Multi-dimensional Darboux type differential systems with quadratic nonlinearities

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Abstract. In the article the \( n \)-dimensional autonomous Darboux type differential systems with nonlinearities of the 2\(^{nd} \) degree are considered. With the aid of theorem on integrating factor the particular invariant \( GL(n, \mathbb{R}) \)-integrals are constructed as well as the first integrals of Darboux type for considered systems. These integrals represent the algebraic curves of the 1\(^{st} \) degree. The recurrence formula of particular invariant \( GL(n, \mathbb{R}) \)-integrals of the Darboux type differential system is found.

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Consider the system of differential equations

\[
\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \equiv P^j(x, a) \quad (j, \alpha, \beta = 1, n; \ n \geq 2),
\]

(1)

where coefficient tensor \( a^j_{\alpha\beta} \) is symmetrical in lower indices, in which the complete convolution holds. The system (1) is considered with the action of the group \( GL(n, \mathbb{R}) \) of center-affine transformations [1], and \( x = (x^1, x^2, ..., x^n) \) is a phase variable vector of the system.

Suppose that system (1) admits \( (n - 1) \)-dimensional commutative Lie algebra with operators

\[
X_\alpha = \xi_\alpha^j(x) \frac{\partial}{\partial x^j} \quad (j = 1, n; \ \alpha = 1, n - 1)
\]

(2)

and

\[
\Lambda = P^j(x, a) \frac{\partial}{\partial x^j} \quad (j = 1, n).
\]

(3)

Consider the determinant constructed on coordinates of operators (2)-(3) as follows

\[
\Delta = \begin{vmatrix}
\xi_1^1 & \xi_2^1 & \xi_3^1 & ... & \xi_n^1 \\
\xi_1^2 & \xi_2^2 & \xi_3^2 & ... & \xi_n^2 \\
\xi_1^3 & \xi_2^3 & \xi_3^3 & ... & \xi_n^3 \\
... & ... & ... & ... & ... \\
\xi_1^{n-1} & \xi_2^{n-1} & \xi_3^{n-1} & ... & \xi_n^{n-1} \\
\xi_1^n & \xi_2^n & \xi_3^n & ... & \xi_n^n
\end{vmatrix}
\]

(4)

From [2] it follows that holds

\( \odot \quad O.V. \ \text{Diaconescu, 2007} \)
Theorem 1. If an-dimensional differential system (1) admits \((n-1)\)-dimensional commutative Lie algebra of operators (2), then the function \(\mu = \frac{1}{\Delta}\), where \(\Delta \neq 0\) from (4), is the integrating factor for Pfaff equations

\[
\sum_{i=1}^{(-1)^{i+j}} \begin{vmatrix}
\xi_1^i & \ldots & \xi_1^{i-1} & \xi_1^{i+1} & \ldots & \xi_1^n \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\xi_j^{i-1} & \ldots & \xi_j^{i-1} & \xi_j^{i+1} & \ldots & \xi_j^n \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
p^i & \ldots & p^{i-1} & p^{i+1} & \ldots & p^n \\
\end{vmatrix} dx^j = 0 \quad (i = \overline{1,n}; \; j = \overline{1,n-1}),
\]

defining a general integral of the system (1).

Following [3], consider system (1) in a "Darboux" like case, i.e. system (1) written in the form

\[
\frac{dx^j}{dt} = a^j_\alpha x^\alpha + 2x^j R(x) \equiv P^j(x,a) \quad (j, \alpha = \overline{1,n}; \; n \geq 2),
\]

where \(R(x) \neq 0\) is a homogeneous linear polynomial with constant coefficients in coordinates of the vector \(x\).

