BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 2(54), 2007, Pages 118–124
ISSN 1024–7696

Quasivarieties of Commutative Solvable Moufang loops

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Abstract. This paper studies the quasivarieties of commutative Moufang loops, whose loops are solvable.

Mathematics subject classification: 20N05.
Keywords and phrases: Quasivariety, loop, solvable, nilpotent, spectrum.

1. The class of solvable commutative Moufang loops is one of the most examined classes of Moufang loops, the main results of which are presented in R. Bruck's monograph. Based on these results and the construction of the direct spectrum, and the limit of this spectrum, the present paper describes some equivalent conditions for a quasivariety of commutative Moufang loops to contain only solvable (nilpotent) loops.

Now we will make an overview of some notions connected with Moufang loops, some of which may be found in [1] or [2].

An algebra $(L, \cdot, ^{-1})$ of $< 2, 1 >$ type, whose elements and main operations satisfy the identities

$$x^{-1} \cdot xy = y = yx \cdot x^{-1}, xy = yx, xy \cdot zx = x(yz \cdot x)$$

is called a *commutative Moufang loop*. For convenience the commutative Moufang loop $(L, \cdot, ^{-1})$ will be marked simply as L.

Let L be a commutative Moufang loop, $x, y, z \in L$ and $X, Y, Z \subseteq L$. The associator of the (x, y, z) triplet is the element

$$[x, y, z] = (x \cdot yz) \cdot (xy \cdot z)^{-1}.$$

We will denote by $[X, Y, Z]$ the subloop generated in L by all the associators $[x, y, z]$ where $x \in X$, $y \in Y$, $z \in Z$. The subloop $[X, L, L]$ is called the mutual *associator* or the mutual *associator subloop* of X in L. In particular, when $X = L$, the subloop $[L, L, L]$ will be named simply *associator* or the *associator subloop* or the *derived subloop* of loop L, which sometimes will be denoted L'. We notice that a certain subloop $H \subseteq L$ is *normal* in L, i.e. the equalities $xH = Hx$, $x \cdot yH = xy \cdot H$hold for any $x, y \in L$, if and only if the subloop H has the property $[H, L, L] \subseteq H$.

Here in after with the help of the mutual associator of the commutative Moufang loop L, we will define two descendent series $L^{(i)}$ and L_i, $i = 0, 1, 2, \dots$ of normal subloops which tend to L. By the definition

$$L^{(0)} = L, L' = [L, L, L], \dots, L^{(n)} = [L^{(n-1)}, L^{(n-1)}, L^{(n-1)}], \dots$$

and

$$L_0 = L, L_1 = [L, L, L], \dots, L_n = [L_{n-1}, L, L], \dots$$

It results that

$$L^{(0)} \supseteq L' \supseteq \dots \supseteq L^{(n)} \supseteq \dots \tag{1}$$

and

$$L_0 \supseteq L_1 \supseteq \dots \supseteq L_n \supseteq \dots \ . \tag{2}$$

Series (1) is called *the series of associative subloops* or *the series of the derived subloops*, while series (2) is called *the central descendent series* of L.

The commutative Moufang loop L is called *nilpotent* if there exists such a number n that $L_n = \{1\}$. The smallest natural number n for which $L_n = \{1\}$ is called *the nilpotence class* of L.

The commutative Moufang loop L is called *solvable* if there exists such a number n that $L^{(n)} = \{1\}$. The smallest natural number n for which $L^{(n)} = \{1\}$ is called *the solvability class* of L.

A class of commutative Moufang loops is called a *quasivariety* (respectively *variety*) if it consists only of those loops in which all the formulae of the same system of quasiidentities (respectively identities) are true.

With the help of the variables $x_1, x_2, \dots$ in the signature of commutative Moufang loops we define by induction the following words $[x_1, \dots, x_{2n+1}]$ and $\delta_n(x_1, \dots, x_{3^n})$:

$$[x_1, x_2, x_3] = (x_1 \cdot x_2x_3)^{-1} \cdot (x_1x_2 \cdot x_3), \delta_1(x_1, x_2, x_3) = [x_1, x_2, x_3]$$

and

$$[x_1, \dots, x_{2n+1}] = [[x_1, \dots, x_{2n-1}], x_{2n}, x_{2n+1}],$$

$$\delta_n = [\delta_{n-1}(x_1, \dots, x_{3^{n-1}}), \delta_{n-1}(x_{3^{n-1}+1}, \dots, x_{2\cdot 3^{n-1}}), \delta_{n-1}(x_{2\cdot 3^{n-1}+1}, \dots, x_{3^n})]$$

for $n \geq 2$.

