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Infinitely many functional pre-complete classes of formulas in the propositional provability intuitionistic logic

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Abstract. We consider the propositional provability intuitionistic logic I^{Δ}, introduced by A.V. Kuznetsov [2]. We prove that there are infinitely many classes of formulas in the calculus of I^{Δ}, which are pre-complete with respect to functional expressibility in I^{Δ}. This result is stronger than an ealier one stated by the author in [1].

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In the present paper we extend the result established in [1] to a wider class of logics, which includes the provability-intuitionistic logic I^{Δ} itself. The last one is the logic of propositional provability-intuitionistic calculus I^{Δ} proposed and formalized by A.V. Kuznetsov [2, 3]. It is based on formulas built in an usual way from propositional variables $p, q, r, \ldots$ (may be indexed) by means of logical connectives $\&, \vee, \supset, \neg, \Delta$. The axioms of I^{Δ} consist of well-known axioms of the intuitionistic propositional calculus and three additional Δ-axioms:

$$(p \supset \Delta p), \tag{1}$$

$$((\Delta p \supset p) \supset p), \tag{2}$$

$$(((p \supset q) \supset p) \supset (\Delta q \supset p)). \tag{3}$$

Its rules of inference consist of traditional rules of *substitution*, and *modus ponens* and *the rule of necessitation* $\frac{A}{\Delta A}$. An extension L of the logic I^{Δ} is defined as usual as any set of formulas which contains the axioms of I^{Δ} and is closed with respect to the rules of inference of the calculus I^{Δ}. In the following let us denote by L any extension of I^{Δ} if other things are not stated. By equivalence of formulas A and B, denoted $A \sim B$, in the logic L we understand the formula $((A \supset B)\&(B \supset A))$.

Let us remind the notion of Δ-pseudo-Boolean algebra [2, 3] as a system of type $\mathfrak{A} =< E; \&, \vee, \supset, \neg, \Delta >$, where $< E; \&, \vee, \supset, \neg >$ is a pseudo-Boolean algebra and operation Δ satisfies the relations

$$x \leq \Delta x, \qquad (\Delta x \supset x) = x, \qquad \Delta x \leq y \vee (y \supset x).$$

These algebras serve as algebraic models for logic I^{Δ} [2, 3]. Valid formulas on the algebra $\mathfrak{A}$ are defined as usual. It is also known that the set of valid formulas on

$\mathfrak{A}$ constitutes an extension of I^{Δ} [3], it is called *the logic of the algebra* $\mathfrak{A}$, and it is denoted by $L\mathfrak{A}$.

We consider the Δ-pseudo-Boolean algebra $\mathfrak{C} =< E; \&, \vee, \supset, \neg, \Delta >$, where E is the chain of elements $0 = \tau_0 < \tau_1 < \cdots < 1$, and for any elements α and β of $\mathfrak{C}$ we assume that $\alpha\&\beta = min(\alpha, \beta)$, $\alpha \vee \beta = max(\alpha, \beta)$, $\alpha \supset \beta = 1$ when $\alpha \leq \beta$, $\alpha \supset \beta = \beta$ when $\alpha > \beta$, $\neg\alpha = \alpha \supset 0$, $\Delta\tau_i = \tau_{i+1}$ for $i = 0, 1, \ldots,$ and $\Delta 1 = 1$.

By a formula realization [5, 6] of the Δ-pseudo-Boolean algebra $\mathfrak{A}$ into the proof (provability) logic L we undestand a mapping f from the algebra $\mathfrak{A}$ into the set of formulas such that if we examine the formulas up to equivalent ones, then f is an isomorphism between the algebra $\mathfrak{A}$ and some subalgebra of the Lindenbaum algebra of the logic L. Let us build a formula realization of the algebra $\mathfrak{C}$ into the logic I^{Δ}. So, we have first of all to map each element of $\mathfrak{C}$ into the set of formulas. Consider the mapping f is as follows:

$$f(0) = 0, \qquad f(1) = 1,$$

$$f(\tau_i) = \Delta^i 0, \qquad i = 1, 2, \ldots$$

Let us prove the following

Lemma 1. *For any elements β and γ of the algebra $\mathfrak{C}$ the next deductions in the I^{Δ} logic take place*

$$\vdash (f(\beta\&\gamma) \sim (f(\beta)\&f(\gamma))), \tag{4}$$
$$\vdash (f(\beta \vee \gamma) \sim (f(\beta) \vee f(\gamma))), \tag{5}$$
$$\vdash (f(\beta \supset \gamma) \sim (f(\beta) \supset f(\gamma))), \tag{6}$$
$$\vdash (f(\neg\beta) \sim \neg f(\beta)), \tag{7}$$
$$\vdash (f(\Delta\beta) \sim \Delta f(\beta)). \tag{8}$$

