of $\mathbb{L}({}_R R)$. We consider R-Cocl as partial ordered set by inclusions and define the lattice operations as follows:

$$\bigwedge_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha = \bigcap_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha, \quad \bigvee_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha = \bigcup_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha,$$

using the obvious fact that the intersection and the union of sets with (a_2) have the same property. In such a way we obtain the complete lattice R-Cocl $(\subseteq, \wedge, \vee)$ with both infinite distributive laws and with extreme elements $\mathbf{1} = \mathbb{L}({}_RR)$ and $\mathbf{0} = \{{}_RR\}$. For every set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ the coclosed set generated by $\mathcal{E}$ is denoted by $\widehat{\mathcal{E}}$ and consists of all majorants of left ideals of $\mathcal{E}$.

Since the lattice R-Cocl is a frame, every element $\mathcal{E} \in R$-Cocl has a unique *pseudocomplement* $\mathcal{E}^*$ (the greatest of such $\mathcal{E}' \in R$-Cocl that $\mathcal{E}' \wedge \mathcal{E} = \{{}_RR\}$). To construct the pseudocomplement $\mathcal{E}^*$, we introduce the following operator (which is dual to the operator $(\ )^\perp$ from §1):

$$\mathcal{E} \subseteq \mathbb{L}({}_RR), \quad \mathcal{E}^\top = \{I \in \mathbb{L}({}_RR) \,|\, I \subseteq J, \quad J \subseteq \mathcal{E} \Rightarrow J = R\},$$

i.e. $\mathcal{E}^\top$ consists of all left ideals of R which have no proper majorants from $\mathcal{E}$. Since ${}_RR$ has no proper majorants, so is clear that ${}_RR \in \mathcal{E}^\top$ for every $\mathcal{E} \subseteq \mathbb{L}({}_RR)$.

Proposition 2.1. *For every coclosed set $\mathcal{E} \in R$-Cocl of left ideals of R the set $\mathcal{E}^\top$ is the pseudocomplement of $\mathcal{E}$ in the lattice R-Cocl, i.e. $\mathcal{E}^\top = \mathcal{E}^*$.*

Proof. From the definition it is obvious that $\mathcal{E}^\perp$ satisfies the condition (a_2). If $\mathcal{E} \in R$-Cocl, then $\mathcal{E} \wedge \mathcal{E}^\top = \{{}_RR\}$.

Let $\mathcal{F} \in R$-Cocl and $\mathcal{E} \wedge \mathcal{F} = \{{}_RR\}$. If $\mathcal{F} \not\subseteq \mathcal{E}^\top$, then there exists $I \in \mathcal{F}$ such that $I \notin \mathcal{E}^\top$, i.e. I has a proper majorant J from $\mathcal{E}$ $(I \subseteq J \neq R, \quad J \in \mathcal{E})$. Since $I \in \mathcal{F}$, from (a_2) it follows that $J \in \mathcal{F}$, i.e. $R \neq J \in \mathcal{E} \wedge \mathcal{F} = \{{}_RR\}$, a contradiction. Therefore $\mathcal{F} \subseteq \mathcal{E}^\top$ for every $\mathcal{F} \in R$-Cocl with $\mathcal{E} \wedge \mathcal{F} = \{{}_RR\}$. This means that $\mathcal{E}^\top$ is the greatest element of R-Cocl with the property $\mathcal{E} \wedge \mathcal{E}^\top = \{{}_RR\}$, i.e. $\mathcal{E}^\top = \mathcal{E}^*$. □

Now we can describe the second pseudocomplement $\mathcal{E}^{\top\top}$ of the set $\mathcal{E} \in R$-Cocl.

Proposition 2.2. *Let $\mathcal{E} \in R$-Cocl. Then:*
$\mathcal{E}^{\top\top} = \{I \in \mathbb{L}({}_RR) \mid$ *every proper majorant of I has a proper majorant from* $\mathcal{E}\}$.

Corollary 2.3. *If $\mathcal{E} \in R$-Cocl, then $\mathcal{E} \subseteq \mathcal{E}^{\top\top}$.*

From the definitions it follows that if $\mathcal{E} \subseteq \mathcal{F}$, then $\mathcal{E}^\top \supseteq \mathcal{F}^\top$ and $\mathcal{E}^{\top\top} \subseteq \mathcal{F}^{\top\top}$.

In continuation we consider the elements of the skeleton of the lattice R-Cocl.

