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Biharmonic curves in Cartan-Vranceanu (2n+1)-dimensional spaces

Dorel Fetcu *

Abstract. Biharmonic curves in Cartan-Vranceanu spaces of dimension 2n+1 are characterized and an example of such curve is given.

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1 Preliminaries

First we should recall some notions and results related to the biharmonic maps between Riemannian manifolds, as they are presented in [6] and in [7].

Harmonic maps $f:(M,g)\to(N,h)$ between a compact Riemannian manifold, (M,g), and a Riemannian manifold, (N,h), are the critical points of the energy functional $E(f)=\frac{1}{2}\int_M|df|^2\nu_g$ and it is proved (in [4]) that the corresponding Euler-Lagrange equation is $\tau(f)=trace\nabla df$, where $\tau(f)$ is called the tension field of f. If the manifold M is not compact f is said to be harmonic if $\tau(f)=0$. The critical points of the bienergy functional $E_2(f)=\frac{1}{2}\int_M|\tau(f)|^2\nu_g$ are called biharmonic maps. In [6] the Euler-Lagrange equation for E_2 is given

$$\tau_2(f)=-\Delta\tau(f)-traceR^N(df,\tau(f))df=0,$$

where $R^N(X,Y)=\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}$. The equation $\tau_2(f)=0$ is called the biharmonic equation. Note that the harmonic maps are also biharmonic. Then the main interest is to find the non-harmonic biharmonic maps, which are called proper biharmonic maps.

2 Cartan-Vranceanu spaces

Let us consider the following two-parameter family of Riemannian metrics, called the Cartan-Vranceanu metrics,

$$ds^2_{l,m}=\sum_{i=1}^{n}\frac{dx_i^2+dy_i^2}{[1+m(x_i^2+y_i^2)]^2}+\Big[dz+\frac{l}{2}\sum_{i=1}^{n}\frac{y_idx_i-x_idy_i}{1+m(x_i^2+y_i^2)}\Big]^2 \qquad (1)$$

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defined on (2n+1)-dimensional manifold M, where $M=\mathbb{R}^{2n+1}$ if $m\geq 0$, and

$$M=\left\{(x_1,y_1,x_2,y_2,...,x_n,y_n,z)\in\mathbb{R}^{2n+1}|x_i^2+y_i^2\lneqq-\frac{1}{m},i=\overline{1,n}\right\}$$

if $m\lneqq 0$. The biharmonic curves in 3-dimensional Cartan-Vranceanu spaces are characterized in [2] and, moreover, their explicit parametrizations is given in the cited paper. For the 3-dimensional case another results are obtained if $m=0$, $l\neq 0$ and if $l=1$, $m\neq 0$. Thus, if $m=0$, $l\neq 0$ then $(M,ds^2_{l,m})$ is the Heisenberg group, $\mathbb{H}_3$, and the biharmonic curves in this space are studied in [1]. If $l=1$, $m\neq 0$ the biharmonic curves are studied in [3]. In (2n+1)-dimensional case, if $m=0$, $l\neq 0$ then $(M,ds^2_{l,m})$ is the generalized Heisenberg group, $\mathbb{H}_{2n+1}$, and a study of biharmonic curves in this space was given in [5].

In the following let us consider a (2n+1)-dimensional Cartan-Vranceanu space $(M,ds^2_{l,m})$, with $m\neq 0$, and the elements of M are of the form $X=(x_1,y_1,x_2,y_2,...$ $...,x_n,y_n,z)$. We can define a global orthonormal frame field on M by

$$E_{2i-1}=F_i\frac{\partial}{\partial x_i}-\frac{ly_i}{2}\frac{\partial}{\partial z},\quad E_{2i}=F_i\frac{\partial}{\partial y_i}+\frac{lx_i}{2}\frac{\partial}{\partial z},\quad E_{2n+1}=\frac{\partial}{\partial z},$$

