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Multi-dimensional Darboux type differential systems with quadratic nonlinearities

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Abstract. In the article the n-dimensional autonomous Darboux type differential systems with nonlinearities of the 2^{nd} degree are considered. With the aid of theorem on integrating factor the particular invariant $GL(n,\mathbb{R})$-integrals are constructed as well as the first integrals of Darboux type for considered systems. These integrals represent the algebraic curves of the 1^{st} degree. The recurrence formula of particular invariant $GL(n,\mathbb{R})$-integrals of the Darboux type differential system is found.

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Consider the system of differential equations

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \equiv P^j(x,a) \quad (j,\alpha,\beta = \overline{1,n};\ n \geq 2), \tag{1}$$

where coefficient tensor $a^j_{\alpha\beta}$ is symmetrical in lower indices, in which the complete convolution holds. The system (1) is considered with the action of the group $GL(n,\mathbb{R})$ of center-affine transformations [1], and $x = (x^1, x^2, ..., x^n)$ is a phase variable vector of the system.

Suppose that system (1) admits $(n-1)$-dimensional commutative Lie algebra with operators

$$X_\alpha = \xi^j_\alpha(x)\frac{\partial}{\partial x^j} \qquad (j=\overline{1,n};\ \alpha = \overline{1,n-1}) \tag{2}$$

and

$$\Lambda = P^j(x,a)\frac{\partial}{\partial x^j} \qquad (j=\overline{1,n}). \tag{3}$$

Consider the determinant constructed on coordinates of operators (2)-(3) as follows

$$\Delta = \begin{vmatrix} \xi^1_1 & \xi^2_1 & \xi^3_1 & ... & \xi^n_1 \\ \xi^1_2 & \xi^2_2 & \xi^3_2 & ... & \xi^n_2 \\ ... & ... & ... & ... & ... \\ \xi^1_{n-1} & \xi^2_{n-1} & \xi^3_{n-1} & ... & \xi^n_{n-1} \\ P^1 & P^2 & P^3 & ... & P^n \end{vmatrix}. \tag{4}$$

From [2] it follows that holds

Theorem 1. *If n-dimensional differential system* (1) *admits* $(n-1)$*-dimensional commutative Lie algebra of operators* (2)*, then the function* $\mu = \frac{1}{\Delta}$*, where* $\Delta \neq 0$ *from* (4)*, is the integrating factor for Pfaff equations*

$$\sum_{i=1} (-1)^{i+j} \begin{vmatrix} \xi_1^1 & \dots & \xi_1^{i-1} & \xi_1^{i+1} & \dots & \xi_1^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \xi_{j-1}^1 & \dots & \xi_{j-1}^{i-1} & \xi_{j-1}^{i+1} & \dots & \xi_{j-1}^n \\ \xi_{j+1}^1 & \dots & \xi_{j+1}^{i-1} & \xi_{j+1}^{i+1} & \dots & \xi_{j+1}^n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ P^1 & \dots & P^{i-1} & P^{i+1} & \dots & P^n \end{vmatrix} dx^i = 0 \quad (i = \overline{1,n};\ j = \overline{1,n-1}), \tag{5}$$

defining a general integral of the system (1)*.*

Following [3], consider system (1) in a "Darboux" like case, i.e. system (1) written in the form

$$\frac{dx^j}{dt} = a_\alpha^j x^\alpha + 2x^j R(x) \equiv P^j(x,a) \quad (j, \alpha = \overline{1,n};\ n \geq 2), \tag{6}$$

where $R(x) \neq 0$ is a homogeneous linear polynomial with constant coefficients in coordinates of the vector x.

According to [4] will treat invariant $GL(n,\mathbb{R})$-integrating factors and invariant $GL(n,\mathbb{R})$-integrals of the system (6) with $n = 2, 3, 4, 5, ...$