According to [4] will treat invariant \(GL(n, \mathbb{R})\)-integrating factors and invariant \(GL(n, \mathbb{R})\)-integrals of the system (6) with \(n = 2, 3, 4, 5, \ldots\)

1. Case \(n = 2\). Will denote the invariants and comitants of the system (1) as follows

\[
I_{1,2} = a^\alpha_{\alpha_1}, \; I_{2,2} = a^\alpha_{\alpha_2}a^\alpha_{\alpha_2}, \; K_{1,2} = a^\alpha_\alpha x^\alpha x^{\alpha_2} \varepsilon_{\alpha_1 \alpha_2}, \\
P_{1,2} = a^\alpha_{\alpha_1}x^\beta, \; P_{2,2} = a^\alpha_{\alpha_2}a^\alpha_{\alpha_2} x^\beta, \; \tilde{K}_{1,2} = a^\alpha_{\beta_1}x^\beta x^{\alpha_2} \varepsilon_{\alpha_1 \alpha_2},
\]

where the first of lower indices for \(I, K, P\) and \(\tilde{K}\) from (7) shows the degree of invariant or comitant with respect to coefficients of the system (1), and the second lower index shows the dimension of the system \((n = 2)\). In [4] it is shown that invariant condition which differs the system (6) from (1) is the following: \(\tilde{K}_{1,2} \equiv 0\).

In the same paper with the aid of Theorem 1 and expressions (7) is proved

Theorem 2. System (1) with \(\tilde{K}_{1,2} \equiv 0\) and \(n = 2\) has the invariant \(GL(2, \mathbb{R})\)-integrating factor \(\mu\) of the form \(\mu^{-1} = K_{1,2} \Phi_{2,2}\), where \(K_{1,2} = 0\) and

\[
\Phi_{2,2} \equiv 8I_{1,2}P_{1,2} - 12P_{2,2} + 3(I_{1,2}^2 - I_{2,2}) = 0
\]

are invariant particular \(GL(2, \mathbb{R})\)-integrals of this system.
2. Case $n = 3$. Following [3] will denote the invariants, comitants and covariants of the system (1) as follows

\[
I_{1,3} = a_{a_1}^2, \quad I_{2,3} = a_{a_2}^3, \quad I_{3,3} = a_{a_3}^4, \\
K_{3,3} = a_{a_1}^2 a_{a_2}^3 a_{a_3}^4 x^{a_1} x^{a_2} x^{a_3} x^{a_4} x^{a_5} x^{a_6} x^{a_7} x^{a_8} x^{a_9} x^{a_{10}} x^{a_{11}}, \\
P_{1,3} = a_{a_1}^2 a_{a_2}^3 x^{a_1}, \quad P_{2,3} = a_{a_2}^3 a_{a_3}^4 x^{a_2}, \quad P_{3,3} = a_{a_3}^4 a_{a_1}^2 a_{a_2}^3 x^{a_3}, \\
\tilde{K}_{1,3} = a_{a_1}^2 a_{a_2}^3 x^{a_1} x^{a_2} x^{a_3} x^{a_4} x^{a_5} x^{a_6} x^{a_7} x^{a_8} x^{a_9} x^{a_{10}} x^{a_{11}},
\]

where the meaning of the lower indices for $I, K, P$ and $\tilde{K}$ is the same, and the vector $x_1 = (x_1^1, x_1^2, x_1^3)$ is cogradient [5] to the phase variable vector $x = (x^1, x^2, x^3)$. The vectors $x$ and $x_1$ are independent. In [3] it is shown that invariant condition which differs the system (6) from (1) is the following: $\tilde{K}_{1,3} \equiv 0$. In the same paper with the aid of Theorem 1 and expressions (8) is proved

**Theorem 3.** System (1) with $\tilde{K}_{1,3} \equiv 0$ and $n = 3$ has the invariant $GL(3, \mathbb{R})$-integrating factor $\mu$ of the form $\mu^{-1} = K_{3,3} \Phi_{3,3}$, where $K_{3,3} = 0$ and

\[
\Phi_{3,3} = 1/3(I_{1,3}^2 - 3I_{1,3}I_{2,3} + 2I_{3,3}) - 3/2(I_{2,3} - I_{1,3})P_{1,3} - 4I_{1,3}P_{2,3} + 4P_{3,3} = 0
\]

are invariant particular $GL(3, \mathbb{R})$-integrals of this system.