Now we notice that the identity $[x_1, \dots, x_{2n+1}] = 1$ defines the variety of commutative Moufang loops of class n, while the identity $\delta_n(x_1, \dots, x_{3^n}) = 1$ defines the variety of commutative solvable Moufang loops of class n.

A collection $\Lambda =< I, L_i, \varphi_{ij} >$, consisting of an oriented set $< I, \leq >$, a family $\{L_i | i \in I\}$ of commutative Moufang loops L_i and a family $\{\varphi_{ij} | i, j \in I, i \leq j\}$ of homomorphisms $\varphi_{ij} : L_i \longrightarrow L_j$ is called a direct spectrum if φ_{ij} is an identical application and $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$ for every $i, j, k \in I, i \leq j \leq k$. We denote by L_∞ the quotient set of the set $\bigcup_{i \in I} L_i x\{i\}$ by the equivalence relation $\equiv$:

$$(a, i) \equiv (b, j) \Longleftrightarrow \exists k \in I (i \leq k \& j \leq k \& \varphi_{ik}(a) = \varphi_{jk}(b)).$$

Let $[a, i]$ be the coset with respect to the equivalence relation $\equiv$ that contains the element $(a, i) \in L_i x\{i\}$. On the set L_∞ we define operations of commutative Moufang loops in the fallowing way:

$$[a, i]^{-1} = [a^{-1}, i], \quad [a, i] \cdot [b, j] = [\varphi_{ik}(a) \cdot \varphi_{jk}(b), k], \quad \textit{where} \quad i \leq k, \quad j \leq k.$$

We can check without any difficulty the correctness of these operations and whether they satisfy the relations of defining the commutative Moufang loop. The obtained commutative Moufang loop will be called *the direct limit over the direct spectrum* Λ and will be denoted by $\varinjlim \Lambda$ [3].

From the definition of $\varinjlim \Lambda$ it results that an atomic formula $u(x_1, \dots, x_n) = v(x_1, \dots, x_n)$ holds true in the commutative Moufang loop $\varinjlim \Lambda$ by replacing x_j with $[a_j, i_j]$, $j = 1, \dots, n$ if and only if there exists an index $k \in I$ for which we have elements $a'_j \in L_k$, $j = 1, \dots, n$ such that $[a_j, \ i_j] = [a'_j, k]$, $j = 1, \dots, n$, and the equality $u(a'_1, \dots, a'_n) = v(a'_1, \dots, a'_n)$ holds in L_k.

Consequently, *every universal formula which is true in each loop of the direct spectrum* Λ *also holds in the limit of the direct spectrum.* Therefore we may also formulate the next proposition:

The quasivariety of commutative Moufang loops is closed relative to the limit of the direct spectrum.

At the same time we notice that the last affirmations result directly from the Tarski-Los theorem [4, p. 174].

2. In this section by *solvable quasivariety* (respectively, *nilpotent*) we will understand the quasivariety of commutative Moufang loops whose loops are solvable (respectively, nilpotent).

Theorem 1. *A quasivariety of commutative Moufang loops is solvable if and only if each of its loops is different from its associator.*

Proof. Indeed, let $\Re$ be a quasivariety of commutative Moufang loops. It is clear that, if $\Re$ is a solvable quasivariety, then for each loop $L \in \Re$ we have $L \neq L'$. Inversely, let $\Re$ be a quasivariety each loop of which is different from its associator, and let us suppose that the quasivariety $\Re$ contains unsolvable loops. Then it is clear that the free loop F of the quasivariety $\Re$ with the countable set $\{x_1, x_2, \dots\}$ of free generators $x_1, x_2, \dots,$ is unsolvable.

We analyze the direct spectrum $\tau =< N, F_i, \varphi_{ij} >$ consisting of the set of integers $N = \{0, 1, 2, \dots\}$, the loop's isomorphic copies F_i of the loop $F_0 = F$ with the sets $\{x^i_1, x^i_2, \dots\}$, $i = 1, 2, \dots,$ of free generators and the homomorphisms φ_{ij} defined in the following way:

$$\varphi_{ij} = \varphi_{j-l_j} \dots \varphi_{i+l_{i+2}} \varphi_{i_{i+1}},$$

where the homomorphisms $\varphi_{k_{k+1}} : F_k \longrightarrow F_{k+1}$, $k = 0, 1, 2, \dots,$ are defined on generators by the equalities

$$\varphi_{0_{k+1}}(x_j) = [x^l_{3j-2}, x^l_{3j-1}, x^l_{3j}],$$

$$\varphi_{k_{k+1}}(x^k_j) = [x^{k+1}_{3j-2}, x^{k+1}_{3j-1}, x^{k+1}_{3j}], \quad j = 1, 2, \dots \ .$$