for $i=\overline{1,n}$, where $F_i=1+m(x_i^2+y_i^2)$. The Levi-Civita connection of the metric $ds^2_{l,m}$ is given by,

$$\begin{cases}\nabla_{E_{2i-1}}E_{2j-1}=2\delta_{ij}my_iE_{2i},\\ \nabla_{E_{2i}}E_{2j}=2\delta_{ij}mx_iE_{2i-1},\\ \nabla_{E_{2i-1}}E_{2j}=\delta_{ij}(-2my_iE_{2i-1}+\frac{l}{2}E_{2n+1}),\\ \nabla_{E_{2i}}E_{2j-1}=\delta_{ij}(-2mx_iE_{2i-1}-\frac{l}{2}E_{2n+1}),\\ \nabla_{E_{2n+1}}E_{2i-1}=\nabla_{E_{2i-1}}E_{2n+1}=-\frac{l}{2}E_{2i},\\ \nabla_{E_{2n+1}}E_{2i}=\nabla_{E_{2i}}E_{2n+1}=\frac{l}{2}E_{2i-1},\\ \nabla_{E_{2n+1}}E_{2n+1}=0,\end{cases}\tag{2}$$

for $i,j=\overline{1,n}$. Also, one obtains

$$\begin{cases}[E_{2i-1},E_{2j-1}]=0,[E_{2i},E_{2j}]=0,\\ [E_{2i-1},E_{2n+1}]=0,[E_{2i},E_{2n+1}]=0,\\ [E_{2i-1},E_{2j}]=\delta_{ij}(2mx_iE_{2i}-2my_iE_{2i-1}+lE_{2n+1}),\end{cases}$$

The curvature tensor field of ∇ is

$$R(X,Y)Z=\nabla_X\nabla_YZ-\nabla_Y\nabla_XZ-\nabla_{[X,Y]}Z,$$

and the Riemann–Christoffel tensor field is

$$R(X,Y,Z,W)=g(R(X,Y)W,Z),$$

where $X,Y,Z,W\in\chi(\mathbb{R}^{2n+1})$. We will use the notations

$$R_{abc}=R(E_a,E_b)E_c,\quad R_{abcd}=R(E_a,E_b,E_c,E_d),$$

where $a, b, c, d = \overline{1, 2n+1}$. Then the non-zero components of the curvature tensor field and of the Riemann-Christoffel tensor field are, respectively

$$\begin{cases} R_{(2i-1)(2j-1)(2k)} = -\frac{l^2}{4}\delta_{jk}E_{2i} + \frac{l^2}{4}\delta_{ik}E_{2j}, \\ R_{(2i-1)(2j)(2j-1)} = \frac{l^2}{4}E_{2i}, \ i \neq j, \\ R_{(2i-1)(2i)(2k-1)} = \delta_{ik}(\frac{l^2}{4} - 4m)E_{2i} + \frac{l^2}{2}E_{2k}, \\ R_{(2i-1)(2j)(2i)} = -\frac{l^2}{4}E_{2j-1}, \ i \neq j, \\ R_{(2i-1)(2i)(2k)} = -\delta_{ik}(\frac{l^2}{4} - 4m)E_{2i-1} - \frac{l^2}{2}E_{2k-1}, \\ R_{(2i-1)(2n+1)(2i-1)} = -\frac{l^2}{4}E_{2n+1}, \\ R_{(2i-1)(2n+1)(2n+1)} = \frac{l^2}{4}E_{2i-1}, \\ R_{(2i)(2j)(2k-1)} = -\frac{l^2}{4}\delta_{jk}E_{2i-1} + \frac{l^2}{4}\delta_{ik}E_{2j-1}, \\ R_{(2i)(2n+1)(2i)} = -\frac{l^2}{4}E_{2n+1}, \\ R_{(2i)(2n+1)(2n+1)} = \frac{l^2}{4}E_{2i}, \end{cases} \tag{3}$$