3. Case $n = 4$. Consider the next invariants, comitants and covariants of the system (1)

\[
I_{1,4} = a_{a_1}^2, \quad I_{2,4} = a_{a_2}^3, \quad I_{3,4} = a_{a_3}^4, \quad I_{4,4} = a_{a_4}^5, \\
K_{4,4} = a_{a_1}^2 a_{a_2}^3 a_{a_3}^4 a_{a_4}^5 x^{a_1} x^{a_2} x^{a_3} x^{a_4} x^{a_5} x^{a_6} x^{a_7} x^{a_8} x^{a_9} x^{a_{10}} x^{a_{11}}, \\
P_{2,4} = a_{a_2}^3 a_{a_3}^4 x^{a_2}, \quad P_{3,4} = a_{a_3}^4 a_{a_4}^5 x^{a_3}, \quad P_{4,4} = a_{a_4}^5 a_{a_1}^2 a_{a_2}^3 a_{a_3}^4 x^{a_4}, \\
\tilde{K}_{1,4} = a_{a_1}^2 a_{a_2}^3 x^{a_1} x^{a_2} x^{a_3} x^{a_4} x^{a_5} x^{a_6} x^{a_7} x^{a_8} x^{a_9} x^{a_{10}} x^{a_{11}},
\]

where the meaning of the lower indices for $I, K, P$ and $\tilde{K}$ is the same, and the vectors $x_1 = (x_1^1, x_1^2, x_1^3, x_1^4)$ and $x_2 = (x_2^1, x_2^2, x_2^3, x_2^4)$ are cogradient to the phase variable vector $x$. One can verify easily that invariant condition which differs the system (6) from (1) is the following: $\tilde{K}_{1,4} \equiv 0$. With the aid of Theorem 1 and expressions (9) it is proved the following

**Theorem 4.** System (1) with $\tilde{K}_{1,4} \equiv 0$ and $n = 4$ has the invariant $GL(4, \mathbb{R})$-integrating factor $\mu$ of the form $\mu^{-1} = K_{6,4} \Phi_{4,4}$, where $K_{6,4} = 0$ and

\[
\Phi_{4,4} = L_{4,4} - 2(4/5L_{3,4}P_{1,4} + L_{2,4}P_{2,4} + L_{1,4}P_{3,4} + P_{4,4}) = 0
\]

are invariant particular $GL(4, \mathbb{R})$-integrals of this system. In (10) we have

\[
L_{1,4} = -I_{1,4}, \quad L_{2,4} = 1/2(I_{1,4}^2 - I_{2,4}), \quad L_{3,4} = 1/6(3I_{1,4}I_{2,4} - 2I_{3,4} - I_{4,4}^2), \\
L_{4,4} = 1/24(8I_{1,4}I_{3,4} - 6I_{4,4} - 6I_{1,4}I_{2,4} + 3I_{2,4}^2 + I_{1,4}^4),
\]

where $I_{k,4}$ ($k = 1, 4$) are from (9).
4. Case $n = 5$. Consider the next invariants, comitants and covariants of the system (1)

\[
\begin{align*}
I_{1,5} &= a_{\alpha_1}^1, \quad I_{2,5} = a_{\alpha_2}^1 a_{\alpha_1}^2, \quad I_{3,5} = a_{\alpha_3}^1 a_{\alpha_4}^2 a_{\alpha_5}^3, \\
I_{4,5} &= a_{\alpha_4}^1 a_{\alpha_5}^2 a_{\alpha_4}^3 a_{\alpha_5}^4, \quad I_{5,5} = a_{\alpha_5}^1 a_{\alpha_3}^2 a_{\alpha_5}^3 a_{\alpha_4}^4 a_{\alpha_5}^5, \\
K_{10,5} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5, \quad I_{5,5} = a_{\alpha_5}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5, \\
P_{1,5} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5, \quad P_{2,5} = a_{\alpha_2}^2 a_{\alpha_1}^3 a_{\alpha_2}^4 a_{\alpha_3}^5, \quad P_{3,5} = a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5, \\
P_{4,5} &= a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_3}^6 a_{\alpha_4}^7 a_{\alpha_5}^8 a_{\alpha_3}^9 a_{\alpha_4}^{10}, \quad P_{5,5} = a_{\alpha_5}^5 a_{\alpha_3}^6 a_{\alpha_4}^7 a_{\alpha_5}^8 a_{\alpha_3}^9 a_{\alpha_4}^{10}, \\
\tilde{K}_{1,5} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_1}^6 a_{\alpha_2}^7 a_{\alpha_3}^8 a_{\alpha_4}^9 a_{\alpha_5}^{10},
\end{align*}
\]

