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On LCA groups in which some closed subgroups have commutative rings of continuous endomorphisms

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Abstract. We describe here the locally compact abelian (LCA) groups all of whose closed polythetic, respectively, copolythetic subgroups have commutative rings of continuous endomorphisms. We also determine the LCA groups all of whose polythetic, respectively, copolythetic quotients by closed subgroups have commutative rings of continuous endomorphisms.

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1 Introduction

The problem of characterizing the abelian groups with commutative endomorphism ring was first considered by T. Szele and J. Szendrei in [11], where, among other things, certain important special cases were completly solved. Later L. C. A. van Leeuwen [7] noted that, if X is an abelian group, then every finitely generated subgroup of X has a commutative endomorphism ring if and only if X is isomorphic to a subgroup of $\mathbb{Q}$ or of $\mathbb{Q}/\mathbb{Z}$.

Inspired by the above mentioned paper of T. Szele and J. Szendrei, we initiated in [10] the study of LCA groups with commutative ring of continuous endomorphisms. In the present paper, we continue in the same direction by extending the L. C. A. van Leeuwen's result to this more general setting. Some other results of this nature will also be established. To be more explicit, we need a couple of definitions.

Definition 1.1. *An LCA group X is said to be*

(i) *polythetic if it contains a dense finitely generated subgroup.*

(ii) *copolythetic if there exists a continuous injective homomorphism from X into a group of the form $\mathbb{T}^n$ for some $n \in \mathbb{N}$.*

Let $\mathcal{L}$ be the class of LCA groups. For $X \in \mathcal{L}$, we let $E(X)$ denote the ring of continuous endomorphisms of X.

Our aim here is to determine the groups $X \in \mathcal{L}$ such that every closed polythetic, respectively, copolythetic subgroup G of X has a commutative ring $E(G)$. We also determine the groups $X \in \mathcal{L}$ such that every polythetic, respectively, copolythetic quotient of X by a closed subgroup G, indicated as usual by X/G, has a commutative ring $E(X/G)$.

2 Notation

Since this paper continues the work of [10], we employ here the notation and terminology introduced there.

In addition, $\mathcal{L}_0$ will denote the subclass of $\mathcal{L}$ consisting of those groups which have a compact open subgroup, and $\dot{+}$ will stand for the algebraic direct sum.

For any $p \in \mathbb{P}$ and $X \in \mathcal{L}$, we let $t_p(X)$ denote the p-primary component of X, and

$$S_0(X) = \{q \in \mathbb{P} \mid t_q(X) \neq \{0\}\}.$$

Given a family $(A_i)_{i\in I}$ of subgroups of X, $\sum_{i\in I} A_i$ will designate the minimal subgroup of X containing $\bigcup_{i\in I} A_i$.

If M is a set, $|M|$ will stand for the cardinality of M.

We shall also use the groups of integers, $\mathbb{Z}$, and of rationals, $\mathbb{Q}$, both taken discrete, and the groups of reals, $\mathbb{R}$, and of reals modulo one, $\mathbb{T}$, both taken with their usual topologies.

3 Polythetic subgroups

In this section we characterize completly the groups $X \in \mathcal{L}$ such that every closed polythetic subgroup G of X has a commutative ring $E(G)$. By use of duality, we obtain also the characterization of those groups $X \in \mathcal{L}$ which have the property that for every closed subgroup G of X such that X/G is copolythetic, the ring $E(X/G)$ is commutative.

We start with some preparatory lemmas.

Lemma 3.1. *Let $X \in \mathcal{L}$. For any $a, b \in X$ such that $a \in k(X)$ and $\overline{\langle a\rangle} \cap \overline{\langle b\rangle} = \{0\}$, we have $\overline{\langle a, b\rangle} \cong \overline{\langle a\rangle} \times \overline{\langle b\rangle}$.*

Proof. Since $\overline{\langle a\rangle}$ is compact, $\overline{\langle a\rangle} + \overline{\langle b\rangle}$ is closed in X [6, (4.4)]. It is then easy to see that $\overline{\langle a, b\rangle} = \overline{\langle a\rangle} + \overline{\langle b\rangle}$, so that $\overline{\langle a, b\rangle} = \overline{\langle a\rangle} \dot{+} \overline{\langle b\rangle}$, and hence $\overline{\langle a, b\rangle} = \overline{\langle a\rangle} \oplus \overline{\langle b\rangle}$ by [1, Proposition 6.5]. □

