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On description of some radical filters of noetherian rings

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Abstract. We describe some radical filters of noetherian rings.

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Let R be a ring. The category of right R-modules will be denoted by $\mathrm{Mod}-R$. Let $L_r(R)$ be the set of all right ideals of the ring R. Let $\mathrm{Spec}_r(R)$ be the set of all maximal right ideals of R. If N is a submodule of a module M we shall write $N \le M$. A set $\mathcal{P} \subseteq R$ is said to be similarly-closed in case

$$\forall p \in \mathcal{P}\ \forall r \in R: \ r \sim p \ \Rightarrow \ r \in \mathcal{P}.$$

Let

$$\mathcal{E}_{\mathcal{P}} := \{I \in L_r(R) \mid (\exists n \in \mathbb{N}\ \exists a_1, a_2, \dots, a_n \in \mathcal{P}: \ I = a_1 a_2 \dots a_n R) \vee I = R\}.$$

We shall say that R is a domain if $\forall a, b \in R \setminus \{0\}: \ ab \in R \setminus \{0\}$.

A ring R is said to be a principal ideal domain in case it is a domain such that every its right ideal is a right principal ideal and every its left ideal is a left principal ideal.

Definition 1. *A right ideal A is similar to a right ideal B if $R/A \cong R/B$. In this case we shall write $A \sim B$.*

Definition 2. *A set $G \subseteq L_r(R)$ is said to be similarly-closed in case $\forall A \in G\ \forall B \in L_r(R): \ A \sim B \ \Rightarrow \ B \in G$.*

Let $[G] = \{P \mid \exists A \in G: \ P \cong R/A\}$ for $G \subseteq \mathrm{Spec}_r(R)$.

Definition 3 [l, 2]. *A set $E \subseteq L_r(R)$is called a radical filter if the following conditions are fulfilled*

Gl. $I \in E$, $I \subseteq J$, $J \in L_r(R) \ \Rightarrow J \in E$.

G2. $I \in E$, $a \in R \ \Rightarrow \ (I : a) \in E$.

G3. $I \in E$, $J \subseteq I$, $J \in L_r(R)$, $\forall a \in I : (J : a) \in E \ \Rightarrow \ J \in E$.

Let

$$E_G = \{I \mid \exists n \in \mathbb{N} \cup \{0\} \exists A_0, A_1, \dots, A_n \in L_r(R): \ I = A_0 \subseteq A_1 \subseteq \dots \subseteq A_n = R \\ \wedge\ \forall i \in \{1, 2, \dots, n\}: \ A_i/A_{i-1} \in [G]\}.$$

Theorem 1. *Let R be a right noetherian ring. If $G \subseteq \mathrm{Spec}_r(R)$ is similarly-closed then E_G is a radical filter of R.*

Proof. Gl. Let $I \in E_G$, $I \subseteq J$, $J \in L_r(R)$. Then

$$\exists n \in N \cup \{0\}\ \exists A_0, A_1, \ldots, A_n \in L_r(R):\ I = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n = R$$

$$\wedge\ \forall i \in \{1, 2, \ldots, n\}:\ A_i/A_{i-1} \in [G] \qquad (*)$$

It follows from this that

$$0 = A_0/I \subseteq A_1/I \subseteq \ldots \subseteq A_n/I = R/I\ \wedge\ \forall i \in \{1, 2, \ldots, n\}:$$

$$(A_i/I)/(A_{i-1}/I) \in [G] \qquad (**)$$

and $0 \subseteq J/I \subseteq R/I$. Now taking this into consideration by Corollary 3.5.3 [4] we have that

$$J/I = B_0/I \subseteq B_1/I \subseteq \ldots \subseteq B_k/I =$$

$$= R/I\ \wedge\ \forall i \in \{1, 2, \ldots, k\}:\ (B_i/I)/(B_{i-1}/I) \in [G]$$

for some $B_0, B_1, \ldots, B_k \in L_r(R)$.

It follows from this that $J \in E_G$.