1. Case $n = 2$**.** Will denote the invariants and comitants of the system (1) as follows

$$\begin{aligned} &I_{1,2} = a_{\alpha_1}^{\alpha_1}, \quad I_{2,2} = a_{\alpha_2}^{\alpha_1} a_{\alpha_1}^{\alpha_2}, \quad K_{1,2} = a_\alpha^{\alpha_1} x^\alpha x^{\alpha_2} \varepsilon_{\alpha_1\alpha_2}, \\ &P_{1,2} = a_{\alpha_1\beta}^{\alpha_1} x^\beta, \quad P_{2,2} = a_{\alpha_2}^{\alpha_1} a_{\alpha_1\beta}^{\alpha_2} x^\beta, \ \widetilde{K}_{1,2} = a_{\beta\gamma}^{\alpha_1} x^\beta x^\gamma x^{\alpha_2} \varepsilon_{\alpha_1\alpha_2}, \end{aligned} \tag{7}$$

where the first of lower indices for I, K, P and $\widetilde{K}$ from (7) shows the degree of invariant or comitant with respect to coefficients of the system (1), and the second lower index shows the dimension of the system ($n = 2$). In [4] it is shown that invariant condition which differs the system (6) from (1) is the following: $\widetilde{K}_{1,2} \equiv 0$. In the same paper with the aid of Theorem 1 and expressions (7) is proved

Theorem 2. *System* (1) *with* $\widetilde{K}_{1,2} \equiv 0$ *and* $n = 2$ *has the invariant* $GL(2,\mathbb{R})$*-integrating factor* μ *of the form* $\mu^{-1} = K_{1,2}\Phi_{2,2}$*, where* $K_{1,2} = 0$ *and*

$$\Phi_{2,2} \equiv 8I_{1,2}P_{1,2} - 12P_{2,2} + 3(I_{1,2}^2 - I_{2,2}) = 0$$

are invariant particular $GL(2,\mathbb{R})$*-integrals of this system.*

2. Case $n = 3$**.** Following [3] will denote the invariants, comitants and covariants of the system (1) as follows

$$\begin{aligned} &I_{1,3} = a^{\alpha_1}_{\alpha_1}, \quad I_{2,3} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1}, \quad I_{3,3} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2}, \\ &K_{3,3} = a^{\beta_1}_{\alpha_1} a^{\beta_2}_{\alpha_2} a^{\alpha_2}_{\alpha_3} x^{\alpha_1} x^{\alpha_3} x^{\beta_3} \varepsilon_{\beta_1\beta_2\beta_3}, \\ &P_{1,3} = a^{\alpha_1}_{\alpha_1\beta} x^{\beta}, \quad P_{2,3} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1\beta} x^{\beta}, \; P_{3,3} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2\beta} x^{\beta}, \\ &\widetilde{K}_{1,3} = a^{\alpha_1}_{\beta\gamma} x^{\beta} x^{\gamma} x^{\alpha_2} x_1^{\alpha_3} \varepsilon_{\alpha_1\alpha_2\alpha_3}, \end{aligned} \tag{8}$$

where the meaning of the lower indices for I, K, P and $\widetilde{K}$ is the same, and the vector $x_1 = (x_1^1, x_1^2, x_1^3)$ is cogradient [5] to the phase variable vector $x = (x^1, x^2, x^3)$. The vectors x and x_1 are independent. In [3] it is shown that invariant condition which differs the system (6) from (1) is the following: $\widetilde{K}_{1,3} \equiv 0$. In the same paper with the aid of Theorem 1 and expressions (8) is proved

Theorem 3. *System* (1) *with* $\widetilde{K}_{1,3} \equiv 0$ *and* $n = 3$ *has the invariant* $GL(3,\mathbb{R})$*-integrating factor* μ *of the form* $\mu^{-1} = K_{3,3}\Phi_{3,3}$*, where* $K_{3,3} = 0$ *and*

$$\Phi_{3,3} \equiv 1/3(I_{1,3}^2 - 3I_{1,3}I_{2,3} + 2I_{3,3}) - 3/2(I_{2,3} - I_{1,3}^2)P_{1,3} - 4I_{1,3}P_{2,3} + 4P_{3,3} = 0$$