Lemma 3.2. *Any group $X \in \mathcal{L}$ with $\{0\} \neq k(X) \neq X$ has a closed polythetic subgroup G such that $E(G)$ is not commutative.*

Proof. Pick any $a \in k(G)$ and $b \in X \setminus k(G)$. Since $\overline{\langle b\rangle} \cong \mathbb{Z}$ [6, (9.1)], we have $\overline{\langle a\rangle} \cap \overline{\langle b\rangle} = \{0\}$, so that $\overline{\langle a, b\rangle} = \overline{\langle a\rangle} \times \overline{\langle b\rangle}$ by Lemma 3.1. We can take $G = \overline{\langle a, b\rangle}$. □

Lemma 3.3. *Let $X \in \mathcal{L}$ and $S = \{p \in \mathbb{P} \mid X_p \neq \{0\}\}$. If every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms, then, for each $p \in S$, either X_p is torsion and $X[p]$ is isomorphic to $\mathbb{Z}(p)$ or X_p is torsionfree and every its compact subgroup is topologically isomorphic to $\mathbb{Z}_p$.*

Proof. Fix any $p \in S$. If X_p were mixed, we could find two elements $a, b \in X_p$ such that $1 < o(a) < \infty$ and $o(b) = \infty$, i.e. such that $\overline{\langle a \rangle} \cong \mathbb{Z}(p^n)$ for some $n \in \mathbb{N}_0$ and $\overline{\langle b \rangle} \cong \mathbb{Z}_p$ [1, Lemma 2.11]. It would then follow from Lemma 3.1 that

$$\overline{\langle a, b \rangle} \cong \mathbb{Z}(p^n) \times \mathbb{Z}_p,$$

which would contradict the hypothesis because $\mathbb{Z}(p^n) \times \mathbb{Z}_p$ is polythetic and $E(\mathbb{Z}(p^n) \times \mathbb{Z}_p)$ is not commutative. Consequently, X_p must be either torsion or torsionfree. In the former case, we conclude from [4, Ch. 2, §4, Théorème 2] that

$$X[p] \cong \mathbb{Z}(p)^{(\alpha)} \times \mathbb{Z}(p)^{\beta}$$

for some cardinal numbers α, β with $\alpha + \beta \geq 1$. But, in view of the hypothesis, X can contain no copy of $\mathbb{Z}(p) \times \mathbb{Z}(p)$. Therefore we must have $\alpha + \beta = 1$, and so $X[p] \cong \mathbb{Z}(p)$. In the second case, let G be a nonzero compact subgroup of X_p. Then $G \cong \mathbb{Z}_p^{\gamma}$ for some nonzero cardinal number γ [6, (24.25)]. Since our hypothesis ensures that X contains no copy of $\mathbb{Z}_p \times \mathbb{Z}_p$, we must have $\gamma = 1$, so $G \cong \mathbb{Z}_p$. □

We now can dispose of the important case of topological primary groups in $\mathcal{L}$.

Theorem 3.4. *Let $p \in \mathbb{P}$. For a topological p-primary group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms.*

(ii) *X is topologically isomorphic with one of the groups $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^n)$ for some $n \in \mathbb{N}$.*

Proof. Assume X is nonzero and satisfies condition (i). By Lemma 3.3, we know that either X is torsion and $X[p]$ is isomorphic to $\mathbb{Z}(p)$ or X is torsionfree and every its compact subgroup is topologically isomorphic to $\mathbb{Z}_p$.

If the former case occurs, we claim that X is isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^n)$ for some $n \in \mathbb{N}$. To see this, it will be enough, in view of L.C.A. van Leeuwen's result mentioned in the introduction, to show that X is discrete. Pick a compact open subgroup V of X. By the structure theorem for torsion compact abelian groups [6, (25.9)], V is topologically isomorphic to a group of the form $\prod_{i \in I} \mathbb{Z}(p^{n_i})$, where the set $\{n_i \mid i \in I\}$ is finite. As

$$V[p] \subset X[p] \cong \mathbb{Z}(p),$$

it follows that I cannot contain more than one element, so V is finite, and hense X is discrete. This establishes the claim.