(11)

where the meaning of lower indices for $I, K, P$ and $\tilde{K}$ is the same, and the vectors $x_i = (x_i^1, x_i^2, x_i^3, x_i^4, x_i^5)$, $(i = 1, 5)$ are cogradient to the phase variable vector $x$. As it is easy to see the invariant condition which differs the system (6) from (1) is the following: $\tilde{K}_{1,5} \equiv 0$. With the aid of Theorem 1 and expressions (11) is proved the following

**Theorem 5.** System (1) with $\tilde{K}_{1,5} \equiv 0$ and $n = 5$ has the invariant $GL(5, \mathbb{R})$-integrating factor $\mu$ of the form $\mu^{-1} = K_{10,5} \Phi_{5,5}$, where $K_{10,5} = 0$ and

\[
\Phi_{5,5} \equiv L_{5,5} - 2(5/6L_{4,5} P_{1,5} + L_{3,5} P_{2,5} + L_{2,5} P_{3,5} + L_{1,5} P_{4,5} + P_{5,5}) = 0
\]

(12)

are invariant particular $GL(5, \mathbb{R})$-integrals of this system. In (12) we have

\[
\begin{align*}
L_{1,5} &= -I_{1,5}, \quad L_{2,5} = 1/2(I_{2,5}^2 - I_{2,5}), \quad L_{3,5} = 1/6(3I_{1,5} I_{2,5} - 2I_{3,5} - I_{1,5}^3), \\
L_{4,5} &= 1/24(8I_{1,5} I_{3,5} - 6I_{4,5} - 6I_{2,5} I_{2,5} + 3I_{2,5}^2 + I_{1,5}^4), \\
L_{5,5} &= -1/120(15\tilde{I}_{1,5} - 10\tilde{I}_{1,5} I_{2,5} + 20\tilde{I}_{2,5} I_{3,5} + 15\tilde{I}_{3,5} I_{2,5}^2 - 30I_{1,5} I_{4,5} - 20I_{2,5} I_{3,5} + 24I_{5,5}),
\end{align*}
\]

where $I_{k,5}$ $(k = 1, 5)$ are from (11).

5. The general case $n \geq 2$.

Write the center-affine invariants, comitants and covariants in general case of system (1) as follows

\[
\begin{align*}
I_{1,n} &= a_{\alpha_1}^1, \quad I_{2,n} = a_{\alpha_2}^1 a_{\alpha_1}^2, \quad I_{3,n} = a_{\alpha_3}^1 a_{\alpha_2}^2 a_{\alpha_1}^3, ..., I_{n,n} = a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 ... a_{\alpha_{n-1}}^n, \\
K_{m,n} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_6}^6 a_{\alpha_7}^7 a_{\alpha_8}^8 ... a_{\alpha_{n-1}} a_{\alpha_n} a_{\alpha_1} a_{\alpha_2} a_{\alpha_3} a_{\alpha_4} a_{\alpha_5} a_{\alpha_6} a_{\alpha_7} a_{\alpha_8} ... a_{\alpha_{n-1}} a_{\alpha_n}, \\
P_{1,n} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_6}^6 a_{\alpha_7}^7 a_{\alpha_8}^8 a_{\alpha_9}^9 ... a_{\alpha_{n-1}} a_{\alpha_n}, \\
P_{2,n} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_6}^6 a_{\alpha_7}^7 a_{\alpha_8}^8 a_{\alpha_9}^9 a_{\alpha_{n-1}} a_{\alpha_n}, \\
P_{3,n} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_6}^6 a_{\alpha_7}^7 a_{\alpha_8}^8 a_{\alpha_9}^9 a_{\alpha_{n-1}} a_{\alpha_n}, \\
P_{n,n} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_6}^6 a_{\alpha_7}^7 a_{\alpha_8}^8 a_{\alpha_9}^9 a_{\alpha_{n-1}} a_{\alpha_n}, \\
\tilde{K}_{1,n} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_6}^6 a_{\alpha_7}^7 a_{\alpha_8}^8 a_{\alpha_9}^9 a_{\alpha_{n-1}} a_{\alpha_n}, \\
\tilde{K}_{2,n} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_6}^6 a_{\alpha_7}^7 a_{\alpha_8}^8 a_{\alpha_9}^9 a_{\alpha_{n-1}} a_{\alpha_n}, \\
\tilde{K}_{3,n} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_6}^6 a_{\alpha_7}^7 a_{\alpha_8}^8 a_{\alpha_9}^9 a_{\alpha_{n-1}} a_{\alpha_n}, \\
\tilde{K}_{n,n} &= a_{\alpha_1}^1 a_{\alpha_2}^2 a_{\alpha_3}^3 a_{\alpha_4}^4 a_{\alpha_5}^5 a_{\alpha_6}^6 a_{\alpha_7}^7 a_{\alpha_8}^8 a_{\alpha_9}^9 a_{\alpha_{n-1}} a_{\alpha_n}, \\
\end{align*}
\]