Let $\varinjlim \tau$ be the direct limit of the direct spectrum τ of commutative Moufang loops. We notice that the commutative Moufang loop $\varinjlim \tau$ is not a unit loop. Indeed, for example the element $[x_1, 0] \in \varinjlim \tau$ is different from the unit element, which in the loop $\varinjlim \tau$ has the form $[1, 0]$, since according to the unsolvability of the loop F,

$$\varphi_{0n}(x_1) = \delta_n(x_1^n, \ldots, x_{3^n}^n) \neq 1$$

for every $n \geq 1$.

According to the above observation, the loop $\varinjlim \tau$ belongs to the quasivariety $\Re$. We show that this loop coincides with associator. Indeed, the commutative Moufang loop $\varinjlim \tau$ is generated by elements of the form

$$[x_i, 0], [x_i^k], \quad i = 1, 2, \ldots; \quad k = 1, 2, \ldots$$

and since

$$[x_i, 0] = [\varphi_{01}(x_i), 1] = [[x_{3i-2}^1, x_{3i-1}^1, x_{3i}^1], 1] =$$
$$= [[x_{3i-2}^1, 1], [x_{3i-1}^1, 1], [x_{3i}^1, 1]] \in (\varinjlim \tau)',$$

we have

$$[x_i^k, k] = [\varphi_{k_{k+1}}(x_i^k), k+1] = [[x_{3i-2}^{k+1}, x_{3i-1}^{k+1}, x_{3i}^{k+1}], k+1] =$$
$$= [[x_{3i-2}^{k+1}, k+1], [x_{3i-1}^{k+1}, k+1], [x_{3i}^{k+1}, k+1]] \in (\varinjlim \tau)'.$$

Thus, generators of the loop $\varinjlim \tau$ belong to the associator $(\varinjlim \tau)'$ and therefore, we have $\varinjlim \tau = (\varinjlim \tau)'$. In such a way the non-identity loop $\varinjlim \tau$ from the quasivariety $\Re$ coincides with its associator, but this contradicts the hypothesis.

This proves the theorem.

A commutative Moufang loop L is called named *approximately-solvable* if for every non-identity element $x \in L$ there exists an homomorphism φ_x which maps L on a solvable loop in such a way that $\varphi_x(x) \neq 1$, which is equivalent to the fact that L contains a collection of normal subloops $\{H_i | i \in I\}$ such that the quotient loop $L/H_i, i \in I$, should be solvable and $\bigcap\limits_{i \in I} H_i = \{1\}$. Therefore according Theorem 2 from [4, p, 74], the commutative Moufang approximately-solvable loop L is a subdirect product of loops $L/H_i, i \in I$ i.e. L is isomorphically included in the topological product

$$\overline{L} = \prod_{i \in I} L/H_i,$$

where each component of this product is a solvable loop, thus $\overline{L} \neq (\overline{L})'$, and from here it results that $L \neq L'$, from where according to Theorem 1 results the following

Corollary 1. *The quasivariety $\Re$ of commutative Moufang loops is solvable if and only if every loop from $\Re$ is approximately-solvable.*

Also we notice that if each loop L of the quasivariety $\Re$ satisfies the condition $\bigcap\limits_{i=1}^{\infty} L^{(i)} =< 1 >$, then $L \neq L'$ and, according to Theorem 1 the quasivariety $\Re$ is solvable. Thus we may formulate the following

Corollary 2. *The quasivariety* $\Re$ *of commutative Moufang loops is solvable if and only if for each loop* $L \in \Re$, $\bigcap_{i=1}^{\infty} L^{(i)} =< 1 >$ *takes place.*

Theorem 2. *The quasivariety* $\Re$ *of commutative Moufang loops is nilpotent if and only if for each loop* $L \in \Re$ *and each non-identity subloop* $H \subseteq L$, $[H, L, L] \neq H$ *takes place.*

Proof. Let $\Re$ be a nilpotent variety, L a loop from $\Re$ and H a subloop of L. If we admit $[H, L, L] = H$, then the inclusion $H \subseteq L$ implies $H \subseteq L_i$ for every $i \geq 1$. From here we obtain $H =< 1 >$.