are invariant particular $GL(3,\mathbb{R})$*-integrals of this system.*

3. Case $n = 4$**.** Consider the next invariants, comitants and covariants of the system (1)

$$\begin{aligned} &I_{1,4} = a^{\alpha_1}_{\alpha_1}, \quad I_{2,4} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1}, \quad I_{3,4} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2}, \; I_{4,4} = a^{\alpha_1}_{\alpha_4} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2} a^{\alpha_4}_{\alpha_3}, \\ &K_{6,4} = a^{\beta_1}_{\alpha_1} a^{\beta_2}_{\alpha_2} a^{\alpha_2}_{\alpha_3} a^{\beta_3}_{\alpha_4} a^{\alpha_4}_{\alpha_5} a^{\alpha_5}_{\alpha_6} x^{\alpha_1} x^{\alpha_3} x^{\alpha_6} x^{\beta_4} \varepsilon_{\beta_1\beta_2\beta_3\beta_4}, \quad P_{1,4} = a^{\alpha_1}_{\alpha_1\beta} x^{\beta}, \\ &P_{2,4} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1\beta} x^{\beta}, \; P_{3,4} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2\beta} x^{\beta}, \; P_{4,4} = a^{\alpha_1}_{\alpha_4} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2} a^{\alpha_4}_{\alpha_3\beta} x^{\beta}, \\ &\widetilde{K}_{1,4} = a^{\alpha_1}_{\beta\gamma} x^{\beta} x^{\gamma} x^{\alpha_2} x_1^{\alpha_3} x_2^{\alpha_4} \varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4}, \end{aligned} \tag{9}$$

where the meaning of the lower indices for I, K, P and $\widetilde{K}$ is the same, and the vectors $x_1 = (x_1^1, x_1^2, x_1^3, x_1^4)$ and $x_2 = (x_2^1, x_2^2, x_2^3, x_2^4)$ are cogradient to the phase variable vector x. One can verify easily that invariant condition which differs the system (6) from (1) is the following: $\widetilde{K}_{1,4} \equiv 0$. With the aid of Theorem 1 and expressions (9) it is proved the following

Theorem 4. *System* (1) *with* $\widetilde{K}_{1,4} \equiv 0$ *and* $n = 4$ *has the invariant* $GL(4,\mathbb{R})$*-integrating factor* μ *of the form* $\mu^{-1} = K_{6,4}\Phi_{4,4}$*, where* $K_{6,4} = 0$ *and*

$$\Phi_{4,4} \equiv L_{4,4} - 2(4/5L_{3,4}P_{1,4} + L_{2,4}P_{2,4} + L_{1,4}P_{3,4} + P_{4,4}) = 0 \tag{10}$$

are invariant particular $GL(4,\mathbb{R})$*-integrals of this system. In* (10) *we have*

$$L_{1,4} = -I_{1,4}, \quad L_{2,4} = 1/2(I_{1,4}^2 - I_{2,4}), \quad L_{3,4} = 1/6(3I_{1,4}I_{2,4} - 2I_{3,4} - I_{1,4}^3),$$

$$L_{4,4} = 1/24(8I_{1,4}I_{3,4} - 6I_{4,4} - 6I_{1,4}^2 I_{2,4} + 3I_{2,4}^2 + I_{1,4}^4),$$

where $I_{k,4}$ $(k = \overline{1,4})$ *are from* (9).

4. Case $n = 5$. Consider the next invariants, comitants and covariants of the system (1)

$$\begin{aligned}
&I_{1,5} = a^{\alpha_1}_{\alpha_1}, \quad I_{2,5} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1}, \quad I_{3,5} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2},\\
&I_{4,5} = a^{\alpha_1}_{\alpha_4} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2} a^{\alpha_4}_{\alpha_3}, \; I_{5,5} = a^{\alpha_1}_{\alpha_5} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2} a^{\alpha_4}_{\alpha_3} a^{\alpha_5}_{\alpha_4},\\
&K_{10,5} = a^{\beta_1}_{\alpha_1} a^{\beta_2}_{\alpha_2} a^{\alpha_2}_{\alpha_3} a^{\beta_3}_{\alpha_4} a^{\alpha_4}_{\alpha_5} a^{\alpha_5}_{\alpha_6} a^{\beta_4}_{\alpha_7} a^{\alpha_7}_{\alpha_8} a^{\alpha_8}_{\alpha_9} a^{\alpha_9}_{\alpha_{10}} x^{\alpha_1} x^{\alpha_3} x^{\alpha_6} x^{\alpha_{10}} x^{\beta_5} \varepsilon_{\beta_1\beta_2\beta_3\beta_4\beta_5},\\
&P_{1,5} = a^{\alpha_1}_{\alpha_1\beta} x^\beta, \quad P_{2,5} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1\beta} x^\beta, \; P_{3,5} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2\beta} x^\beta,\\
&P_{4,5} = a^{\alpha_1}_{\alpha_4} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2} a^{\alpha_4}_{\alpha_3\beta} x^\beta, P_{5,5} = a^{\alpha_1}_{\alpha_5} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2} a^{\alpha_4}_{\alpha_3} a^{\alpha_5}_{\alpha_4\beta} x^\beta,\\
&\widetilde{K}_{1,5} = a^{\alpha_1}_{\beta\gamma} x^\beta x^\gamma x^{\alpha_2} x_1^{\alpha_3} x_2^{\alpha_4} x_3^{\alpha_5} \varepsilon_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5},
\end{aligned} \tag{11}$$