$$\begin{cases} R_{(2i-1)(2j-1)(2i)(2j)} = -\frac{l^2}{4}, \ i \neq j, \\ R_{(2i-1)(2j)(2j-1)(2i)} = -\frac{l^2}{4}, \ i \neq j, \\ R_{(2i)(2i-1)(2i-1)(2i)} = \frac{3l^2}{4} - 4m, \\ R_{(2i)(2i-1)(2j-1)(2j)} = \frac{l^2}{2}, \ i \neq j, \\ R_{(2n+1)(2i-1)(2i-1)(2n+1)} = -\frac{l^2}{4}, \\ R_{(2n+1)(2i)(2i)(2n+1)} = -\frac{l^2}{4}, \end{cases} \tag{4}$$

for $i, j, k = \overline{1, n}$.

3 Biharmonic curves in (2n+1)-dimensional Cartan–Vranceanu spaces

Let $\gamma : I \subset \mathbb{R} \to (M, ds^2_{l,m})$ be a non-inflexionar curve, parametrized by its arc length. Let $\{T, N_1, ..., N_{2n}\}$ be the Frenet frame in $(M, ds^2_{l,m})$ defined along γ, where $T = \gamma'$ is the unit tangent vector field of γ, N_1 is the unit normal vector field of γ, with the same direction as $\nabla_T T$ and the vectors $N_1, ..., N_{2n}$ are the unit vectors obtained from the following Frenet equations for γ.

$$\begin{cases} \nabla_T T & = & \chi_1 N_1 \\ \nabla_T N_1 & = & -\chi_1 T + \chi_2 N_2 \\ \dots & \dots & \dots \\ \nabla_T N_{2n-1} & = & -\chi_{2n-2} N_{2n-2} + \chi_{2n-1} N_{2n} \\ \nabla_T N_{2n} & = & -\chi_{2n-1} N_{2n-1} \end{cases} \tag{5}$$

where $\chi_1 = \|\nabla_T T\| = \|\tau(\gamma)\|$, and $\chi_2 = \chi_2(s), ..., \chi_{2n} = \chi_{2n}(s)$ are real valued functions, named the curvatures of γ, where s is the arc length of γ.

In [2] is proved the following result

Proposition 3.1. *Let $\gamma : I \subset \mathbb{R} \to (N^n, h)$, $n \geq 2$, be a curve parametrized by arc length from an open interval of $\mathbb{R}$ into a Riemannian manifold (N, g). Then γ is* *biharmonic if and only if*

$$\begin{cases} \chi_1\chi_1' = 0 \\ \chi_1'' - \chi_1^3 - \chi_1\chi_2^2 + \chi_1 R(T,N_1,T,N_1) = 0 \\ 2\chi_1'\chi_2 + \chi_1\chi_2' + \chi_1 R(T,N_1,T,N_2) = 0 \\ \chi_1\chi_2\chi_3 + \chi_1 R(T,N_1,T,N_3) = 0 \\ \chi_1 R(T,N_1,T,N_k) = 0,\ k = \overline{4,n}. \end{cases} \tag{6}$$

Using Proposition 3.1 and equations (4), after a straightforward computation, one obtains

Theorem 3.2. *Let $\gamma : I \subset \mathbb{R} \to (M, ds^2_{l,m})$ be a curve parametrized by its arc length. Then γ is a proper biharmonic curve if and only if*

$$\begin{cases} \chi_1 \in \mathbb{R}\setminus\{0\}, \\ \chi_1^2 + \chi_2^2 = -\eta_1, \\ \chi_2' = \eta_2, \\ \chi_2\chi_3 = \eta_3, \\ \eta_k = 0,\ k = \overline{4,2n}, \end{cases} \tag{7}$$

with η_k, $k = \overline{1,2n}$, given by

$$\eta_1 = -R(T,N_1,T,N_1) = -4m\sum_{i=1}^{n}(T_{2i-1}N_1^{2i} - T_{2i}N_1^{2i-1})^2 + \tag{8}$$