Definition 2. *The set of left ideals $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ will be called* **conatural** *if it is of the form $\mathcal{E}^\top$, where $\mathcal{E} \in R$-Cocl.*

We denote by R-Conat the family of all conatural sets of left ideals of R. To describe the sets of such type we consider the following condition:

(CN) *If the left ideal $I \in \mathbb{L}({}_RR)$ has the property:*

$$(*) \begin{cases} \text{for every proper majorant } J \text{ of } I \ \ (I \subseteq J \neq R) \text{ there exists a} \\ \text{(proper) majorant of a left ideal } L \text{ from } \mathcal{E} \ \ (\exists\, L \in \mathcal{E},\ L \subseteq K \neq R) \end{cases}$$

then $I \in \mathcal{E}$.

We can express this condition in other form. Namely, for every set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ we have $\mathcal{E} \subseteq \widehat{\mathcal{E}} \subseteq (\widehat{\mathcal{E}})^{\top\top}$, where $\widehat{\mathcal{E}}$ as above is the closure of $\mathcal{E}$ under majorants. From the description of the second pseudocomplement (Prop. 2.2) it is clear that the hypothesis $(*)$ of (CN) means that $I \in (\widehat{\mathcal{E}})^{\top\top}$ and the condition (CN) is equivalent to the relation $(\widehat{\mathcal{E}})^{\top\top} \subseteq \mathcal{E}$ (which means that $(\widehat{\mathcal{E}})^{\top\top} = \mathcal{E}$). If $\mathcal{E} \in R$-Cocl, then $\widehat{\mathcal{E}} = \mathcal{E}$ and the condition (CN) is equivalent to $\mathcal{E}^{\top\top} \subseteq \mathcal{E}$ (i.e. $\mathcal{E}^{\top\top} = \mathcal{E}$).

Proposition 2.4. *For the set $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ of left ideals of R the following conditions are equivalent:*

1) *$\mathcal{E}$ is conatural;*
2) *$\mathcal{E}$ satisfies the condition (CN);*
3) *$\mathcal{E} \in R$-Cocl and $\mathcal{E}^{\top\top} \subseteq \mathcal{E}$.*

Proof. 1) $\Rightarrow$ 2). If $I \in R$-Conat, then $\mathcal{E} = \mathcal{F}^{\top}$ for some $\mathcal{F} \in R$-Cocl. Let I satisfy the hypotheses $(*)$ of condition (CN), i.e. $I \in (\widehat{\mathcal{E}})^{\top\top}$. We will prove that $I \in \mathcal{E} = \mathcal{F}^{\top}$.

Suppose the contrary: $I \notin \mathcal{E} = \mathcal{F}^{\top}$. From the definition of $\mathcal{F}^{\top}$ it follows that I has a proper majorant J from $\mathcal{F}$ $(I \subseteq J \neq R,\ J \in \mathcal{F})$. Since $I \in (\widehat{\mathcal{E}})^{\top\top}$, this proper majorant J has a proper majorant K $(J \subseteq K \neq R)$, which contains a left ideal L from $\mathcal{E}$ $(L \subseteq K \neq R,\ L \in \mathcal{E})$. Since $\mathcal{E} = \mathcal{F}^{\top}$ is coclosed, from $L \in \mathcal{E}$ it follows that $K \in \mathcal{E} = \mathcal{F}^{\top}$, where $K \neq R$. From the other hand, the condition $J \in \mathcal{F} \in R$-Cocl implies $K \in \mathcal{F}$ and we have $R \neq K \in \mathcal{F} \wedge \mathcal{F}^{\top} = \{{}_RR\}$, a contradiction. Therefore $I \in \mathcal{E} = \mathcal{F}^{\top}$.

2) $\Rightarrow$ 3). If $\mathcal{E}$ satisfies the condition (CN), then $(\widehat{\mathcal{E}})^{\top\top} = \mathcal{E}$, hence $\mathcal{E} \in R$-Cocl, $\mathcal{E} = \widehat{\mathcal{E}}$ and $\mathcal{E} = \mathcal{E}^{\top\top}$.

3) $\Rightarrow$ 1). From 3) it follows that $\mathcal{E} = \mathcal{E}^{\top\top}$, therefore $\mathcal{E}$ is conatural. □

As we mentioned above, every conatural set satisfies the condition (a_2) (i.e. it is coclosed). In continuation we will show some new properties of conatural sets of left ideals of R.

In the study of classes of modules $\mathcal{K} \subseteq R$-Mod the closeness under extensions is „translated" in the language of $\mathbb{L}({}_RR)$ by the following condition on the set of left ideals $\mathcal{E} = \mathbf{\Gamma}(\mathcal{K})$ (see [7]):

(a_5) If $I \in \mathcal{E},\ J \in \mathbb{L}({}_RR),\ J \subseteq I$ and $(J : i) \in \mathcal{E}\ \forall\, i \in I$, then $J \in \mathcal{E}$.