where the meaning of lower indices for I, K, P and $\widetilde{K}$ is the same, and the vectors $x_i = (x_i^1, x_i^2, x_i^3, x_i^4, x_i^5)$, $(i = \overline{1,3})$ are cogradient to the phase variable vector x. As it is easy to see the invariant condition which differs the system (6) from (1) is the following: $\widetilde{K}_{1,5} \equiv 0$. With the aid of Theorem 1 and expressions (11) is proved the following

Theorem 5. *System* (1) *with* $\widetilde{K}_{1,5} \equiv 0$ *and* $n = 5$ *has the invariant* $GL(5,\mathbb{R})$-*integrating factor* μ *of the form* $\mu^{-1} = K_{10,5}\Phi_{5,5}$, *where* $K_{10,5} = 0$ *and*

$$\Phi_{5,5} \equiv L_{5,5} - 2(5/6 L_{4,5} P_{1,5} + L_{3,5} P_{2,5} + L_{2,5} P_{3,5} + L_{1,5} P_{4,5} + P_{5,5}) = 0 \tag{12}$$

are invariant particular $GL(5,\mathbb{R})$-*integrals of this system. In* (12) *we have*

$$L_{1,5} = -I_{1,5}, \quad L_{2,5} = 1/2(I_{1,5}^2 - I_{2,5}), \quad L_{3,5} = 1/6(3I_{1,5}I_{2,5} - 2I_{3,5} - I_{1,5}^3),$$

$$L_{4,5} = 1/24(8I_{1,5}I_{3,5} - 6I_{4,5} - 6I_{1,5}^2 I_{2,5} + 3I_{2,5}^2 + I_{1,5}^4),$$

$$L_{5,5} = -1/120(I_{1,5}^5 - 10I_{1,5}^3 I_{2,5} + 20I_{1,5}^2 I_{3,5} + 15I_{1,5}I_{2,5}^2 - 30I_{1,5}I_{4,5} - 20I_{2,5}I_{3,5} + 24I_{5,5}),$$

where $I_{k,5}$ $(k = \overline{1,5})$ *are from* (11).

5. The general case $n \geq 2$.

Write the center-affine invariants, comitants and covariants in general case of system (1) as follows

$$\begin{aligned}
&I_{1,n} = a^{\alpha_1}_{\alpha_1}, \quad I_{2,n} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1}, \quad I_{3,n} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2}, ..., I_{n,n} = a^{\alpha_1}_{\alpha_n} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2} ... a^{\alpha_n}_{\alpha_{n-1}},\\
&K_{m,n} = a^{\beta_1}_{\alpha_1} a^{\beta_2}_{\alpha_2} a^{\alpha_2}_{\alpha_3} a^{\beta_3}_{\alpha_4} a^{\alpha_4}_{\alpha_5} a^{\alpha_5}_{\alpha_6} a^{\beta_4}_{\alpha_7} a^{\alpha_7}_{\alpha_8} a^{\alpha_8}_{\alpha_9} a^{\alpha_9}_{\alpha_{10}} ... a^{\alpha_{m-1}}_{\alpha_m} x^{\alpha_1} x^{\alpha_3} x^{\alpha_6} x^{\alpha_{10}} ... x^{\alpha_m} x^{\beta_n} \varepsilon_{\beta_1...\beta_n},\\
&P_{1,n} = a^{\alpha_1}_{\alpha_1\beta} x^\beta, \quad P_{2,n} = a^{\alpha_1}_{\alpha_2} a^{\alpha_2}_{\alpha_1\beta} x^\beta, \; P_{3,n} = a^{\alpha_1}_{\alpha_3} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2\beta} x^\beta, ...,\\
&P_{n,n} = a^{\alpha_1}_{\alpha_n} a^{\alpha_2}_{\alpha_1} a^{\alpha_3}_{\alpha_2} ... a^{\alpha_n}_{\alpha_{n-1}\beta} x^\beta,\\
&\widetilde{K}_{1,n} = a^{\beta_1}_{\alpha\beta} x^\alpha x^\beta x^{\beta_2} x_1^{\beta_3} x_2^{\beta_4} ... x_{n-2}^{\beta_n} \varepsilon_{\beta_1\beta_2...\beta_n},\\
&(\alpha, \alpha_1, \alpha_2, ..., \alpha_m, ..., \alpha_n, \beta, \beta_1, \beta_2, ..., \beta_n = \overline{1,n}; \quad m = \tfrac{n(n-1)}{2}; \quad n \geq 2)
\end{aligned} \tag{13}$$