In the latter case, fix a compact open subgroup W of X and a topological group isomorphism f from W onto $\mathbb{Z}_p$. Also let η denote the canonical injection of $\mathbb{Z}_p$ into $\mathbb{Q}_p$. Since $\mathbb{Q}_p$ is divisible and W is open in X, $\eta \circ f$ extends to a continuous open homomorphism h from X into $\mathbb{Q}_p$ [6, (A.7)]. Pick any $x \in \ker(h)$. Since X is a topological p-primary group, $p^m x \in W$ for sufficiently large $m \in \mathbb{N}$. We then have

$$p^m x \in \ker(f) = \{0\},$$

so $x = 0$ because X is torsionfree. It follows that h induces a topological isomorphism of X onto an open subgroup of $\mathbb{Q}_p$, and hence X is topologically isomorphic with either $\mathbb{Q}_p$ or $\mathbb{Z}_p$. This proves that (i) implies (ii).

The converse implication is clear in view of [7] and the fact that every nontrivial closed subgroup in the groups $\mathbb{Z}_p$ and $\mathbb{Q}_p$ is topologically isomorphic with $\mathbb{Z}_p$. $\square$

As a consequence, we obtain the solution to the considered problem in the case of topological torsion groups in $\mathcal{L}$.

Corollary 3.5. *For a topological torsion group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms.*

(ii) *For each $p \in S(X)$, X_p is topologically isomorphic with one of the groups $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}$.*

Proof. Assume (i). Since $X \in \mathcal{L}$ is a topological torsion group, we have

$$X \cong \prod_{p \in S(X)} (X_p; U_p),$$

where, for each $p \in S(X)$, X_p is the topological p-primary component of X and U_p is a compact open subgroup of X_p [1, Theorem 3.13]. Pick any $s \in S(X)$, and let G be a closed polythetic subgroup of X_s. Further, letting

$$\eta_s : X_s \to \prod_{p \in S(X)} (X_p; U_p)$$

denote the canonical injection, put $G' = \eta_s(G)$. Then G' is a closed polythetic subgroup of $\prod_{p \in S(X)}(X_p; U_p)$, so that $E(G')$ must be commutative. Since $E(G) \cong E(G')$, $E(G)$ is commutative as well. It follows that every closed polythetic subgroup of X_s has a commutative ring of continuous endomorphisms, so that, by Theorem 3.4, X_s is topologically isomorphic to one of the groups $\mathbb{Q}_s$, $\mathbb{Z}_s$, $\mathbb{Z}(s^\infty)$ or $\mathbb{Z}(s^{n_s})$ for some $n_s \in \mathbb{N}$.

Assume (ii), and let G be a closed polythetic subgroup of X. Since $k(G) = G$, G is compact [1, (5.40)(c)]. It follows that for each $p \in S(G)$, G_p is a compact subgroup of X_p, so that G_p is topologically isomorphic with either $\mathbb{Z}_p$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}$. Since, by [10, Lemma 3.4],

$$E(G) \cong \prod_{p \in S(G)} E(G_p),$$

we conclude that $E(G)$ is commutative. □

The next lemma shows that the case of groups in $\mathcal{L}$ consisting entirely of compact elements and having a nonzero connected component reduces to the case of compact and connected groups.

Lemma 3.6. *Let X be a group in $\mathcal{L}$ such that $c(X) \neq \{0\}$ and $k(X) = X$. If every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms, then X is compact and connected.*

Proof. We shall show first that $X_p \subset c(X)$ for all $p \in \mathbb{P}$. For this purpose, fix any $p \in \mathbb{P}$ and let C_p denote the topological p-primary component of $c(X)$. As is well known, C_p is dense in $c(X)$ [1, Corollary 4.18(a)], so that C_p and hence X_p is nonzero. In view of Lemma 3.3, we distinguish two cases.

Case (a): X_p is torsion and $X[p]$ is isomorphic to $\mathbb{Z}(p)$. Then $X[p]$ has no nontrivial subgroups, so that

$$X[p] = C_p[p] \subset c(X).$$

To apply induction, assume $X[p^k] \subset c(X)$ for some $k \in \mathbb{N}_0$, and choose any $a \in X[p^{k+1}]$. Since $pa \in X[p^k] \subset c(X)$ and since $c(X)$ is divisible, there exists $c \in c(X)$ such that $pa = pc$, and so

$$b = a - c \in X[p] \subset c(X),$$

whence $a = b + c \in c(X)$. As $a \in X[p^{k+1}]$ was arbitrarily chosen, $X[p^{k+1}] \subset c(X)$. Consequently, $X[p^i] \subset c(X)$ for all $i \in \mathbb{N}_0$, and hence

