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Linear convolution of criteria in the vector p-center problem *

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Abstract. We investigate a linear convolution of criteria and possibility of its application for finding Pareto set in the vector variant of the well-known combinatorial p-center problem. The polynomial algorithm which transforms any vector p-center problem to a solvable problem with the same Pareto set is proposed. An example which illustrates the work of algorithm is performed.

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1 Introduction

Wide and important class of the best choice problems is the vector (multicriteria) problems in which the quality of decision making is estimated by several criteria at once. One of the main method of vector optimization is scalaring. Scalaring is a process of transforming vector problem of finding best alternatives to a scalar problem with aggregated (generalized) criterion which is a convolution of criteria. Such convolution of criteria usually depends on parameters. The central concept in vector optimization is a Pareto principle of optimality. And an important method of finding Pareto-optimal solutions (efficient alternatives) is based on linear convolution of criteria. But this approach cannot always guarantees to find the whole Pareto set. In these cases we say that the vector problem is unsolvable by ALC. Many classes of the vector problems which are solvable by ALC were found by Koopmans, Karlin, Geoffrion, Kuhn, Tucker, Saaty and others. The history review of this question was presented in [1] (see also [2]).

We investigate the possibility of application of ALC to finding all Pareto set in vector variant of the well-known combinatorial p-center problem, i. e. the problem of best locating p facilities (see, for example, [3–5]). In this paper it is shown that there exist (see, theorem 1) vector p-center problems which are unsolvable by ALC. Analogous results for various kinds of vector discrete optimization problems (salesman problem, optimal spanning tree, perfect matching and others) were obtained earlier in [6–11].

Using the well-known sufficient condition of solvability [12] we build an algorithm, which transforms any vector p-center problem to an equivalent solvable problem.

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Earlier in the works [12–15] similar algorithms were built for the vector trajectory problems with another kinds of partial criteria.

2 Basic definitions and notations

We consider the vector (s-criteria) variant of the p-center problem. Let us use the following definitions:

$N_m = \{1, 2, \ldots, m\}$ is the set of possible locating points of facilities (equipment, warehouses, providers etc.),

N_n is the set of clients,

$D_k = [d_{ij}^k] \in \mathbf{R}^{m\times n}$ is a matrix of costs connected with delivery of product from point $i \in N_m$ to point $j \in N_n$ by criterion $k \in N_s$.

The vector $D = (D_1, D_2, \ldots, D_s)$, composed from matrices of costs, is called a system of costs.

Let $1 \le p \le m-1$ and T be some system of nonempty subsets (p-centers) of the set N_m such that

$$|t| = p, \quad \forall t \in T.$$

As usual (see, for example, [10–14]), all the elements of set T are called trajectories.

We define the vector function on T:

$$f(t, D) = (f_1(t, D_1), f_2(t, D_2), \ldots, f_s(t, D_s)),$$

where

$$f_k(t, D_k) = \max_{j\in N_n} \min_{i\in t} d_{ij}^k \to \min_{t\in T}, \quad k \in N_s.$$

The s-criteria (vector) $m \times n$-dimensional p-center problem is understood as the problem of finding Pareto set (the set of efficient trajectories)

$$P^s(T, D) = \{t \in T : \ \forall\, t' \in T \ \ (t \underset{D}{\not\succ} t')\},$$

where $\underset{D}{\not\succ}$ is the negation of binary relation $\underset{D}{\succ}$, which specifies the Pareto principle of optimality:

$$t \underset{D}{\succ} t' \ \Leftrightarrow\ f(t, D) \ge f(t', D) \ \&\ f(t, D) \ne f(t', D).$$

This problem is denoted by $Z^s_{m\times n}(T, D)$. Scalar (single-criterion) problem $Z^1_{m\times n}(T, D)$, $D \in \mathbf{R}^{m\times n}$, can be interpreted as an extremal problem on graphs or on networks. It consists in locating p facilities and assigning clients to them in order to minimize the maximum distance between a client and the facility to which it is assigned. If we want to optimize the location of p facilities by several criteria then it leads us to the above multicriteria variant of the p-center problem.

Following [7–9], the problem $Z^s_{m\times n}(T, D)$, $s \ge 2$, is called solvable by ALC if

$$P^s(T, D) = \Xi^s(T, D),$$

where

$$\Xi^s(T,D) = \bigcup_{\lambda \in \Lambda^s} T(\lambda),$$

$$\Lambda^s = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s) : \sum_{k=1}^{s} \lambda_k = 1, \quad \lambda_k > 0, \quad k \in N_s\},$$

$$T(\lambda) = \operatorname{Argmin}\{\langle \lambda, f(t,D) \rangle : t \in T\}$$

and $\langle \lambda, f(t,D) \rangle = \sum\limits_{k=1}^{s} \lambda_k f_k(t, D_k)$ is a linear convolution of criteria $f_k(t, D_k),\ k \in N_s$.