G2. Let $I \in E_G$, $a \in R$. Then we have (*). Hence (**). But $0 \subseteq (aR + I)/I \subseteq R/I$. Now taking this into consideration by Corollary 3.5.3 [4] we have that

$$0 = C_0/I \subseteq C_1/I \subseteq \ldots \subseteq C_t/I =$$

$$= (aR + I)/I\ \wedge\ \forall i \in \{1, 2, \ldots, t\}:\ (C_i/I)/(C_{i-1}/I) \in [G]$$

for some $C_0, C_1, \ldots, C_t \in L_r(R)$.

It is obviour that $R/(I : a) \cong (aR + I)/I$. Taking this into account we obtain that $(I : a) \in E_G$.

G3. Let $I \in E_G$, $J \subseteq I$, $J \in L_r(R)$, $\forall a \in I : (J : a) \in E_G$. It is obvious that R/J is noetherian as a factor-module of the right noetherian module R. Hence I/J is also noetherian as a submodule of R/J. It is clear that $\sum\limits_{a \in I} (aR + J)/J = I/J$. Since I/J is finitely generated as a noetherian module,

$$(a_1R + J)/J + (a_2R + J)/J + \ldots + (a_sR + J)/J = I/J$$

for some $\{a_1, a_2, \ldots, a_s\} \subseteq I$.

Then $I/J \cong \Big(\bigoplus\limits_{h=1}^{s} (a_hR + J)/J\Big)/S$ for some submodule S of $\bigoplus\limits_{h=1}^{s} (a_hR + J)/J$.

Since

$$\forall h \in \{1, 2, \ldots, s\} : (a_hR + J)/J \cong R/(J : a_h)\ \wedge\ (J : a_h) \in E_G,$$

$$\forall h \in \{1, 2, \ldots, s\}\ \exists \{B_{h,i} \mid B_{h,i} \leq (a_hR + J)/J,\ i \in \{0, 1, \ldots, p_h\}\} :\ 0 = B_{h,0} \subseteq$$

$$\subseteq B_{h,1} \subseteq \ldots \subseteq B_{h,p_h} = (a_hR+J)/J \ \wedge\ \forall i \in \{1,2,\ldots,p_h\}:\ B_{h,i}/B_{h,i-1} \in [G].$$

It follows from this that

$$0 = B_{1,0} \subseteq B_{1,1} \subseteq \ldots \subseteq B_{1,p_1} \oplus B_{2,1} \subseteq \ldots \subseteq B_{1,p_1} \oplus B_{2,p_2} \subseteq \ldots$$

$$\ldots \subseteq B_{1,p_1} \oplus B_{2,p_2} \oplus \ldots \oplus B_{s,p_s} = \bigoplus_{h=1}^{s}(a_hR+J)/J$$

is a composition series for $\bigoplus_{h=1}^{s}(a_hR+J)/J$ with all factors belonging to $[G]$. Since $0 \subseteq S \subseteq \bigoplus_{h=1}^{s}(a_hR+J)/J$, by Corollary 3.5.3 [4], there exists a composition series $0 \subseteq L_0/S \subseteq L_1/S \subseteq \ldots \subseteq L_u/S = \Big(\bigoplus_{h=1}^{s}(a_hR+J)/J\Big)/S$ with all factors belonging to $[G]$ (where $S \le L_0 \le L_1 \le \ldots \le L_u = \bigoplus_{h=1}^{s}(a_hR+J)/J$). Now taking into account $I/J \cong \Big(\bigoplus_{h=1}^{s}(a_hR+J)/J\Big)/S$, it is easy to see that there exists a composition series $0 = M_0/J \subseteq M_1/J \subseteq \ldots \subseteq M_u/J = I/J$ with all factors belonging to $[G]$. But since $I \in E_G$, there exists a composition series $0 = N_0/I \subseteq N_1/I \subseteq \ldots \subseteq N_v/I = R/I$ with all factors belonging to $[G]$. Therefore $0 = M_0/J \subseteq M_1/J \subseteq \ldots \subseteq M_u/J \subseteq N_1/J \subseteq \ldots \subseteq N_v/J = R/J$ is a composition series with all factors belonging to $[G]$. Therefore $J \in E_G$. □

