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Identities with permutations associated with quasigroups isotopic to groups

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Abstract. In this note we select a class of identities with permutations including three variables in a quasigroup $(Q,\cdot)$ each of which provides isotopy of this quasigroup to a group and describe a class of identities in a primitive quasigroup $(Q,\cdot,\backslash,/)$ each of which is sufficient for the quasigroup $(Q,\cdot)$ to be isotopic to a group. From these results it follows that in the identity of V. Belousov [6] characterizing a quasigroup isotopic to a group (to an abelian group) two from five (one of four) variables can be fixed.

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1 Introduction

The quasigroups isotopic to groups (to abelian groups) form an important class of quasigroups which arise by the research of many questions of the quasigroup theory, related systems and are used in different applications. The well known subclasses of such quasigroups are medial quasigroups, linear (over groups) of distinct types quasigroups and T-quasigroups. These quasigroups are connected with the nuclei and the center of quasigroups [1, 2].

The quasigroups isotopic to groups (to abelian groups), their subclasses and identities, leading to them, were investigated in many articles. Such quasigroups arose by the study of balanced identities involving three variables [3–5] and an arbitrary number $n \geq 3$ of variables [6]. Four quasigroups associated by the low of general associativity are quasigroups isotopic to the same group [7, 8].

In [9, 10] the automorphism groups of quasigroups isotopic to groups and of special forms of quasigroups were considered. In [6] an identity involving five (four) variables in a primitive quasigroup $(Q,\cdot,\backslash,/)$ which is necessary and sufficient for the quasigroup $(Q,\cdot)$ to be isotopic to a group (to an abelian group) was established. Quasigroups isotopic to groups were also studied and some properties of identities for a quasigroup $(Q,\cdot)$ to be isotopic to a group were given in [11–13].

In this note we select a class of identities with permutations including three variables in a quasigroup $(Q,\cdot)$ each of which provides isotopy of this quasigroup to a group and describe an infinite class of identities in a primitive quasigroup $(Q,\cdot,\backslash,/)$ each of which is sufficient for the quasigroup $(Q,\cdot)$ to be isotopic to a group. From these results it follows that in the identity (2) of V. Belousov [6] (in the identity (4) of F. Sokhatsky) which characterizes a quasigroup isotopic to a group two from five (one of four) variables can be fixed and in the identity (3) characterizing quasigroups isotopic to abelian groups one of four variables can be fixed.

2 Necessary notions and results

At first we recall some notions and results which will be used below. The rest needed notions can be found in [14].

Theorem 1 [7]. *If four quasigroups A_i, $i = 1, 2, 3, 4$, given on the same set Q, are connected by the general associative low:*

$$A_1(A_2(x, y), z) = A_3(x, A_4(y, z)), \tag{1}$$

then all these quasigroups are isotopic to the same group.

Lemma 1 [6]. *If in a group $(Q, +)$ for any $x, y \in Q$ the equality $\alpha x + \beta y = \gamma y + \delta x$ holds where $\alpha, \beta, \gamma, \delta$ are some fixed permutations of Q, then the group $(Q, +)$ is abelian.*

Lemma 2 [6]. *If a loop B is isotopic to a quasigroup A, then the isotopy has the form $B = A_{a,b}$, where $A_{a,b} = A^T$, $T = (R_a^{-1}, L_b^{-1}, 1)$, $R_a x = A(x, a)$, $L_b x = A(b, x)$.*

Note that in this case the element $A(b, a)$ is the identity of the loop B, the isotopy T is called principal and the permutations R_a, L_b are called *translations* of the quasigroup A.

Theorem of Albert [14]. *If a loop $(Q, \circ)$ is isotopic to a group $(Q, +)$, then it is a group isomorphic to the group $(Q, +)$.*

We recall that an identity $w_1 = w_2$ in a quasigroup $(Q, \cdot)$ is called balanced if every variable occurs on the left side and on the right side exactly one time. Such identity is called of the I-st kind if the variables are ordered equally, is called of the II-d kind otherwise and it is noncancellable if the following two conditions are satisfied: $xy \in w_1$ implies $xy \in w_2$ or $x * y = yx \in w_2$; if w_1 has the form xu or ux, then w_2 has the form xv or vx where x is a free element and u, v are words in $(Q, \cdot)$ [6]. The number of variables in a balanced identity is the length of this identity. In [6] the following important result is proved.