Inversely, let us consider that for every loop $L \in \Re$ and every non-identity subloop $H \subseteq L$ takes place $[H, L, L] \neq H$. We assume that the quasivariety $\Re$ is not nilpotent, and then it is clear that the free commutative Moufang loop F of the quasivariety $\Re$ with the countable set $\{x_1, x_2, \dots\}$ of free generators $x_1, x_2, \dots$ is not nilpotent.

We analyze the direct spectrum $\tau =< N, F_i, \varphi_{ij} >$, consisting of the integers set $N = \{0, 1, 2, \dots\}$, the isomorphic copies F_i of the loop $F_0 = F$ with the sets $\{x_1^i, x_2^i, \dots\}$, $i = 1, 2, \dots$, of free generators and the homomorphisms φ_{ij} defined as follows:

$$\varphi_{01}(x_1) = [x_1^1, x_2^1, x_3^1], \quad \varphi_{i_{i+1}}(x_1^i) = [x_1^{i+1}, x_2^{i+1}, x_3^{i+1}],$$

$$\varphi_{01}(x_j) = x_j^1, \quad \varphi_{i_{i+1}}(x_j^i) = x_j^{i+1}, \quad j = 2, 3, \dots \ .$$

We denote by $L = \varinjlim \tau$ and let H be the normal subloop of the commutative Moufang loop L generated by the elements

$$[x_1, 0], [x_1^1, 1], \dots, [x_1^n, n], \dots \ .$$

We notice that in the loop L the following relations are true

$$[x_1, 0] = [[x_1^1, x_2^1, x_3^1], 1] = [[x_1^1, 1], [x_2^1, 1], [x_3^1, 1]],$$

$$[x_1^i, i] = [[x_1^{i+1}, x_2^{i+1}, x_3^{i+1}], i + 1] =$$
$$= [[x_1^{i+1}, i + 1], [x_2^{i+1}, i + 1], [x_3^{i+1}, i + 1]], \quad i = 1, 2, \dots,$$

so the generators of H belong to the associator group $[H, L, L]$ too, therefore $H \subseteq [H, L, L]$. But the subloop H is normal in the loop L, that is why $[H, L, L] \subseteq H$ from where we obtain the contradiction equality $H = [H, L, L]$.

This completes the proof of the theorem.

A commutative Moufang loop L is called *approximately-nilpotent* if for every non-identity element $x \in L$ there exists an homorphomism φ_x which maps L on a nilpotent loop in such a way that $\varphi_x(x) \neq 1$ which is equivalent with the fact that L contains a collection of normal subloops $\{H_i | i \in I\}$ such that the quotient loop $L/H_i, i \in I$, should be nilpotent and $\bigcap_{i \in I} H_i = \{1\}$. Therefore according to Theorem

2 from [4, p. 74], the nilpotent-approximately commutative Moufang loop L is a subdirect product of the cartesian product

$$\overline{L} = \prod_{\substack{x \in L \\ x \neq 1}} L/H_x,$$

where each component of this product is a nilpotent loop, and thus for every normal subloop $\overline{H}$ of the loop $\overline{L}$ the strict inclusion $[\overline{H}, \overline{L}, \overline{L}] \subset \overline{H}$ takes place. Consequently, normal subloops of the loop L posses the same property. From where, according to Theorem 2 the following

Corollary 3. *The quasivariety $\Re$ of commutative Moufang loops is nilpotent if and only if every loop from $\Re$ is approximately-nilpotent.*

We notice also that if each loop L of the quasivariety $\Re$ satisfies the condition $\bigcap_{i=1}^{\infty} L_i =< 1 >$, then it results that the loop L is approximately-nilpotent and, according to Corollary 3, the quasivariety $\Re$ is solvable. In the same way we may formulate the next

Corollary 4. *The quasivariety $\Re$ of commutative Moufang loops is solvable if and only if $\bigcap_{i=1}^{\infty} L^{(i)} =< 1 >$ holds for any loop $L \in \Re$.*

According to R. Bruck and Slaby's results [1] (see also Manin [5] and Smith [6]), the commutative Moufang loop with n generators is (central-) nilpotent. Since the works of J. Smith [7], G. Malbos [8] and L. Beneteau [9] demonstrated that the exact upper border of the nilpotence class of the free commutative 3-periodic Moufang loops with n free generators is $n-1$, it follows that the class of all 3-periodic commutative Moufang loops is not nilpotent. Therefore from the proof of Theorem 2 results

Corollary 5. *There exist 3-periodic commutative Moufang loops which contain subloops that coincide with their associator in the given loop.*

Remark. Since every variety is a quasivariety the affirmations of the proved theorems and their corollaries in the case of varieties of commutative Moufang loops are also true.

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Received September 29, 2006

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