Proposition 2.5. *Every conatural class $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ of left ideals of R satisfies the condition (a_5).*

Proof. Let us have the situation from the hypothesis of (a_5), i.e. $I \in \mathcal{E}$, $J \subseteq I$ and $(J : i) \in \mathcal{E}$ for every $i \in I$. Since $\mathcal{E} \in R$-Conat, we have $\mathcal{E} = \mathcal{E}^{\top\top}$, therefore to obtain the relation $J \in \mathcal{E}$ it is sufficient to prove that $J \in \mathcal{E}^{\top\top}$, i.e. that every proper majorant of J has a proper majorant from $\mathcal{E}$.

Let K be an arbitrary proper majorant of J $(J \subseteq K \neq R)$. The following two cases are possible.

a) $I + K = R$. Then every element $r \in R$ has the form $r = i + k$, $i \in I$, $k \in K$, and we have:

$$(K : r) = (K : (i + k)) \subseteq (K : i) \cap (K : k) = (K : i) \cap R = (K : i).$$

Since $(J : i) \subseteq (K : i)$ and $(J : i) \in \mathcal{E}$ by assumption, from the condition (a_2) we obtain $(K : i) \in \mathcal{E}$. In particular, for $r = 1_R$ we have:

$$K = (K : r) \subseteq (K : i) \in \mathcal{E},$$

where $i \notin K$, since $K \neq R$ (if $i \in K$ and $i + k = 1 \in K$, then $R = K$). So we obtain that every proper majorant K of J has a proper majorant $(K : i) \in \mathcal{E}$, therefore $J \in \mathcal{E}^{\top\top} = \mathcal{E}$.

b) $I + K \neq R$. Then from $I \in \mathcal{E}$ and condition (a_2) we have $I + K \in \mathcal{E}$. So the proper majorant K of J has a proper majorant $I + K \in \mathcal{E}$, therefore in this case also $J \in \mathcal{E}^{\top\top} = \mathcal{E}$. □

The closeness of the class $\mathcal{K} \subseteq R$-Mod under essential extensions is „translated" in the language of $\mathbb{L}({}_RR)$ by condition (a_6) (see [7]). Now we consider the dualization of this condition:

$$(a_6^*) \quad \text{If} \;\; I \in \mathcal{E}, \;\; J \subseteq I \;\; \text{and} \;\; I/J \subseteq^0 R/J, \;\; \text{then} \;\; J \in \mathcal{E}.$$

The relation ${}_RN \subseteq^0 {}_RM$ means that ${}_RN$ is a *coessential* (superfluous) submodule of ${}_RM$, i.e. if $N + L = M$ then $L = M$. In our case the relation $I/J \subseteq^0 R/J$ means that if $I + K = R$ and $K \supseteq J$, then $K = R$ (where $I, J, K \in \mathbb{L}({}_RR)$).

Proposition 2.6. *Every conatural class $\mathcal{E} \subseteq \mathbb{L}({}_RR)$ of left ideals of R satisfies the condition (a_6^*).*

Proof. Let $I \in \mathcal{E}, \; J \subseteq I$ and $I/J \subseteq^0 R/J$. Since $\mathcal{E} \in R$-Conat, it is sufficient to prove that $J \in \mathcal{E}^{\top\top}$.

Let K be a proper majorant of J. Then $I + K \neq R$, since in the case $I + K = R \;\; (K \supseteq J)$ from the relation $I/J \subseteq^0 R/J$ it follows that $K = R$, a contradiction. So we obtain that $I + K$ is a proper majorant of K and, since $I \in \mathcal{E}$, from condition (a_2) it follows that $I + K \in \mathcal{E}$. Therefore, every proper majorant of J has a proper majorant from $\mathcal{E}$. This means that $J \in \mathcal{E}^{\top\top} = \mathcal{E}$. □

3 R-Conat as a lattice

Now we transform the family R-Conat of all conatural sets of left ideals of R in a lattice and will indicate some properties of this lattice. For that the following remark is useful.

Lemma 3.1. *The intersection of every family of conatural sets of $\mathbb{L}({}_RR)$ is a natural set.*

Proof. Let $\mathcal{E} = \bigcap\limits_{\alpha \in \mathfrak{A}} \mathcal{E}_\alpha$, where $\mathcal{E}_\alpha \in R$-conat for every $\alpha \in \mathfrak{A}$. Then $\mathcal{E}$ is coclosed, therefore it is sufficient to prove that $\mathcal{E}^{\top\top} \subseteq \mathcal{E}$. Let $I \in \mathcal{E}^{\top\top}$, i.e. every proper majorant J of I has a proper majorant K from $\mathcal{E}$. Then for every $\alpha \in \mathfrak{A}$ we have $K \in \mathcal{E}_\alpha$, so $I \in \mathcal{E}_\alpha^{\top\top} = \mathcal{E}_\alpha$ and $I \in \bigcap\limits_{\alpha \in \mathfrak{A}} \mathcal{E}_\alpha = \mathcal{E}$. □

For every set $\mathcal{E} \in R$-Cocl we denote by $\xi(\mathcal{E})$ the conatural class generated by $\mathcal{E}$ (i.e. the intersection of all conatural sets of $\mathbb{L}({}_R R)$ containing $\mathcal{E}$).