Theorem 2 [6]. *A quasigroup $(Q, \cdot)$ with a noncancellable balanced identity is isotopic to a group.*

The primitive quasigroup $(Q, \cdot, \backslash, /)$ corresponds to every quasigroup $(Q, \cdot)$, where the inverse operations $(\backslash) = (\cdot)^{-1}$ and $(/) = {}^{-1}(\cdot)$ for the operation $(\cdot)$ are defined as follows: $x \backslash y = z \Leftrightarrow x \cdot z = y$, $x/y = z \Leftrightarrow zy = x$ [14].

In the work [6] V.D. Belousov proved that a quasigroup $(Q,\cdot)$ is isotopic to a group, to an abelian group if and only if in the primitive quasigroup $(Q,\cdot,\backslash,/)$ the following identity respectively holds:

$$((x(y\backslash z))/u)v = x(y\backslash((z/u)v)) \tag{2}$$

$$x\backslash(y(u\backslash v)) = u\backslash(y(x\backslash v)). \tag{3}$$

F.N. Sokhatsky has characterized the quasigroups isotopic to groups by the identity including four variables $(Q,\cdot,\backslash,/)$:

$$((x(u\backslash z))/u)v = x(u\backslash((z/u)v)). \tag{4}$$

It is identity (2) by $y = u$.

3 Identities with permutations and quasigroups isotopic to groups

Let $(Q,\cdot)$, $(Q,*)$ be quasigroups where $x * y = y \cdot x$. Consider the following identities with permutations (or simply, identities):

$$\beta_1(\beta_2(\beta_3 x \otimes_1 \beta_4 y) \otimes_2 \beta_5 z) = \beta_6 x \otimes_3 \beta_7(\beta_8 y \otimes_4 \beta_9 z),$$

where β_i, $i = 1, 2, ..., 9$ (shortly, $i \in \overline{1,9}$), are permutations of Q, $(\otimes_k) = (\cdot)$ or $(\otimes_k) = (*)$, $k \in \overline{1,4}$. Each such identity can be reduced by change of variables to the following identity with the smaller number of permutations:

$$\alpha_1(\alpha_2(x \otimes_1 y) \otimes_2 z) = \alpha_3 x \otimes_3 \alpha_4(\alpha_5 y \otimes_4 \alpha_6 z), \tag{5}$$

where α_i, $i \in \overline{1,6}$, are permutations of Q, $(\otimes_k) = (\cdot)$ or $(\otimes_k) = (*)$, $k \in \overline{1,4}$.

This identity is a special case of the generalized identity of associativity (1) in which $A_1(u,z) = \alpha_1(u \otimes_2 z)$, $A_2(x,y) = \alpha_2(x \otimes_1 y)$, $A_3(x,v) = \alpha_3 x \otimes_3 v$, $A_4(y,z) = \alpha_4(\alpha_5 y \otimes_4 \alpha_6 z)$. By Theorem 1 all these quasigroups are isotopic to the same group.

A special case of (5) is an identity of the form

$$\alpha_2(x \otimes_1 y) \otimes_2 z = \alpha_3 x \otimes_3 \alpha_4(\alpha_5 y \otimes_4 \alpha_6 z), \tag{6}$$

where one of the operations $(\otimes_2)$, $(\otimes_3)$ is $(\cdot)$, another operation is $(*)$.

Theorem 3. *If a quasigroup $(Q,\cdot)$ satisfies the identity (5) (the identity (6)) for some permutations α_i, $i \in \overline{1,6}$ ($i \in \overline{2,6}$), then it is isotopic to a group (to an abelian group).*

If a quasigroup $(Q,\cdot)$ is isotopic to a group (to an abelian group), then it satisfies an identity of the form (5) (of the form (6)) for some suitable permutations α_i, $i \in \overline{1,6}$ ($i \in \overline{2,6}$).

Proof . Let in a quasigroup $(Q,\cdot)$ the identity (5) hold for some permutations, then by Theorem 1 the quasigroup A_1 and, consequently, the quasigroup $(Q,\otimes_2) = (Q,\cdot)$ or $(Q,\otimes_2) = (Q,*)$ is isotopic to a group $(Q,+)$. But if $(Q,*)$ is isotopic to a group, then $(Q,\cdot)$ is also isotopic to this group (if $x*y=\gamma^{-1}(\alpha x+\beta y)$, then $y\cdot x=\gamma^{-1}I(I\beta y+I\alpha x)$ where $Ix=-x$, $-x$ is the inverse element for x in the group).