Proof. Let us consider arbitrary elements β and γ of the algebra $\mathfrak{C}$. Let prove first the relation (4). If $\beta = 1$ or $\gamma = 1$ then the statement (4) is obvious. Let $\beta \neq 1$ and $\gamma \neq 1$. Since elements β and γ are arbitrary we can consider that $\beta = \tau_i$ and $\gamma = \tau_j$, where $i < j$. Then $\beta\&\gamma = \tau_i\&\tau_j = \tau_i$. So,

$$f(\beta\&\gamma) = f(\beta) = f(\tau_i) = \Delta^i 0. \tag{9}$$

The axiom (1) admits the following generalization

$$\vdash p \supset \Delta^k p, \text{ where } k = 0, 1, 2, \ldots \tag{10}$$

Taking into consideration the last relation (10) we have the following sequence of equalities

$$(f(\beta)\&f(\gamma)) = (f(\tau_i)\&f(\tau_j)) = \Delta^i 0\&\Delta^j 0 = \Delta^i 0. \tag{11}$$

So the relation (4) follows from (10) and (11).

The proof of the statement (5) is analogous to the proof of the deduction (4).

Let us prove the relation (6). We will consider two possible cases for elements β and γ, when a) $\beta \leq \gamma$, and b) $\beta > \gamma$.

Let $\beta \leq \gamma$. Then obviously $\beta \supset \gamma = 1$, and $f(\beta \supset \gamma) = f(1) = 1 = (p \supset p)$. If $\beta \neq 1$ and $\gamma \neq 1$, then we can consider as above that there exist i and j, where $i, j = 0, 1, 2, \ldots$ and $i \leq j$, such that $\beta = \Delta^i 0$ and $\gamma = \Delta^j 0$. Thus we have that $f(\beta) = \Delta^i 0$, $f(\gamma) = \Delta^j 0$, $(f(\beta) \supset f(\gamma)) = \Delta^i 0 \supset \Delta^j 0 = 1$.

Now let $\beta > \gamma$. Then $\beta \supset \gamma = \gamma$, and $f(\beta \supset \gamma) = f(\gamma)$. Obviously, $\gamma \neq 1$. So, there exists an integer positive j such that $\gamma = \Delta^j 0$, and thus $f(\beta \supset \gamma) = \Delta^j 0$. Now, let us evaluate $f(\beta) \supset f(\gamma)$. If $\beta = 1$, then, obviously, $f(\beta) \supset f(\gamma) = 1 \supset \Delta^j 0 = \Delta^j 0$. Suppose $\beta \neq 1$. Then there exists an $i > j$ such that $\beta = \Delta^i 0$, and $f(\beta) \supset f(\gamma) = \Delta^i 0 \supset \Delta^j 0 = \Delta^j 0$.

Now let us to prove the relation (7). If $\beta = 1$ then $f(\beta) = 1$, $\neg\beta = 0$, $\neg f(\beta) = \neg 1 = 0$. Suppose $\beta \neq 1$. Then there is an i such that $\beta = \Delta^i 0$. Obviously, $\neg\beta = 0$, and $\neg f(\beta) = \neg\Delta^i 0 = 0$.

Finally, let us look at the last statement (8) of the lemma. If $\beta = 1$ then (8) follows obviously. Suppose $\beta \neq 1$. Then there exists an i such that $\beta = \Delta^i 0$, and $f(\Delta\beta) = f(\Delta\Delta^i 0) = f(\Delta^{i+1} 0) = \Delta^{i+1} 0$, $\Delta f(\beta) = \Delta f(\Delta^i 0) = \Delta\Delta^i 0 = \Delta^{i+1} 0$. Comparing the last two sequences of statements we conclude (8). Lemma is proved.

Next lemma is a generalization of the previous lemma.

Lemma 2. *For any formula $F(p_1, \ldots, p_n)$ of the provability-intuitionistic logic I^Δ and for any elements $\beta_1, \ldots, \beta_n$ of the algebra $\mathfrak{C}$ the following relation is true in I^Δ*

$$\vdash f(F[p_1/\beta_1, \ldots, p_n/\beta_n]) \sim F[p_1/f(\beta_1), \ldots, p_n/f(\beta_n)].$$

The proof can be easily done by induction over the structure of the formula F and using the relations proved in Lemma 1.