Lemma 2. *Let R be a domain. If $\forall i \in \{1,2,\ldots,n\}: a_i \in R\backslash\{0\} \wedge a_iR$ is a maximal right ideal of R, then*

$$L_0 \subseteq L_1 \subseteq \ldots \subseteq L_{n-1} \subseteq L_n$$

is a composition series for L_n, where

$$L_s := a_1a_2\ldots a_{n-s}R/a_1a_2\ldots a_nR,\ s \in \{0,1,\ldots,n-1\},\ L_n := R/a_1a_2\ldots a_nR.$$

Proof. It is clear that

$$L_{s+1}/L_s \cong R/a_{n-s}R,\ s \in \{0,1,\ldots,n-1\}. \qquad \square$$

Therefore we have the following proposition.

Proposition 3. *Let R be a right noetherian domain and let $\mathcal{P}$ be a similarly-closed subset of R such that $0 \notin \mathcal{P}$. If $G \subseteq \mathrm{Spec}_r(R)$ is a similarly-closed set containing the set $\{aR \mid a \in \mathcal{P}\}$, then the radical filter E_G contains the set $\mathcal{E}_{\mathcal{P}}$.*

Let R be a principal ideal domain. An element $p \in R$ is said to be an atom in case $p \neq 0 \ \wedge\ p \notin U(R) \ \wedge\ (\forall a,b \in R:\ (p = ab \ \Rightarrow\ a \in U(R) \ \vee\ b \in U(R)))$. The set of all atoms of R will be denoted by Ω_R .

Taking into account Theorem 1 and the proof of Lemma 1 we obtain (see [3, p.69–70] and [4, Corollary 3.5.3]).

Corollary 4 [5]. *Let R be a principal ideal domain. If $\mathcal{P} \subseteq \Omega_R$ is a similarly-closed set then $\mathcal{E}_{\mathcal{P}}$ is a radical filter.*

Corollary 5. *Let R be a Dedekind domain. If $G \subseteq \mathrm{Spec}_r(R)$ is similarly-closed then $T_G := \{I \in L_r(R) \mid (\exists n \in \mathbb{N}\ \exists I_1, I_2, \ldots, I_n \in G :\ I = I_1 I_2 \ldots I_n)\ \vee\ I = R\}$ is a radical filter of R.*

Proof. It is obvious that

$$L_0 \subseteq L_1 \subseteq \ldots \subseteq L_{n-1} \subseteq L_n$$

is a composition series for L_n, where

$$L_s := I_1 I_2 \ldots I_{n-s}/I_1 I_2 \ldots I_n,\ s \in \{0, 1, \ldots, n-1\},\ L_n := R/I_1 I_2 \ldots I_n,$$

because

$$L_{s+1}/L_s \cong R/I_{n-s},\ s \in \{0, 1, \ldots, n\}$$

(see Proposition 17 [6]).

Now apply Theorem l and Corollary 3.5.3 [4]. □

References

[1] Mishina A.P., Skorniakov L.A. *Abelian groups and modules.* Providence, Rhode Island.: Amer. Math. Soc, 1976.

[2] Kashu A.I. *Radicals and torsions in modules.* Kishinev, Stiintsa, 1983 (in Russian).

[3] Jacobson N. *The theory of rings.* Surveys of the Amer. Math. Soc., vol. 2, Providence, 1942.

[4] Kasch F. *Moduln and ringe.* Stuttgart, B.G. Teubner, 1977.

[5] Horbachuk O., Komarnytskyi M. *Radical filters in principal ideal domains.* DAN USSR, Fiz.-Mat. i Tehn. Nauki, 1977, **2**, A, p. 103–104. (in Russian).

[6] Dummit D.S., Foot R.M. *Abstract algebra.* Prentice Hall, Upper Saddle River, N.J. 07458, 1999.

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