$$X_p = \bigcup_{i \in \mathbb{N}_0} X[p^i] \subset c(X).$$

Case (b): X_p is torsionfree and every its nonzero compact subgroup is topologically isomorphic to $\mathbb{Z}_p$. Pick an arbitrary nonzero $x \in X_p$. We assert that $\overline{\langle x \rangle} \cap c(X) \neq \{0\}$. For if not, then choosing any nonzero $x' \in C_p$ we would certainly have $\overline{\langle x \rangle} \cap \overline{\langle x' \rangle} = \{0\}$. By Lemma 3.1, this would imply that

$$\overline{\langle x, x' \rangle} \cong \overline{\langle x \rangle} \times \overline{\langle x' \rangle},$$

which contradicts the hypothesis because $\overline{\langle x \rangle} \cong \mathbb{Z}_p \cong \overline{\langle x' \rangle}$. Consequently, we must have $\overline{\langle x \rangle} \cap c(X) \neq \{0\}$, and hence

$$\overline{\langle x \rangle} \cap c(X) = p^m \overline{\langle x \rangle}$$

for some $m \in \mathbb{N}$. In particular $p^m x \in c(X)$, and so $p^m x = p^m z$ for some $z \in c(X)$, because of the divisibility of $c(X)$. Then $x - z \in t(X_p) = \{0\}$, and hence $x = z \in c(X)$. Since $x \in X_p$ was arbitrary, we have $X_p \subset c(X)$. Thus either case leads us to the conclusion that $X_p \subset c(X)$.

We now are ready to show that X is compact and connected. Let

$$\mathcal{V} = \{V \mid V \quad \text{is a compact open subgroup of} \quad X\}.$$

Since $k(X) = X$, it is clear that $X = \bigcup_{V \in \mathcal{V}} V$. We will be done if we show that every $V \in \mathcal{V}$ coincides with $c(X)$. To this end, pick an arbitrary $V \in \mathcal{V}$ and let r_0 denote the torsionfree rank of the discrete group V^*. We have $r_0 \neq 0$, since otherwise it would follow that V^* is torsion, and so V would be totally desconnected [6, (24.26)], which is however impossible because $c(X) \subset V$. By [5, §16] or [9, §3], there exists an injective homomorphism $f : \mathbb{Z}^{(r_0)} \to V^*$ such that $V^*/\operatorname{im}(f)$ is torsion. The adjoint mapping f^* is then a continuous open homomorphism from V onto $\mathbb{T}^{r_0}$ [6, (24.40)]. Letting $K = \ker(f^*)$, it follows that $V/K \cong \mathbb{T}^{r_0}$ [3, Ch. 3, §2, Proposition 24], and hence V/K is connected. Also, since

$$K = A(V, \operatorname{im}(f)) \cong (V^*/\operatorname{im}(f))^*$$

[6, (24.38) and (23.25)] and $V^*/\operatorname{im}(f)$ is torsion, K is totally disconnected [6, (24.26)]. It follows that $K \cong \prod_{p \in S(K)} K_p$ [1, Proposition 3.10], and thus

$$K = \overline{\sum_{p \in S(K)} K_p}$$

[6, (6.2)], whence $K \subset c(X)$ because by the above every K_p is contained in $c(X)$. Taking account of [6, (5.34)], we then have

$$V/c(X) \cong (V/K)/(c(X)/K),$$

so that $V/c(X)$ must be connected. Since $V/c(X)$ is certainly totally disconnected [6, (7.3)], this can occur only when $V = c(X)$, and the proof is complete. $\square$

We now consider the case of compact and connected groups in $\mathcal{L}$.

Theorem 3.7. *Let $X \in \mathcal{L}$ be compact and connected. The following statements are equivalent:*

(i) *Every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms.*

(ii) *X is topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup.*

Proof. Assume X is nonzero and satisfies (i). If $t(X) = \{0\}$, then $X \cong (\mathbb{Q}^*)^\alpha$ for some cardinal number $\alpha \geq 1$ [6, (25.8)]. We must have $\alpha = 1$, since otherwise X would contain a copy of $\mathbb{Q}^* \times \mathbb{Q}^*$, which is a contradiction because $\mathbb{Q}^* \times \mathbb{Q}^*$ is polythetic [1, (5.40)(b)] but $E(\mathbb{Q}^* \times \mathbb{Q}^*)$ is not commutative.