Thus, the problem $Z^s_{m\times n}(T,D)$ is solvable if for any efficient trajectory $t^* \in P^s(T,D)$ there exists a vector $\lambda^* \in \Lambda^s$ such that

$$\langle \lambda^*, f(t^*,D) \rangle = \min\{\langle \lambda^*, f(t,D) \rangle : t \in T\},$$

i. e. any trajectory $t^* \in P^s(T,D)$ can be found as a solution of a scalar minimization problem with function which is a linear convolution of partial criteria with an appropriate vector $\lambda^* \in \Lambda^s$.

Otherwise, if there exists a trajectory $t^* \in P^s(T,D)$ such that for any vector $\lambda \in \Lambda^s$ the inequality

$$\langle \lambda, f(t^*,D) \rangle > \min\{\langle \lambda, f(t,D) \rangle : t \in T\}$$

holds, then the problem $Z^s_{m\times n}(T,D)$ is called unsolvable by ALC. It is evident, that in this case we have

$$\Xi^s(T,D) \subset P^s(T,D).$$

3 Insolubility

The set of trajectories T is called primitive if the following two conditions hold:

1) there exist three pairwise different trajectories $t_1,\ t_2$ and t_3 such that

$$i \in t_i \setminus t^0, \quad i = 1, 2, 3,$$

where

$$t^0 = \bigcup_{1 \le r_1 < r_2 \le 3} (t_{r_1} \cap t_{r_2}),$$

2) for any trajectory $t \in T \setminus \{t_1, t_2, t_3\}$ the equality

$$t \cap N_3 = \emptyset$$

holds.

Thus, in the case, where the set T is primitive, the number of possible locating points of facilities $m \ge 4$.

Theorem 1. *For any primitive set of trajectories T there exists a system of costs D such that the p-center problem $Z^s_{m\times n}(T,D),\ p \ge 1,\ s \ge 2,\ m \ge 4,\ n \ge 1,$ is unsolvable by ALC.*

Proof. First we consider the case where $s = 2$. Let $t_1,\ t_2,\ t_3$ be the three trajectories of set T described above. Let the matrices $D_k = [d_{ij}^k] \in \mathbf{R}^{m\times n}$, $k = 1,\ 2$, have the following form

$$D_1 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ a & a & \dots & a \\ b & b & \dots & b \\ c & c & \dots & c \\ \dots & \dots & \dots & \dots \\ c & c & \dots & c \end{pmatrix}, \quad D_2 = \begin{pmatrix} a & a & \dots & a \\ 0 & 0 & \dots & 0 \\ b & b & \dots & b \\ c & c & \dots & c \\ \dots & \dots & \dots & \dots \\ c & c & \dots & c \end{pmatrix},$$

where $c > a > b > a/2 > 0$.

Then, taking into account the primitivity of set T, we obtain the vector evaluations of trajectories of T:

$$f(t_1, D) = (0, a),$$

$$f(t_2, D) = (a, 0),$$

$$f(t_3, D) = (b, b),$$

$$f(t, D) = (c, c) \quad \forall\, t \in T \setminus \{t_1, t_2, t_3\}.$$

Therefore we have $P^2(T, D) = \{t_1, t_2, t_3\}$ and for any vector $\lambda \in \Lambda^2$ the following relations hold

$$\langle \lambda, f(t_3, D)\rangle = b > a/2 \geq \min\{\langle \lambda, f(t_i, D)\rangle : i = 1, 2\} \geq \min\{\langle \lambda, f(t, D)\rangle : t \in T\}. \tag{1}$$

It follows that the theorem is valid in the case $s = 2$.

Finally, the theorem ($s > 2$) can be proved if the matrices D_1 and D_2 are the same as before, and

$$D_k = D_2, \quad k \in\ 3, 4, \dots, s.$$

As a result we obtain the following vector evaluations of trajectories:

$$f(t_1, D) = (0, a, a, \dots, a) \in \mathbf{R}^s,$$

$$f(t_2, D) = (a, 0, 0, \dots, 0) \in \mathbf{R}^s,$$

$$f(t_3, D) = (b, b, b, \dots, b) \in \mathbf{R}^s,$$

$$f(t, D) = (c, c, \dots, c) \in \mathbf{R}^s \quad \forall\, t \in T \setminus \{t_1, t_2, t_3\}.$$

Therefore we have $P^s(T, D) = \{t_1, t_2, t_3\}$ and any vector $\lambda \in \Lambda^s$ satisfies relations (1).