$$+\frac{3l^2}{4}\Big[\sum_{i=1}^{n}(T_{2i}N_1^{2i-1} - T_{2i-1}N_1^{2i})\Big]^2 - \frac{l^2}{4}(T_{2n+1}^2 + (N_1^{2n+1})^2)$$

and

$$\eta_k = -R(T,N_1,T,N_k) = \tag{9}$$

$$= -4m\sum_{i=1}^{n}(T_{2i-1}N_1^{2i} - T_{2i}N_1^{2i-1})(T_{2i-1}N_k^{2i} - T_{2i}N_k^{2i-1}) +$$

$$+\frac{3l^2}{4}\sum_{i=1}^{n}(T_{2i}N_1^{2i-1} - T_{2i-1}N_1^{2i})\sum_{i=1}^{n}(T_{2i}N_k^{2i-1} - T_{2i-1}N_k^{2i}) - \frac{l^2}{4}N_1^{2n+1}N_k^{2n+1},$$

for $k = \overline{2,2n}$, where $T = \sum_{a=1}^{2n+1} T_a E_a$ and $N_k = \sum_{a=1}^{2n+1} N_k^a E_a$.

From the second equation of (7) follows immediately

Corollary 3.3. *If $l = 0$ and $m \leq 0$ then all biharmonic curves of $(M, ds^2_{l,m})$ are geodesics.*

In order to find a proper biharmonic curve $\gamma : I \subset \mathbb{R} \to (M, ds^2_{l,m})$, $\gamma = (x_1(s), y_1(s), x_2(s), y_2(s), ..., x_n(s), y_n(s), z(s))$, let us suppose that the components of its tangent vector $T(s) = \gamma'(s)$ are $T_{2i-1}(s) = \frac{\cos\beta_i(s)\sin\alpha}{\sqrt{n}}$, $T_{2i}(s) = \frac{\sin\beta_i(s)\sin\alpha}{\sqrt{n}}$, $T_{2n+1}(s) = \cos\alpha$, for $i = \overline{1,n}$, where β_i are smooth functions, s being the arc length of γ, and $\alpha \in (0, \pi)$ is a constant. Working this way is suggested by the fact that T is a unitary vector field and by the paper [2], where it is proved that, in dimension 3, the tangent vector is of this form for all proper biharmonic curves of Cartan–Vranceanu spaces.

The covariant derivative of the vector field T is given by

$$\nabla_T T = \sum_{i=1}^{n} [(T'_{2i-1} - 2my_i T_{2i} T_{2i-1} + 2mx_i T_{2i}^2 + lT_{2i} T_{2n+1}) E_{2i-1} +$$

$$+ (T'_{2i} - 2mx_i T_{2i} T_{2i-1} + 2my_i T_{2i-1}^2 - lT_{2i-1} T_{2n+1}) E_{2i}] + T'_{2n+1} E_{2n+1} =$$

$$= \sum_{i=1}^{n} \frac{\sin\alpha}{\sqrt{n}} (-A_i \sin\beta_i E_{2i-1} + A_i \cos\beta_i E_{2i}),$$

where

$$A_i = \beta'_i - 2mx_i \frac{\sin\beta_i \sin\alpha}{\sqrt{n}} + 2my_i \frac{\cos\beta_i \sin\alpha}{\sqrt{n}} - l\cos\alpha.$$

Next, assume that $A_i = A$, for any $i = \overline{1, n}$ (that is the values of A_i's are the same for all indices). It follows, from the first Frenet equation, that χ_1 is given by

$$\chi_1 = \|\nabla_T T\| = |A \sin\alpha|.$$

Suppose that $A \sin\alpha \gneq 0$. Then

$$\chi_1 = A \sin\alpha \tag{10}$$

and

$$N_1 = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} (-\sin\beta_i E_{2i-1} + \cos\beta_i E_{2i}).$$