On the basis of Lemma 2 we conclude that examining formulas up to equivalent ones in the logic I^Δ the mapping f is an isomorphism between the algebra $\mathfrak{C}$ and some subalgebra of the Lindenbaum's algebra of the logic I^Δ. This fact means that f is a formula realization of the algebra $\mathfrak{C}$ into the logic I^Δ. We see from the definition of this formula realization that it puts into correspondence only unary formulas to elements of $\mathfrak{C}$. So, we get next theorem as a consequence from Lemma 1 and Lemma 2.

Theorem 1. *The mapping f defined above is a formula realization of the algebra $\mathfrak{C}$ into the provability-intuitionistic logic I^Δ.*

Next lemma ilustrates a usefull property of the formula realization f defined above.

Lemma 3. *The formula realization f of the algebra $\mathfrak{C}$ into the logic I^Δ puts into correspondence to any element β of $\mathfrak{C}$ such unary formula $f(\beta)$ that the equality holds*

$$f(\beta)[\gamma] = \beta. \tag{12}$$

Proof. Really, let element β be someone from $\mathfrak{C}$. Then, obviously, it is either equal to the unit 1 of the algebra, or there exists an index k such that $\beta = \tau_k$. Recalling that $f(\tau_k) = \Delta^k(p\&\neg p)$ we get that the result of substitution $\Delta^k(p\&\neg p)[\gamma]$ does not depend on the element γ, and, moreover,

$$\Delta^k(p\&\neg p)[\gamma] = \tau_k.$$

The last relation together with the fact that element β is an arbitrary element of $\mathfrak{C}$ ensure us the validness of (12) from the lemma. The lemma is proved.

Definitions regarding expressibility in logics were proposed by A.V. Kuznetsov [7, 8, 9]. A system of formulas Σ is said to be *complete (with respect to functional expessibility) in the logic* L if any formula of the language of logic L can be obtained from variables and formulas of Σ applying a finite number of times the following two rules: the weak rule of substitution and the rule of replacement by equivalent formula in L. The system of formulas Σ is called *pre-complete (with respect to functional expressibility) in the logic* L if it is incomplete in L, and for any formula F, which is not expressible in L via Σ the system $\Sigma \cup \{F\}$ is complete in L. They say the formula $F(p_1, \dots, p_n)$ *conserves on the algebra* $\mathfrak{A}$ *the predicate* $R(x_1, \dots, x_m)$ if for any elements $\alpha_{ij} \in \mathfrak{A}$ $(i = 1, \dots, m; j = 1, \dots, n)$ the relations

$$R[\alpha_{11}, \dots, \alpha_{m1}], \dots, R[\alpha_{1n}, \dots, \alpha_{mn}]$$

imply the truth of the predicate

$$R[F[\alpha_{11}, \dots, \alpha_{1n}], \dots, F[\alpha_{m1}, \dots, \alpha_{mn}]].$$

In the following let L be any extension of the provability-intuitionistic logic I^{Δ}, which satisfies the relation $I^{\Delta} \subseteq L \subseteq L\mathfrak{C}$. Let us denote by K_i the class of all formulas that preserves on the algebra $\mathfrak{C}$ the predicate $x \leq \tau_i$ $(i = 0, 1, \dots)$.

Lemma 4. *The classes $K_0, K_1, \dots$ are two by two distinct with respect to set inclusion.*

Really, suppose $r < s$ and let us consider classes K_r and K_s. Can be checked the relation

$$(\Delta^{r+1}0 \supset p) \in K_r \setminus K_s, \quad \Delta^s 0 \in K_s \setminus K_r.$$

So, the classes $K_0, K_1, \dots$ are two by two distinct with respect to the set inclusion.

Lemma 5. *Let f be the formula realization of the algebra $\mathfrak{C}$ into the logic I^{Δ} that was defined ealier. Then for any element β of this algebra and for any $j = 0, 1, 2, \dots$ the formula $f(\beta)$ belongs to the class K_j if and only if the folowing relation holds*

$$\beta \leq \Delta^j 0. \tag{13}$$

Proof. Let element β satisfy the relation (13) from the lemma. We have to show that the formula $f(\beta)$ conserves the relation R_j on the algebra $\mathfrak{C}$. Let an arbitrary element $\gamma \in \mathfrak{C}$ conserve the predicate R_j on $\mathfrak{C}$, i.e. $R_j(\gamma)$ holds. Then, using (13) and the equality $f(\beta)[\gamma] = \beta$ already proved ealier in Lemma 3, we obtain the relation $f(\beta)[\gamma] \leq \Delta^j 0$, i.e. the relation $R_j(f(\beta)[\gamma]$. So, we get that $f(\beta) \in K_j$.