Theorem 1 has been proved.

4 Algorithm

Each of five stages of algorithm Ψ, which builds a transformed system of costs $\tilde{D}$ consists of s steps ($s \geq 2$).

Stage 1. Step $k \in N_s$. For any number $j \in N_n$ we sort all the elements d_{ij}^k, $i \in N_m$, of j-th column of matrix D_k:

$$d_{i_1 j}^k \geq d_{i_2 j}^k \geq \ldots \geq d_{i_m j}^k.$$

Stage 2. Step $k \in N_s$. We delete all the elements of the following sequence from matrix D_k

$$d_{i_1 j}^k,\ d_{i_2 j}^k, \ldots,\ d_{i_q j}^k, \quad j \in N_n, \tag{2}$$

where $q = p - 1$.

Stage 3. Step $k \in N_s$. We sort the rest of elements of matrix D_k in ascending order:

$$b_1^k \leq b_2^k \leq \ldots \leq b_u^k, \tag{3}$$

where $u = n(m - p + 1)$. Of course, all the elements stay on the same places in matrix D_k.

Stage 4. Step $k \in N_s$. We transform the elements of (3) by the following recurring formula

$$\tilde{b}_r^k = \begin{cases} b_r^k, & \text{if } r = 1, 2, \\ \tilde{b}_{r-1}^k, & \text{if } \Delta^k(r, r-1) = 0,\ r = 3, 4, \ldots, u, \\ b_r^k + s(\tilde{b}_{r-1}^k - \tilde{b}_1^k) + \tilde{b}_1^k, & \text{if } \Delta^k(r, r-1) > 0,\ r = 3, 4, \ldots, u, \end{cases} \tag{4}$$

where $\Delta^k(v, w) = b_v^k - b_w^k$. As a result we obtain $\tilde{b}_1^k,\ \tilde{b}_2^k, \ldots,\ \tilde{b}_u^k$.

Stage 5. Step $k \in N_s$. Instead of elements of sequence (2), deleted on the step 2, we write the number $\tilde{b}_u^k + 1$, i. e. for any $j \in N_n$ we set

$$\tilde{d}_{i_1 j}^k = \tilde{d}_{i_2 j}^k = \ldots = \tilde{d}_{i_q j}^k = \tilde{b}_u^k + 1.$$

As a result of the work of the algorithm Ψ the system of costs D is replaced by $\tilde{D} = (\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_s)$, where $\tilde{D}_k = [\tilde{d}_{ij}^k]_{m \times n}$.

Remark 1. It is easy to see that for any number $k \in N_s$ and for any trajectory t, the inequality $f_k(t, \tilde{D}_k) < \tilde{b}_u^k + 1$ holds.

5 Substantiation of algorithm

First we prove that the obtained vector problem $Z_{m \times n}^s(T, \tilde{D})$ is solvable by ALC. To prove this we use the known sufficient condition of solvability for the vector discrete problems [12] and formulate it in the form convenient for us. Let us introduce a new definition.

For any natural numbers $s \geq 2$ and $h \geq 1$ the set composed of h pairwise different numbers is called (s, h)-regular if after sorting these numbers in ascending order

$$a_1 < a_2 < \ldots < a_h$$

under $h \geq 3$ the inequalities

$$s \cdot \delta(r+1, 1) \leq \delta(r+2, 1), \;\; r \in N_{h-2}$$

hold, where $\delta(u, v) = a_u - a_v$.

Remark 2. It is evident that for any $s \geq 2$ the set composed of one or two different numbers is $(s, 1)$- or $(s, 2)$-regular, respectively.

In these terms for any s-criteria discrete problem Z^s

$$f_k(t) \to \min_{t \in T}, \quad k \in N_s,$$

where $s \geq 2$, $f_k(t) \in \mathbf{R}$, $|T| < \infty$, the following known sufficient condition of solvability is valid.

Theorem 2. *([12]) If for any number $k \in N_s$ the set, composed of $h(k)$ different values of k-th partial criterion $f_k(t)$ on the set T, is $(s, h(k))$-regular, then the problem Z^s is solvable by ALC.*

It is easy to see that for any number $k \in N_s$ the set of $h(k)$ different numbers of sequence

$$\tilde{b}_1^k, \; \tilde{b}_2^k, \ldots, \; \tilde{b}_u^k,$$

obtained as a result of the Stage 4 of algorithm Ψ, is $(s, h(k))$-regular. Hence, taking into account Remark 1, we conclude that the set of $h'(k)$ $(h'(k) \leq h(k))$ different values of the k-th criterion $f_k(t, \tilde{D}_k)$ on the set T is $(s, h'(k))$-regular, and therefore in view of Theorem 2 the problem $Z^s_{m \times n}(T, \tilde{D})$ is solvable by ALC.