Lemma 3.2. *If $\mathcal{E} \in R$-Cocl, then $\xi(\mathcal{E}) = \mathcal{E}^{\top\top}$.*

Proof. Since $\mathcal{E} \in R$-Cocl, we have $\mathcal{E} \subseteq \mathcal{E}^{\top\top}$ and it is obvious that the set $\mathcal{E}^{\top\top}$ is conatural, therefore $\xi(\mathcal{E}) \subseteq \mathcal{E}^{\top\top}$. From the other hand, if $\mathcal{F} \in R$-Conat and $\mathcal{E} \subseteq \mathcal{F}$, then $\mathcal{E}^{\top\top} \subseteq \mathcal{F}^{\top\top} = \mathcal{F}$, hence $\mathcal{E}^{\top\top} \subseteq \xi(\mathcal{F})$. □

The mentioned facts permit us to transform R-Conat in to a lattice:

$$(R\text{-Conat}, \subseteq, \wedge, \vee),$$

where $(\subseteq)$ is the inclusion and the lattice operations are defined as follows:

$$\bigwedge_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha = \bigcap_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha, \quad \bigvee_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha = \xi(\bigcup_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha).$$

This lattice has the extreme elements $\mathbf{1} = \mathbb{L}({}_R R)$ and $\mathbf{0} = \{{}_R R\}$. From Lemma 3.2 the characterization of the unions in the lattice R-Conat immediately follows.

Proposition 3.3. *For every family $\{\mathcal{E}_\alpha \,|\, \alpha \in \mathfrak{A}\}$ of conatural sets of left ideals of R we have*:

$$\bigvee_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha = \big(\bigcup_{\alpha\in\mathfrak{A}} \mathcal{E}_\alpha\big)^{\top\top} = \big\{I \in \mathbb{L}({}_R R) \mid \forall J \supseteq I,\ J \neq R,\ \exists K \supseteq I,\ K \neq R \text{ such that } K \in \mathcal{E}_\alpha \text{ for some } \alpha \in \mathfrak{A}\big\}.$$

By definition, the set $\mathcal{F} \subseteq \mathbb{L}({}_R R)$ is conatural if $\mathcal{F} = \mathcal{E}^{\top}$ for some $\mathcal{E} \in R$-Cocl, where $\mathcal{E}^{\top}$ is the pseudocomplement of $\mathcal{E}$ in the lattice R-Cocl (Prop. 2.1). From the general results of lattice theory (see, for example, [8, Ch. V, §10 and 11]; [9, Ch. I, §6]) follows

Corollary 3.4. *The lattice R-Conat is the skeleton of the lattice R-Cocl: R-Conat $= Sk$ (R-Cocl). R-Conat is a boolean lattice.*

The following statement answers the question: when R-Cocl is a boolean lattice?

Proposition 3.5. *The following conditions on ring R are equivalent:*

1) *R-Cocl is a boolean lattice;*
2) *R-Cocl $=$ R-Conat;*
3) *${}_R R$ is a simple module (i.e. $\mathbb{L}({}_R R) = \{0, {}_R R\}$).*

Proof. 1) ⇔ 2) is obvious.

2) ⇒ 3). If R-Cocl $=$ R-Conat, then for every $\mathcal{E} \in R$-Cocl we have $\mathcal{E} = \mathcal{E}^{\top\top}$. We consider in $\mathbb{L}({}_R R)$ the following special set:

$$\mathcal{E} = Max\,({}_R R) \cup \{{}_R R\},$$

where $Max\,({}_R R)$ is the set of all maximal left ideals of R. Then $\mathcal{E} \in R$-Cocl and by 2) we have $\mathcal{E}^{\top\top} \subseteq \mathcal{E}$. This means that if $I \in \mathbb{L}({}_R R)$ has the property: „every proper majorant of I has a proper majorant from $Max\,({}_R R)$", then $I \in \mathcal{E}$. Since every proper left ideal of R is contained in a maximal left ideal, it is clear that in our case every proper left ideal is maximal, hence ${}_R R$ is simple. □

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