\[
(\alpha, \alpha_1, \alpha_2, ..., \alpha_m, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_n = 1/n; \quad m = n(n-1)/2; \quad n \geq 2)
\]
where $\varepsilon_{\beta_1,\beta_2,\beta_3...\beta_n}$ is a unit $n$-vector, and the vectors $x_i = (x_i^1, x_i^2, ..., x_i^n)$, $(i = 1, n - 2)$ are independent cogradient vectors [5] to $x$.

**Remark 1.** System (1) with $\tilde{K}_{1,n} \equiv 0$ has the form (6), where $R(x) = \frac{1}{n+1} P_{1,n}$.

Will call the systems written in the form (6) a Darboux type differential system (analogically to the case when $n = 2$ in [4]).

As it is easy to see the center-affine invariant condition differ the system (6) from (1). Indeed, it is true that for system (6) with $\tilde{K}_{1,n} \equiv 0$, we have $P_{1,n} = (n + 1)R(x)$.

One can verify that the next theorem generalizes cases 1-4

**Theorem 6.** System (1) with $\tilde{K}_{1,n} \equiv 0$ and $n = 2, 3, 4, 5$ has the invariant $GL(n, \mathbb{R})$-integrating factor $\mu$ of the form

$$\mu^{-1} = K_{m,n} \Phi_{n,n},$$

where $K_{m,n} = 0$ and

$$\Phi_{n,n} \equiv L_{n,n} - 2(\frac{n}{n + 1} L_{n-1,n} P_{1,n} + L_{n-2,n} P_{2,n} + L_{n-3,n} P_{3,n} + ... + L_{1,n} P_{n-1,n} + P_{n,n}) = 0$$

are invariant particular $GL(n, \mathbb{R})$-integrals of this system, and $L_{i,n}$ ($i = 1, n$) are the coefficients of characteristic equation of the system (1) as follows

$$\lambda^n + L_{1,n} \lambda^{n-1} + L_{2,n} \lambda^{n-2} + ... + L_{n-1,n} \lambda + L_{n,n} = 0$$

and they can be expressed though the invariants from (13) by the recurrence formula

$$L_{i,n} = -\frac{1}{i}(I_{i,n} + I_{i-1,n} L_{1,n} + I_{i-2,n} L_{2,n} + ... + I_{1,n} L_{i-1,n}) \quad (i = 1, n).$$

With the aid of the cases 1-4 it is easy to verify that holds the next

**Theorem 7.** System (1) with $\tilde{K}_{1,n} \equiv 0$ and $n = 2, 3, 4, 5$ has the first invariant $GL(n, \mathbb{R})$-integral of Darboux type [6] as follows

$$K_{m,n}^{-1} \Phi_{n,n}^n = C$$

if and only if $I_{1,n} = 0$, where $K_{m,n}$, $K_{1,n}$, $I_{1,n}$ are from (13), and $\Phi_{n,n}$ is from (14).

The proof of Theorem 7 for system (6) follows from the equation

$$\Lambda(K_{m,n}^{-1} \Phi_{n,n}^n) = -I_1 K_{m,n}^{-1} \Phi_{n,n}^n,$$

where $\Lambda$ is from (3).

**Remark 2.** There exists the assumption that Theorems 6 and 7 hold for $n \geq 6$.

One can verify that holds

**Remark 3.** Expression $K_{m,n} = 0$ from (13) is the invariant particular $GL(n, \mathbb{R})$-integral for linear system $\frac{dx}{dt} = a_\alpha x^\alpha$ ($\alpha = 1, n$).
References


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