Taking into account algorithm Ψ, we conclude that for any two trajectories t and t' the following formula

$$t \underset{D}{\succ} t' \Leftrightarrow t \underset{\tilde{D}}{\succ} t',$$

holds, which implies that the vector problems $Z^s_{m \times n}(T, D)$ and $Z^s_{m \times n}(T, \tilde{D})$ are equivalent, i. e. the equality $P^s(T, D) = P^s(T, \tilde{D})$ is valid.

It is easy to see that the complexity of Stages 1 and 2 of algorithm Ψ is $O(snm \log_2 m)$. Since the Stage 3 is a multi-way merging of ordered numerical sequences then the complexity of Stage 3 is $O(sn(m-p+1) \log_2 n)$ (see, for example [16]). The complexity of Stages 4 and 5 of algorithm Ψ are $O(sn(m-p+1))$ and $O(sn(p-1))$ respectively.

Summarizing the said above, we conclude that the following theorem holds.

Theorem 3. *The algorithm Ψ transforms any vector p-center problem $Z^s_{m \times n}(T, D)$, $s \geq 2$, to the equivalent p-center problem $Z^s_{m \times n}(T, \tilde{D})$, which is solvable by ALC, moreover the complexity of algorithm Ψ is $O(sn(m \log_2 m + (m-p+1) \log_2 n))$.*

In view of Remark 2 Theorem 3 implies

Corollary 4. *The problem $Z^s_{m\times n}(T,D)$ is solvable by ALC if for any $k \in N_s$ the inequality $h(k) \le 2$ holds.*

In the partial case, where $p = 1$, the complexity of algorithm Ψ is $O(smn \log_2 mn)$.

6 Example

We consider 2-criteria 2-center problem with 5×2-dimension, i. e. $s = 2,\ p = 2,\ m = 5,\ n = 2$. Let $T = \{t_1, t_2, t_3, t_4\},\ t_1 = \{1,2\},\ t_2 = \{2,3\},\ t_3 = \{3,4\},\ t_4 = \{4,5\},\ D = (D_1, D_2)$,

$$D_1 = \begin{pmatrix} 4 & 0 \\ 0 & 6 \\ 6 & 4 \\ 10 & 2 \\ 8 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2 & 6 \\ 1 & 10 \\ 0 & 4 \\ 8 & 0 \\ 10 & 3 \end{pmatrix}.$$

Then the problem $Z^2_{5\times 2}(T,D)$ has the following vector evaluations of trajectories

$$f(t_1, D) = (0, 6),$$

$$f(t_2, D) = (4, 4),$$

$$f(t_3, D) = (6, 0),$$

$$f(t_4, D) = (8, 8).$$

It is easy to see that $P^2(T,D) = \{t_1, t_2, t_3\}$, and the problem $Z^2_{5\times 2}(T,D)$ is unsolvable, because for any vector $\lambda \in \Lambda^2$ we have

$$\langle \lambda, f(t_2, D)\rangle = 4 > 3 \ge \min\{\langle \lambda, f(t_i, D)\rangle : i \in N_4\},$$

i. e. $t_2 \notin \Xi^2(T,D)$.

Using algorithm Ψ, we transform the system of costs D to $\tilde{D} = \{\tilde{D_1}, \tilde{D_2}\}$.

Stage 1. Let us sort all the elements of each column for each matrix D_1 and D_2 in ascending order

$$D_1 = \begin{pmatrix} 4^{\mathbf{2}} & 0^{\mathbf{1}} \\ 0^{\mathbf{1}} & 6^{\mathbf{5}} \\ 6^{\mathbf{3}} & 4^{\mathbf{4}} \\ 10^{\mathbf{5}} & 2^{\mathbf{3}} \\ 8^{\mathbf{4}} & 1^{\mathbf{2}} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2^{\mathbf{3}} & 6^{\mathbf{4}} \\ 1^{\mathbf{2}} & 10^{\mathbf{5}} \\ 0^{\mathbf{1}} & 4^{\mathbf{3}} \\ 8^{\mathbf{4}} & 0^{\mathbf{1}} \\ 10^{\mathbf{5}} & 3^{\mathbf{2}} \end{pmatrix}.$$