Now suppose $t(X) \neq \{0\}$, and pick any $p \in \mathbb{P}$ with $t_p(X) \neq \{0\}$. We know from Lemma 3.3 that $X_p = t_p(X)$ and $X[p] \cong \mathbb{Z}(p)$. Since X_p is dense in X [1, Corollary 4.18(a)], it follows that $X = \overline{t_p(X)}$. Also, since X is divisible [6, (24.25)], $t_p(X)$ is divisible too, so that $t_p(X)$ is algebraically isomorphic to $\mathbb{Z}(p^\infty)$ [9, Lemma, p. 33]. To see that X is topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup, it will be enough in view of [6, (24.11)] and Pontryagin duality theorem to show that X^* is isomorphic to a subgroup of $\mathbb{Q}$, i. e. that X^* is of rank 1. Pick any nonzero $\gamma \in X^*$. Since $o(\gamma) = \infty$, we will be done if we show that $X^*/\langle\gamma\rangle$ is torsion [9, Proposizione 1, p. 23]. But

$$(X^*/\langle\gamma\rangle)^* \cong A(X, \langle\gamma\rangle)$$

[6, (23.25)], so that in order to show that X is topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup, it will suffice to show that $A(X, \langle\gamma\rangle)$ is totally disconnected [6, (24.26)]. Assume not, and let $C = c(A(X, \langle\gamma\rangle))$. By [1, Corollary 4.18(a)], C_p is then a nonzero subgroup of $X_p = t_p(X)$, so that $C_p = t_p(C)$, and hence C_p is divisible. As $t_p(X)$ is algebraically isomorphic to $\mathbb{Z}(p^\infty)$, we must have $C_p = t_p(X)$, and so

$$X = \overline{t_p(X)} = \overline{C_p} = c(A(X, \langle\gamma\rangle)),$$

whence $X = A(X, \langle\gamma\rangle)$, which contradicts our assumption that $\gamma \neq 0$. Consequently, X must be topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup.

For the converse, assume (ii). It follows from [6, (24.11)] that X^* is topologically isomorphic to a subgroup of $\mathbb{Q}$. Let G be a closed polythetic subgroup of X. Since $G^* \cong X^*/A(X^*, G)$ and since every quotient of $\mathbb{Q}$ by a nonzero subgroup is isomorphic to a divisible subgroup of $\mathbb{Q}/\mathbb{Z}$, we conclude that G^* is isomorphic either to a subgroup of $\mathbb{Q}$ or to a subgroup of $\mathbb{Q}/\mathbb{Z}$. By the result of [7] mentioned in the introduction, it follows that in either case $E(G^*)$ is commutative, so that in view of [10, Lemma 3.1] $E(G)$ is commutative as well. The proof is complete. □

We are now ready to prove the main theorem of this section, which describes the groups $X \in \mathcal{L}$ such that every closed polythetic subgroup G of X has a commutative ring $E(G)$.

Theorem 3.8. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every closed polythetic subgroup of X has a commutative ring of continuous endomorphisms.*

(ii) *X is topologically isomorphic to one of the groups:*
(1) $\mathbb{R}$, (2) *a subgroup of* $\mathbb{Q}$, (3) *a quotient of* $\mathbb{Q}^*$ *by a closed subgroup,* (4) $\prod_{p \in S(X)}(X_p; U_p)$, *where, for each* $p \in S(X)$, X_p *is topologically isomorphic to either* $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$, *or* $\mathbb{Z}(p^{n_p})$ *for some* $n_p \in \mathbb{N}$, *and* U_p *is a compact open subgroup of* X_p.

Proof. Assume (i). If $X \notin \mathcal{L}_0$, we can write $X = V \oplus Y$, where V, Y are closed subgroups of X such that $V \cong \mathbb{R}^d$ for some $d \in \mathbb{N}_0$ and $Y \in \mathcal{L}_0$ [6, (24.30)]. Since V

is of course polythetic [1, (5.40)(e)], $E(V)$ must be commutative, so that we must have $d = 1$. Also, since $X \neq k(X)$, we deduce from Lemma 3.2 that $k(X) = \{0\}$, so Y is discrete and V is open in X. If Y were not the zero group, it would follow that, for every nonzero $a \in Y$, $V + \langle a \rangle$ is an open and hence closed subgroup of X satisfying $V + \langle a \rangle \cong \mathbb{R} \times \mathbb{Z}$. This is a contradiction because $\mathbb{R} \times \mathbb{Z}$ is polythetic [1, (5.40)(f)] and $E(R \times \mathbb{Z})$ is not commutative. Consequently, in this case $X \cong \mathbb{R}$.