Conversely, since the quasigroup $(Q,\cdot)$ is isotopic to a group, then the primitive quasigroup $(Q,\cdot,\backslash,/)$ satisfies the identity (2) (or (4)).

But (2) can be written with the help of translations, taking into account that $x/u=R_u^{-1}x$, $u\backslash x=L_u^{-1}x$, as follows:

$$R_u^{-1}(x\cdot L_y^{-1}z)\cdot v=x\cdot L_y^{-1}(R_u^{-1}z\cdot v),\ R_u^{-1}(x\cdot z)\cdot v=x\cdot L_y^{-1}(R_u^{-1}L_yz\cdot v).$$

Putting in the last equality $u=a$, $y=b$, where a,b are some fixed elements of Q, we obtain an identity of the form (5) with $\alpha_2=R_a^{-1}$, $\alpha_1=\alpha_3=\alpha_6=\varepsilon$, $\alpha_4=L_b^{-1}$, $\alpha_5=R_a^{-1}L_b$, where ε is the identity permutation.

Let now a quasigroup $(Q,\cdot)$ satisfy the identity (6) with $(\otimes_2)=(\cdot),(\otimes_3)=(*)$, which is a particular case of (5), so the quasigroup is isotopic to a group. If $(Q,+)$: $x+y=R_a^{-1}x\cdot L_b^{-1}y$ is a loop principally isotopic to the quasigroup $(Q,\cdot)$, then by Albert's Theorem $(Q,+)$ is a group and $x\cdot y=R_ax+L_by$. Pass in (6) to the group operation:

$$R_a\alpha_2(x\otimes_1 y)+L_bz=R_a\alpha_4(\alpha_5y\otimes_4\alpha_6z)+L_b\alpha_3x$$

and put in this equality $y=c$, where c is an arbitrary element of Q, then we obtain the following identity with permutations in the group: $\beta_1x+L_bz=\beta_2z+\alpha_3'x$, where $\beta_1x=R_a\alpha_2R_c^{\otimes_1}x$, $\beta_2z=R_a\alpha_4L_{\alpha_5c}^{\otimes_4}z$, $R_c^{\otimes_1}x=x\otimes_1c$, $L_{\alpha_5c}^{\otimes_4}z=\alpha_5c\otimes_4z, \alpha_3'x=L_b\alpha_3x$. By Lemma 2 the group $(Q,+)$ is abelian.

In the similar way one can prove this fact for the identity (6) with $(\otimes_2)=(*),(\otimes_3)=(\cdot)$.

Conversely, if a quasigroup $(Q,\cdot)$ is isotopic to an abelian group, then the primitive quasigroup $(Q,\cdot,\backslash,/)$ satisfies the identity (3). Write this identity with the help of translations by $x=b$ and transform it in the following way:

$$L_b^{-1}(y\cdot L_u^{-1}v)=L_u^{-1}(y\cdot L_b^{-1}v),\ u\cdot L_b^{-1}(y\cdot v)=y\cdot L_b^{-1}(u\cdot v),$$

$$L_b^{-1}(y\cdot v)*u=y\cdot L_b^{-1}(v*u).$$

It is the identity with permutations (6) by $(\otimes_2)=(*),(\otimes_3)=(\cdot)$. □

Let us illustrate Theorem 3 by the following examples.

Let a quasigroup $(Q,\cdot)$ satisfy the identity of mediality: $xy\cdot uv=xu\cdot yv$, then $xy\cdot R_vu=xu\cdot R_vy$, $xy\cdot u=(x\cdot R_v^{-1}u)\cdot R_vy$ or $(y*x)\cdot u=R_vy*(x\cdot R_v^{-1}u)$. By the fixed $v=a$ this identity has the form (6), so the quasigroup $(Q,\cdot)$ by Theorem 3 is isotopic to an abelian group (it is the known fact by Theorem of Toyoda [14]).