Stage 2. Delete from each column of matrices D_1 and D_2 one ($q = p - 1 = 1$) maximal number:

$$D_1 = \begin{pmatrix} 4^{\mathbf{2}} & 0^{\mathbf{1}} \\ 0^{\mathbf{1}} & \\ 6^{\mathbf{3}} & 4^{\mathbf{4}} \\ & 2^{\mathbf{3}} \\ 8^{\mathbf{4}} & 1^{\mathbf{2}} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2^{\mathbf{3}} & 6^{\mathbf{4}} \\ 1^{\mathbf{2}} & \\ 0^{\mathbf{1}} & 4^{\mathbf{3}} \\ 8^{\mathbf{4}} & 0^{\mathbf{1}} \\ & 3^{\mathbf{2}} \end{pmatrix}.$$

Stage 3. Sort the rest of elements of each matrices D_1 and D_2 in ascending order, and put them in matrices B_1 and B_2, respectively:

$$D_1 = \begin{pmatrix} 4^{\mathbf{2}} & 0^{\mathbf{1}} \\ 0^{\mathbf{1}} & \\ 6^{\mathbf{3}} & 4^{\mathbf{4}} \\ & 2^{\mathbf{3}} \\ 8^{\mathbf{4}} & 1^{\mathbf{2}} \end{pmatrix}, \quad D_2 = \begin{pmatrix} 2^{\mathbf{3}} & 6^{\mathbf{4}} \\ 1^{\mathbf{2}} & \\ 0^{\mathbf{1}} & 4^{\mathbf{3}} \\ 8^{\mathbf{4}} & 0^{\mathbf{1}} \\ & 3^{\mathbf{2}} \end{pmatrix},$$

$$B^1 = (b_1^1, b_2^1, \dots, b_8^1) = (0, 0, 1, 2, 4, 4, 6, 8),$$

$$B^2 = (b_1^2, b_2^2, \dots, b_8^2) = (0, 0, 1, 2, 3, 4, 6, 8).$$

Stage 4. Transform the numbers of matrices B_1 and B_2 according to the formula (4):

$$\tilde{B^1} = (\tilde{b}_1^1, \tilde{b}_2^1, \dots, \tilde{b}_8^1) = (0, 0, 1, 4, 12, 12, 30, 68),$$

$$\tilde{B^2} = (\tilde{b}_1^2, \tilde{b}_2^2, \dots, \tilde{b}_8^2) = (0, 0, 1, 4, 11, 26, 58, 124).$$

Stage 5. In each column of matrix D_1 on the deleted number place (Stage 2) we put the number

$$\tilde{b}_u^1 + 1 = \tilde{b}_8^1 + 1 = 69,$$

and in each column of matrix D_2 we put the number

$$\tilde{b}_u^2 + 1 = \tilde{b}_8^2 + 1 = 125.$$

As a result we obtain the problem $Z_{5\times 2}^2(T, \tilde{D})$, where

$$\tilde{D_1} = \begin{pmatrix} 12 & 0 \\ 0 & 69 \\ 30 & 12 \\ 69 & 4 \\ 68 & 1 \end{pmatrix}, \quad \tilde{D^2} = \begin{pmatrix} 4 & 58 \\ 1 & 125 \\ 0 & 26 \\ 124 & 0 \\ 125 & 11 \end{pmatrix}.$$

Since

$$f(t_1, \tilde{D}) = (0, 58),$$

$$f(t_2, \tilde{D}) = (12, 26),$$

$$f(t_3, \tilde{D}) = (30, 0),$$

$$f(t_4, \tilde{D}) = (68, 124),$$

we have $P^2(T,D) = P^2(T,\tilde{D})$, i. e. the problems $Z^2_{5\times 2}(T,D)$ and $Z^2_{5\times 2}(T,\tilde{D})$ are equivalent. Moreover, suppose

$$\lambda^1 = (0.9, 0.1),$$
$$\lambda^2 = (0.6, 0.4),$$
$$\lambda^3 = (0.1, 0.9),$$

we obtain

$$\langle \lambda^1, f(t_1,\tilde{D})\rangle = \min\{\langle \lambda^1, f(t_i,\tilde{D})\rangle : i \in N_4\} = 5.8,$$
$$\langle \lambda^2, f(t_2,\tilde{D})\rangle = \min\{\langle \lambda^2, f(t_i,\tilde{D})\rangle : i \in N_4\} = 17.6,$$
$$\langle \lambda^3, f(t_3,\tilde{D})\rangle = \min\{\langle \lambda^3, f(t_i,\tilde{D})\rangle : i \in N_4\} = 3.$$

Hence, $P^2(T,\tilde{D}) = \Xi^2(T,\tilde{D})$, i. e. the problem $Z^2_{5\times 2}(T,\tilde{D})$ is solvable by ALC.

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