Now let $X \in \mathcal{L}_0$. In view of Lemma 3.2, we must have either $k(X) = \{0\}$ or $k(X) = X$. If the former case occurs, X is discrete and torsionfree, and hence it is isomorphic to a subgroup of $\mathbb{Q}$, by the result of [7] mentioned in the introduction. Assume the latter. If $c(X) = \{0\}$, it follows from Corollary 3.5 that

$$X \cong \prod_{p \in S(X)} (X_p; U_p),$$

where, for each $p \in S(X)$, X_p is topologically isomorphic to one of the groups $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}$, and U_p is a compact open subgroup of X_p. Finally, if $c(X) \neq \{0\}$, we conclude from Lemma 3.6 that X is compact and connected, and hence X is topologically isomorphic to a quotient of $\mathbb{Q}^*$ by a closed subgroup, according to Theorem 3.7.

The converse is clear. □

Dualizing the preceding theorem, we obtain the description of groups in $\mathcal{L}$ all of whose copolythetic quotients by closed subgroups have commutative rings of continuous endomorphisms.

Corollary 3.9. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every copolythetic quotient of X by a closed subgroup has a commutative ring of continuous endomorphisms.*

(ii) *X is topologically isomorphic to one of the groups:*
(1) $\mathbb{R}$, (2) *a subgroup of* $\mathbb{Q}$, (3) *a quotient of* $\mathbb{Q}^*$ *by a closed subgroup,* (4) $\prod_{p \in S(X)} (X_p; U_p)$, *where, for each* $p \in S(X)$, X_p *is topologically isomorphic to either* $\mathbb{Q}_p$, $\mathbb{Z}_p$, $\mathbb{Z}(p^\infty)$, *or* $\mathbb{Z}(p^{n_p})$ *for some* $n_p \in \mathbb{N}$, *and* U_p *is a compact open subgroup of* X_p.

4 Copolythetic subgroups

In the present section, we characterize the groups $X \in \mathcal{L}$ such that every closed copolythetic subgroup G of X has a commutative ring $E(G)$. By utilizing duality, we also get the description of those groups $X \in \mathcal{L}$ which have the property that for each closed subgroup G of X such that X/G is polythetic, the ring $E(X/G)$ is commutative.

We will obtain these results as a consequence of a number of lemmas. The first two of these establish, for certain particular types of groups, some necessary conditions.

Lemma 4.1. *Let X be a group in $\mathcal{L}$ such that every its closed copolythetic subgroup has a commutative ring of continuous endomorphisms. If $X = A \oplus Y$, where A is topologically isomorphic with either $\mathbb{R}$ or $\mathbb{T}$, then Y is torsionfree.*

Proof. If $t(Y)$ were nonzero, it would contain a copy of $\mathbb{Z}(p)$ for some $p \in \mathbb{P}$. Since A is topologically isomorphic with either $\mathbb{R}$ or $\mathbb{T}$, it would then follow from Lemma 3.1 that X contains a copy of either $\mathbb{Z} \times \mathbb{Z}(p)$ or $\mathbb{Z}(p) \times \mathbb{Z}(p)$, a contradiction. $\square$

Lemma 4.2. *Let X be a group in $\mathcal{L}$ with $t(X) \neq \{0\}$. If every closed copolythetic subgroup of X has a commutative ring of continuous endomorphisms, then $k(X) = X$ and $X[p] \cong \mathbb{Z}(p)$ for all $p \in S_0(X)$.*

Proof. Pick any $p \in S_0(X)$. By [4, Ch. 2, §4, Théorème 2], we have

$$X[p] \cong \mathbb{Z}(p)^{(\alpha)} \times \mathbb{Z}(p)^{\beta}$$

for some cardinal numbers α, β with $\alpha + \beta \geq 1$. Since X cannot contain copies of $\mathbb{Z}(p) \times \mathbb{Z}(p)$, we must have $\alpha + \beta = 1$, and so $X[p] \cong \mathbb{Z}(p)$. If there existed $x \in X \setminus k(X)$, it would then follow from Lemma 3.1 that X contains a copy of $\mathbb{Z} \times \mathbb{Z}(p)$, contradicting the hypothesis. $\square$

We continue with two simple lemmas that will be usefull in the sequel.