where $\varepsilon_{\beta_1\beta_2\beta_3...\beta_n}$ is a unit n-vector, and the vectors $x_i = (x_i^1, x_i^2, ..., x_i^n)$, $(i = \overline{1, n-2})$ are independent cogradient vectors [5] to x .

Remark 1. System (1) with $\widetilde{K}_{1,n} \equiv 0$ has the form (6), where $R(x) = \frac{1}{n+1}P_{1,n}$.

Will call the systems written in the form (6) a Darboux type differential system (analogically to the case when $n = 2$ in [4]).

As it is easy to see the center-affine invariant condition differ the system (6) from (1). Indeed, it is true that for system (6) with $\widetilde{K}_{1,n} \equiv 0$, we have $P_{1,n} = (n+1)R(x)$.

One can verify that the next theorem generalizes cases 1-4

Theorem 6. *System* (1) *with* $\widetilde{K}_{1,n} \equiv 0$ *and* $n = 2, 3, 4, 5$ *has the invariant* $GL(n, \mathbb{R})$*-integrating factor* μ *of the form*

$$\mu^{-1} = K_{m,n}\Phi_{n,n},$$

where $K_{m,n} = 0$ *and*

$$\Phi_{n,n} \equiv L_{n,n} - 2(\frac{n}{n+1}L_{n-1,n}P_{1,n} + L_{n-2,n}P_{2,n} + L_{n-3,n}P_{3,n} + ... + L_{1,n}P_{n-1,n} + P_{n,n}) = 0 \tag{14}$$

are invariant particular $GL(n, \mathbb{R})$*-integrals of this system, and* $L_{i,n}$ $(i = \overline{1, n})$ *are the coefficients of characteristic equation of the system* (1) *as follows*

$$\lambda^n + L_{1,n}\lambda^{n-1} + L_{2,n}\lambda^{n-2} + ... + L_{n-1,n}\lambda + L_{n,n} = 0 \tag{15}$$

and they can be expressed though the invariants from (13) *by the recurrence formula*

$$L_{i,n} = -\frac{1}{i}(I_{i,n} + I_{i-1,n}L_{1,n} + I_{i-2,n}L_{2,n} + ... + I_{1,n}L_{i-1,n}) \quad (i = \overline{1, n}). \tag{16}$$

With the aid of the cases 1-4 it is easy to verify that holds the next

Theorem 7. *System (1) with* $\widetilde{K}_{1,n} \equiv 0$ *and* $n = 2, 3, 4, 5$ *has the first invariant* $GL(n, \mathbb{R})$*-integral of Darboux type [6] as follows*

$$K_{m,n}^{-1}\Phi_{n,n}^n = C \tag{17}$$

if and only if $I_{1,n} = 0$*, where* $K_{m,n}$*,* $\widetilde{K}_{1,n}, I_{1,n}$ *are from (13), and* $\Phi_{n,n}$ *is from (14).*

The proof of Theorem 7 for system (6) follows from the equation

$$\Lambda(K_{m,n}^{-1}\Phi_{n,n}^n) = -I_1K_{m,n}^{-1}\Phi_{n,n}^n,$$

where Λ is from (3).

Remark 2. There exists the assumption that Theorems 6 and 7 hold for $n \geq 6$.

One can verify that holds

Remark 3. Expression $K_{m,n} = 0$ from (13) is the invariant particular $GL(n, \mathbb{R})$-integral for linear system $\frac{dx^j}{dt} = a^j_\alpha x^\alpha \quad (\alpha = \overline{1, n})$.

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