Let $(...((x_1\cdot x_2)x_3)...)x_k=(x_1x_2...x_k)$. In [14] the following balanced identities were considered: $(xy_0y_1...y_n)=(xy_{\theta0}y_{\theta1}...y_{\theta n})$ where θ is a permutation of the set $\{0,1,...,n\}$, $\theta n\neq n$ and it is proved (passing to a group, see Theorem 2) that a quasigroup $(Q,\cdot)$ with this identity is isotopic to an abelian group. It can also be proved directly using Theorem 3. Indeed, let $\theta n=k\neq n$, then $(xy_0y_1...y_n)=(xy_{\theta0}...y_n...y_k)$. Fix all variables in this identity except x,y_k,y_n, then we obtain the following identity with permutations: $\alpha_1(\alpha_2 x \cdot y_k) \cdot y_n = \alpha_3(\alpha_4 x \cdot y_n) \cdot y_k$ or $\alpha_1(y_k * x) \cdot y_n = y_k * \alpha_3(\alpha'_4 x \cdot y_n)$ where $\alpha'_4 x = \alpha_4 \alpha_2^{-1} x$. It is an identity of the form (6) with $(\otimes_2) = (\cdot), (\otimes_3) = (*)$. By Theorem 3 the quasigroup $(Q, \cdot)$ is isotopic to an abelian group.

From Theorem 3 a number of useful corollaries easy follows.

Corollary 1. *A quasigroup $(Q, \cdot)$ is isotopic to a group (to an abelian group) if and only if in it the identity (2) (the identity (3)) holds for some arbitrarily fixed $u = a, y = b$, in particular, for $u = y = a$ (for $x = a$), $a, b \in Q$.*

Proof. If a quasigroup $(Q, \cdot)$ is isotopic to a group (to an abelian group), then the identity (2) (the identity (3)) holds for any fixed elements $u = a$, $y = b$ of Q (for any fixed element $x = a$). Conversely, the identity (2) (the identity (3)) passes in the identity with permutations of the form (5) (of the form (6)) for any fixed elements $u = a$, $y = b$ (for any fixed $x = a$). So the quasigroup is isotopic to a group (to an abelian group) by Theorem 3. □

We say that an identity in a primitive quasigroup $(Q, \cdot, \backslash, /)$ has *the property* (A) (*the property* $(\overline{A})$) if it is possible to select three variables (for example, x, y, z) such that by an arbitrary fixing of the rest variables it has the form (5) (the form (6)).

The following statement is a direct corollary of Theorem 3.

Theorem 4. *If a primitive quasigroup $(Q, \cdot, \backslash, /)$ satisfies an identity with the property (A) (with the property $(\overline{A})$), then the quasigroup $(Q, \cdot)$ is isotopic to a group (to an abelian group).*

If a quasigroup $(Q, \cdot)$ is isotopic to a group (to an abelian group), then in the primitive quasigroup $(Q, \cdot, \backslash, /)$ an identity with the property (A) (with the property $(\overline{A})$) holds.

Note that some properties of identities guaranteeing isotopy of a quasigroup to a group (to an abelian group) were pointed by F. Sokhatsky in [11–13] on the language of special concepts.

Taking into account the previous results it is easy to show that there exists infinite number of identities in a primitive quasigroups $(Q, \cdot, \backslash, /)$ fulfilling of one of which provides that the quasigroup $(Q, \cdot)$ is isotopic to a group. Describe some method of obtaining such identities.

Let $(Q, \cdot)$ be a quasigroup, $L_t x = t \cdot x, R_t x = x \cdot t$, $M = \{u, v, w, ...\}$ be a finite set of variables not containing x, y, z. Put in an identity of the form (5) (of the form (6)) each of non-identical permutations α_i, $i \in \overline{1,6}$ by a product $S_{t_1}^{\epsilon_1} S_{t_2}^{\epsilon_2} ... S_{t_{n_i}}^{\epsilon_{n_i}}$ where $S_{t_i} = R_{t_i}$ or $S_{t_i} = L_{t_i}$, $\epsilon_i = \pm 1$, n_i are some natural numbers, t_i are arbitrary words with some variables from M with respect to the signature $(\cdot, \backslash, /)$ of a primitive quasigroup $(Q, \cdot, \backslash, /)$. It is true the following

Theorem 5. *Let in a primitive quasigroup $(Q, \cdot, \backslash, /)$ an identity obtained from (5) (from (6)) by the described above method hold, then the quasigroup $(Q, \cdot)$ is isotopic to a group (to an abelian group).*

Proof. Use Theorem 4 taking into account that in this case in the primitive quasigroup an identity with the property (A) (with the property $(\overline{A})$) holds. $\square$

Corollary 2. *For any natural number $n \geq 4$ there exists an identity in the signature $(\cdot, \backslash, /)$, involving n variables, fulfillment of which in a primitive quasigroup $(Q, \cdot, \backslash, /)$ is sufficient for the quasigroup $(Q, \cdot)$ be isotopic to a group (to an abelian group).*

Proof. For finding an identity of length n a set M involving $n-3$ variables and the given above method can be used. $\square$

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