Lemma 4.3. *If $G \in \mathcal{L}_0$ is copolythetic and contains no copy of $\mathbb{T}$, then G is discrete.*

Proof. By the definition of copolythetic groups, there exists for some $n \in \mathbb{N}_0$ an injective $h \in H(G, \mathbb{T}^n)$. Let K be a compact open subgroup of G. Then $h(K)$ is closed in $\mathbb{T}^n$, and hence $h(K)$ is topologically isomorphic to a group of the form $\mathbb{T}^m \times F$, where m is an integer satisfying $0 \leq m \leq n$ and F is a direct sum of at most $n - m$ finite cyclic groups [3, Ch. VII, §1, Proposition 11]. Since h is injective, its restriction to K establishes a topological isomorphism of K onto $h(K)$ [2, Ch. I, §9, Théorème 2, Corollaire 2], and so

$$K \cong \mathbb{T}^m \times F.$$

As G does not contain copies of $\mathbb{T}$, we must have $m = 0$, so that K is finite, and hence G is discrete. $\square$

Lemma 4.4. *Let K be a closed subgroup of an abelian topological group Y such that $K = A \oplus B$ for some subgroups A, B of K. For any closed subset C of A, $C + B$ is closed in Y.*

Proof. Let φ denote the canonical projection of K onto A. We have $C+B = \varphi^{-1}(C)$, so that $C + B$ is closed in K and hence in Y. $\square$

The following two lemmas establish, for certain particular types of groups, some sufficient conditions.

Lemma 4.5. *If X is a torsionfree group in $\mathcal{L}$ all of whose nonzero discrete subgroups are of rank one, then every closed copolythetic subgroup of X has a commutative ring of continuous endomorphisms.*

Proof. Let G be a nonzero closed copolythetic subgroup of X. If $G \notin \mathcal{L}_0$, we can write $G = A \oplus B$, where $A \cong \mathbb{R}^d$ for some $d \in \mathbb{N}_0$ and $B \in \mathcal{L}_0$. We must have $d = 1$, since otherwise G would contain discrete subgroups of rank greater than one. Further, being torsionfree, X contains no copy of $\mathbb{T}$. As B is clearly copolythetic, it then follows from Lemma 4.3 that B is discrete, so that either $B = \{0\}$ or B is of rank one. But the latter is impossible. To see this, assume the contrary and pick any nonzero $a \in A$. Since $\langle a \rangle$ is closed in A, it follows from Lemma 4.4 that $\langle a \rangle + B$ is closed in X. As $\langle a \rangle + B$ is countable, we deduce from [8, Corollary, p. 23] that $\langle a \rangle + B$ is discrete, a contradiction because $\langle a \rangle + B$ is, in our case, of rank two. Thus $B = \{0\}$, and hence $E(G)$ is commutative.

In case $G \in \mathcal{L}_0$, it follows again from Lemma 4.3 that G is discrete, and hence of rank one, so that $E(G)$ is commutative.

Lemma 4.6. *If $X = A \times Y$, where $A \cong \mathbb{T}$ and Y is a torsionfree group in $\mathcal{L}$ with $k(Y) = Y$, then every closed copolythetic subgroup of X has a commutative ring of continuous endomorphisms.*

Proof. Since $\mathbb{T}$ is compact, it is clear that $X = k(X)$. We also have

$$X[p] \cong A[p] \times Y[p] \cong \mathbb{T}[p] \cong \mathbb{Z}(p)$$

for all $p \in \mathbb{P}$. It follows in particular that X cannot contain copies of $\mathbb{T} \times \mathbb{T}$. Indeed, if there existed a closed subgroup K of X satisfying $K \cong \mathbb{T} \times \mathbb{T}$, then, picking any $p \in \mathbb{P}$, we would have

$$K[p] \cong \mathbb{T}[p] \times \mathbb{T}[p] \cong \mathbb{Z}(p) \times \mathbb{Z}(p),$$

contardicting the fact that $X[p] \cong \mathbb{Z}(p)$.

Now, fix an arbitrary nonzero closed copolythetic subgroup G of X. If G contains no copy of $\mathbb{T}$, it follows from Lemma 4.3 that G is discrete, so $G = k(G) = t(G)$. Since

$$t(X) \cong t(A) \times t(Y) \cong t(\mathbb{T}),$$

we conclude that G is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$, and so $E(G)$ is commutative by [11, Theorem 1].