Conversely, let element β do not satisfy the condition (13). Then, since elements of $\mathfrak{C}$ form a chain, we get

$$\beta > \Delta^j 0.$$

After that, using again the above mentioned equality (12) we get the inequality

$$f(\beta)[\gamma] > \Delta^j 0,$$

which is equivalent to the fact that $R_j(f(\beta)[\gamma])$ is false. Subsequently, the formula $f(\beta)$ does not belong to the class K_j. The lemma is proved.

Lemma 6. *Let L be any logic such that $I^{\Delta} \subseteq L \subseteq L\mathfrak{C}$. Then the classes of formulas $K_0, K_1, \ldots$ are pre-complete with respect to expressibility in the logic L.*

Really, according to Lemma 4, no one of these classes is complete with respect to expressibility in the logic L. Let us prove that for any $j = 0, 1, \ldots$ the class K_j is pre-complete in L. Let $B(p_1, \ldots, p_n)$ be any formula which does not belong to the class K_j. Then, according to the definition of the class K_j there exist elements $\beta_1, \ldots, \beta_n$ of the algebra $\mathfrak{C}$ such that for any $i = 1, \ldots, n$ we have

$$\beta_i \leq \Delta^j 0,$$

but

$$B[\beta_1, \ldots, \beta_n] \leq \Delta^j 0$$

is false. Since all elements of $\mathfrak{C}$ form a chain we get the strict inequality on the algebra $\mathfrak{C}$

$$\Delta^j 0 < B[\beta_1, \ldots, \beta_n],$$

which implies the relation

$$\Delta^{j+1} 0 \leq B[\beta_1, \ldots, \beta_n].$$

The last statement implies also

$$(\Delta^{j+1} 0 \supset B[\beta_1, \ldots, \beta_n]) = 1.$$

In view of the above defined formula realization f of the algebra $\mathfrak{C}$ in the logic I^{Δ}, the last formula conducts us to the relation

$$\vdash f(\Delta^{j+1} 0 \supset B[\beta_1, \ldots, \beta_n]) \sim f(1).$$

Applying Lemma 2 to the left part of the above equivalence, we get that

$$\vdash (f(\Delta^{j+1} 0) \supset f(B[\beta_1, \ldots, \beta_n])) \sim f(1).$$

Reminding ourselves that $f(\Delta^{j+1}0)$ is the formula $\Delta^{j+1}0$ and $f(1)$ is 1, and applying once again Lemma 2 to formula B, we get the deduction

$$\vdash (\Delta^{j+1}0 \supset B[f(\beta_1), \dots, f(\beta_n)]) \sim 1.$$

The left hand side of the last equivalence can be represented as

$$\vdash (\Delta^{j+1}0 \supset \pi)[\pi/B[f(\beta_1), \dots, f(\beta_n)]] \sim 1.$$

Let us note that the formula $(\Delta^{j+1}0 \supset \pi)$ belongs to the class K_j. Apart from this, since elements β_i, when $i = 1, \dots, n$, satisfy the condition $\beta_i \leq \Delta^{j+1}0$, the formulas $f(\beta_i)$ also belong, according to Lemma 5, to the class K_j. That is why the last deduction shows that the formula 1 is expressible in the logic I^Δ by means of the formula B and of the formulas of the class K_j. We shall show that any formula F is expressible in I^Δ via system $K_j \cup \{1\}$. Let F do not contain the variable π. Then it is sufficient to take the formula $F\&\pi$, which belongs to the class K_j, and to use the following fact

$$F \sim (F\&\pi)[\pi/1].$$

Thus it is proved that the system $K_j \cup \{B\}$ is complete in I^Δ, and the more so, it is also complete in the logic L. So, lemma is proved.

Now we can state the folowing results.

Theorem 2. *Let L be any logic such that $I^\Delta \subseteq L \subseteq L\mathfrak{C}$. The classes $K_0, K_1, \dots$ constitute a numerable collection of distinct two by two pre-complete in the logic L classes of formulas.*

In 1956 A.V. Kuznetsov [10, 11] have proved that for any finite-valued logic L there exists an algorithm which permits to recognize whether a system of formulas is complete with respect to functional expressibility in L. He have shown that there exists theoretically a finite collection of pre-complete classes of formulas in that logic. Unfortunately, the proposed algorithm is very computationally inefficient. Next theorem establishes that even such algorithm is impossible for any logic L that satisfies the condition $I^\Delta \subseteq L \subseteq L\mathfrak{C}$.

Theorem 3. *Let L be any logic such that $I^\Delta \subseteq L \subseteq L\mathfrak{C}$. The traditional formulation of the theorem of completeness with respect to functional expressibility in terms of a finite collection of pre-complete classes of formulas in the logic L does not exist.*

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