The system $T = \gamma'$ is equivalent with

$$\begin{cases} \dfrac{x'_i}{1 + m(x_i^2 + y_i^2)} = \dfrac{\cos\beta_i \sin\alpha}{\sqrt{n}}, \\ \dfrac{y'_i}{1 + m(x_i^2 + y_i^2)} = \dfrac{\sin\beta_i \sin\alpha}{\sqrt{n}}, \quad i = \overline{1, n}, \\ z' = \cos\alpha + \frac{l}{2} \dfrac{\sin\alpha}{\sqrt{n}} \sum_1^n (x_i \sin\beta_i - y_i \cos\beta_i). \end{cases} \tag{11}$$

Assume that $\beta'_i \neq 0$, for any $i = \overline{1, n}$. By derivation of (10), taking into account that χ_1 must to be constant, one obtains

$$\beta''_i = \frac{2mx_i x'_i + 2my_i y'_i}{1 + m(x_i^2 + y_i^2)} \cdot \beta'_i, \quad i = \overline{1, n}.$$

From the last equations we have

$$b_i \beta'_i = 1 + m(x_i^2 + y_i^2), \quad i = \overline{1, n},$$

where b_i are constants. If we take $b_i = b$ to be independent of i, one obtains

$$\begin{cases} x_i(s) = \dfrac{b\cos\beta_i\sin\alpha}{\sqrt{n}}, \\ y_i(s) = -\dfrac{b\sin\beta_i\sin\alpha}{\sqrt{n}}, \quad i=\overline{1,n}, \\ z(s) = (\cos\alpha)s + \dfrac{lb}{2n}(\sin^2\alpha)s. \end{cases} \tag{12}$$

Again using the facts that χ_1 is a constant and the terms A_i do not depend on i it follows that β_i' must be constants which values are the same for all indices. Hence

$$\beta_i' = C = \frac{1+m(x_i^2+y_i^2)}{b} = \frac{n+mb^2\sin^2\alpha}{bn},\ i=\overline{1,n}.$$

Thus

$$\beta_i(s) = \frac{n+mb^2\sin^2\alpha}{bn}\cdot s + d_i, \tag{13}$$

where d_i are constants.

From expressions of χ_1, T and N_1 we have, after a straightforward computation,

$$\nabla_T N_1 + \chi_1 T = \sum_{i=1}^{n} \frac{B\cos\alpha}{\sqrt{n}}(\cos\beta_i E_{2i-1} + \sin\beta_i E_{2i}) +$$

$$+\Big(\frac{l}{2}\sin\alpha + A\sin\alpha\cos\alpha\Big)E_{2n+1},$$

where $B = (-\frac{1}{b} + \frac{mb}{n}\sin^2\alpha + l\cos\alpha)\cos\alpha - \frac{l}{2} = -A\cos\alpha - \frac{l}{2}$. From the second Frenet equation we have $\chi_2^2 = \|\nabla_T N_1 + \chi_1 T\|^2 = B^2\cos^2\alpha + (\frac{l}{2}\sin\alpha + A\sin\alpha\cos\alpha)^2$. It follows that χ_2 is a constant. Now, since $-\eta_1 = (\frac{4m}{n} - l^2)\sin^2\alpha + \frac{l^2}{4}$ and $\eta_k = 0$, $k \geq 2$, the curve γ is biharmonic and non-geodesic if and only if the second and the fourth equations of (7) hold. From the second equation, after a straightforward computation, one obtains that

$$A^2 + Al\cos\alpha - (\frac{4m}{n} - l^2)\sin^2\alpha = 0. \tag{14}$$

Assume that $m \gneq 0$. If $l^2 + (\frac{16m}{n} - 5l^2)\sin^2\alpha \gneq 0$ then, solving equation (14), one obtains

$$A = \frac{-l\cos\alpha \pm \sqrt{l^2 + (\frac{16m}{n} - 5l^2)\sin^2\alpha}}{2},$$

and

$$b = \frac{-\Big(nl \pm \sqrt{l^2 + (\frac{16m}{n} - 5l^2)\sin^2\alpha}\Big)}{4m\sin^2\alpha} \pm \tag{15}$$

$$\pm\frac{\sqrt{\Big(nl\cos\alpha \pm \sqrt{l^2 + (\frac{16m}{n} - 5l^2)\sin^2\alpha}\Big)^2 + 16nm\sin^2\alpha}}{4m\sin^2\alpha}.$$