Suppose next that G contains closed subgroups topologically isomorphic to $\mathbb{T}$. Since for any cardinal number ν the group $\mathbb{T}^\nu$ is splitting in $\mathcal{L}$ [6, (25.31)(b)], we can write $G = B \oplus C$ for some closed subgroups B, C of X with $C \cong \mathbb{T}$. Now, B must be torsionfree since otherwise X would contain a copy of $\mathbb{Z}(p) \times \mathbb{Z}(p)$ for some $p \in \mathbb{P}$, in contradiction with the fact that $X[p] \cong \mathbb{Z}(p)$. Moreover, since X cannot contain copies of $\mathbb{T} \times \mathbb{T}$, B contains no copy of $\mathbb{T}$. As B is clearly copolythetic, it follows from Lemma 4.3 that B is discrete. Therefore

$$B = k(B) = t(B) = \{0\},$$

so $G \cong \mathbb{T}$, and hence $E(G)$ is commutative. □

We now combine the above results to obtain the desired description of groups in $\mathcal{L}$ all of whose copolythetic subgroups have a commutative ring of continuous endomorphisms.

Theorem 4.7. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every closed copolythetic subgroup of X has a commutative ring of continuous endomorphisms.*

(ii) *X satisfies one of the following three conditions:*

(1) *X is torsionfree and every its nonzero discrete subgroup is of rank one.*

(2) *$X \cong \mathbb{T} \times Y$, where Y is a torsionfree group in $\mathcal{L}$ with $k(Y) = Y$.*

(3) *X contains no copy of $\mathbb{T}$, $k(X) = X$, $t(X) \neq \{0\}$, and $X[p] \cong \mathbb{Z}(p)$ for all $p \in S_0(X)$.*

Proof. Assume (i). We consider first the case when X is torsionfree. Pick an arbitrary nonzero discrete subgroup G of X, and let M be a maximal free subset of G. Then

$$\langle M \rangle \cong \bigoplus_{x \in M} \langle x \rangle$$

[9, Proposizione 1, p. 23]. Note also that every subgroup of G, being discrete, is closed in X [6, (5.10)]. If G were not of rank one, we would have $|M| > 1$, so that X would contain a copy of $\mathbb{Z} \times \mathbb{Z}$, contradicting the hypothesis. Thus G must be of rank one. As G was chosen arbitrarily among the nonzero discrete subgroups of X, we conclude that in this case X satisfies (1).

Next suppose that $t(X) \neq \{0\}$. It follows from Lemma 4.2 that $k(X) = X$ and $X[p] \cong \mathbb{Z}(p)$ for all $p \in S_0(X)$. Therefore, if X contains no copy of $\mathbb{T}$, we are led to (3). If, on the other hand, X contains a closed subgroup $A \cong \mathbb{T}$, we can write $X = A \oplus Y$ for some closed subgroup Y of X. Then, for each $p \in S_0(X)$, we have

$$X[p] \cong A[p] \oplus Y[p] \quad \text{and} \quad A[p] \cong \mathbb{T}[p] \cong \mathbb{Z}(p),$$

so that, in view of the above mentioned fact that $X[p]$ is simple, $Y[p] = \{0\}$. It follows that $t(Y) = \{0\}$, and hence X satisfies (2).

Now assume (ii). If X satisfies (1), then Lemma 4.5 shows that (i) holds. In case X satisfies (2), the validity of (i) follows from Lemma 4.6. Finally, suppose that X satisfies (3), and let G be a closed copolythetic subgroup of X. It follows from Lemma 4.3 that G is discrete, so $G = k(G) = t(G)$, and hence

$$G \cong \bigoplus_{p \in S_0(G)} G_p.$$

Since, clearly, $G_p[p] = X[p] \cong \mathbb{Z}(p)$ for all $p \in S_0(G)$, we conclude that G is isomorphic to a subgroup of $\bigoplus_{p \in S_0(G)} \mathbb{Z}(p^\infty)$, so that $E(G)$ is commutative by [11, Theorem 1]. □

By use of duality, we obtain the description of groups in $\mathcal{L}$ all of whose polythetic quotients by closed subgroups have commutative rings of continuous endomorphisms.

Corollary 4.8. *For a group $X \in \mathcal{L}$, the following statements are equivalent:*

(i) *Every polythetic quotient of X by a closed subgroup has a commutative ring of continuous endomorphisms.*

(ii) *X satisfies one of the following three conditions:*

(1) *X is densely divisible and every its compact quotient by a proper closed subgroup is of dimension one.*

(2) *$X \cong \mathbb{Z} \times Y$, where Y is a densely divisible and totally disconnected group in $\mathcal{L}$.*

(3) *X is a totally disconnected group with no quotients by closed subgroups topologically isomorphic to $\mathbb{Z}$, $\bigcap_{p \in \mathbb{P}} \overline{pX} \neq X$, and $X/\overline{pX} \cong \mathbb{Z}(p)$ for all $p \in \mathbb{P}$ such that $\overline{pX} \neq X$.*

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