Since $\chi_1 \neq 0$ one obtains $A \neq 0$. Then $\frac{4m}{n} - l^2 \neq 0$.

For the values founded for b, from the third Frenet equation, it follows that $\chi_3 = 0$, and then the fourth equation of (7) holds.

We obtained

Proposition 3.4 *Let $(M, ds^2_{l,m})$ be a (2n+1)-dimensional Cartan-Vranceanu space such that $m \gneq 0$ and $\frac{4m}{n} - l^2 \neq 0$. Let $\gamma : I \subset \mathbb{R} \to (M, ds^2_{l,m})$,*

$$\gamma = (x_1(s), y_1(s), x_2(s), y_2(s), ..., x_n(s), y_n(s), z(s)),$$

be a curve parametrized by its arc length, given by

$$\begin{cases} x_i(s) = \dfrac{b\cos\beta_i \sin\alpha}{\sqrt{n}}, \\ y_i(s) = -\dfrac{b\sin\beta_i \sin\alpha}{\sqrt{n}}, \quad i = \overline{1,n}, \\ z(s) = (\cos\alpha)s + \dfrac{lb}{2n}(\sin\alpha)^2 s. \end{cases}$$

where $\alpha \in (0, \pi)$, β_i are given by (13), b is given by (15) and $l^2 + (\frac{16m}{n} - 5l^2)\sin^2\alpha \geq 0$. Then γ is a biharmonic curve and it is not a geodesic.

References

[1] Caddeo R., Oniciuc C., Piu P. *Explicit formulas for non-geodesic biharmonic curves of the Heisenberg group.* E-version, arXiv:math.DG 0311221 v1, 2003.

[2] Caddeo R., Montaldo S., Oniciuc C., Piu P. *The clasification of biharmonic curves of Cartan-Vranceanu 3-dimensional spaces.* E-version, arXiv:math.DG 0510435 v1, 2005.

[3] Cho J.T., Inoguchi J., Lee J.-E. *Biharmonic curves in 3-dimensional Sasakian space form.* Preprint, 2004.

[4] Eells J., Lemaire L. *Selected topics in harmonic maps.* Am. Math. Soc, Conf. Board Math. Sci., 1983, **50**, p. 83.

[5] Fetcu D. *Biharmonic curves in the generalized Heisenberg group.* Beiträge Algebra Geom, 2005, **46**,N 2, p. 513–521.

[6] Jiang G.Y. *2-harmonic maps and their first and second variational formulas.* Chinese Ann. Math. Ser. A7, 1986, **4**, p. 389–402.

[7] Montaldo S., Oniciuc C. *A short survey on biharmonic maps between riemannian manifolds.* arXiv:math.DG/0510636 v1, 2005.

[8] Oniciuc C. *Tangency and harmonicity properties.* Geometry Balkan Press, E-version, 2003, p. 87 (in Roumanian).

[9] Urakawa H. *Calculus of Variation and Harmonic Maps.* American Mathematical Society, Providence, Rhode Island, Translations of Mathematical Monographs, 1993, viii, p. 251.

[10] Vranceanu G. *Leçons de géométrie différentielle.* Ed. Acad. Rep. Pop. Roum., 1957, **I**, Bucharest.

"Gh. Asachi" Technical University of Iasi
Romania

E-mail: *dfetcu@math.tuiasi.ro*

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