BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(53), 2007, Pages 46–58
ISSN 1024–7696

Existence and uniqueness results for a class of nonlinear differential problems

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Abstract. We investigate the existence and uniqueness of the strong and weak solutions to a nonlinear differential system with boundary conditions and initial data.

Mathematics subject classification: 34G20, 35L50, 39A10, 47H05.

Keywords and phrases: Nonlinear differential system, boundary condition, maximal monotone operator, Cauchy problem, strong solution, weak solution.

1 Introduction

Let H be a real Hilbert space with the scalar product $< \cdot, \cdot >$ and the associated norm $\| \cdot \|$. We shall investigate the nonlinear differential system

$$\text{(S)} \qquad \begin{cases} \dfrac{du_n}{dt}(t) + \dfrac{v_n(t) - v_{n-1}(t)}{h} + c_n A(u_n(t)) \ni f_n(t), \\ \\ \dfrac{dv_n}{dt}(t) + \dfrac{u_{n+1}(t) - u_n(t)}{h} + d_n B(v_n(t)) \ni g_n(t), \end{cases}$$

$$n = 1, 2, \ldots, \quad 0 < t < T, \ \text{in} \ H,$$

with the boundary condition

$$\text{(BC)} \qquad v_0(t) \in -\alpha(u_1(t)), \ \ 0 < t < T$$

and the initial data

$$\text{(IC)} \qquad u_n(0) = u_{n0}, \ \ v_n(0) = v_{n0}, \ \ n = 1, 2, \ldots,$$

where $c_n > 0$, $d_n > 0$, $\forall\, n = 1, 2, \ldots$, $h > 0$, and A, B, α are multivalued operators in H which satisfy some assumptions.

This problem is a discrete version with respect to x (with $H = I\!R$) of the hyperbolic problem

$$(\mathrm{S})_0 \qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{\partial v}{\partial x}(t,x) + A(u(t,x)) \ni f(t,x), \\[2ex] \dfrac{\partial v}{\partial t}(t,x) + \dfrac{\partial u}{\partial x}(t,x) + B(v(t,x)) \ni g(t,x), \end{cases}$$

$$x > 0, \;\; t > 0, \;\; \text{in } I\!R$$

with the boundary condition

$$(\mathrm{BC})_0 \qquad v(t,0) \in -\overline{\alpha}(u(t,0)), \;\; t > 0$$

and the initial data

$$(\mathrm{IC})_0 \qquad u(0,x) = u_0(x), \;\; v(0,x) = v_0(x), \;\; x > 0.$$

The above problem $(\mathrm{S})_0$+$(\mathrm{BC})_0$+$(\mathrm{IC})_0$ has applications in electrotechnics (the propagation phenomena in electrical networks) and mechanics (the variable flow of a fluid) [5, 6, 13]. The system $(\mathrm{S})_0$ for $x \in (0,1)$ or $x \in (0,\infty)$, subject to various boundary conditions has been studied by many authors: V. Barbu, V. Iftimie, G. Moroşanu, R. Luca, etc (see the papers [2, 3, 7, 9, 12]). The problem (S)+(BC)+(IC) is a generalization of the problem studied in [10], where the operator $\alpha : H \to H$ is everywhere defined and single-valued. The methods used in this paper to prove the maximal monotonicity of the operators $\mathcal{A}$ and $\mathcal{A}+\mathcal{B}$ (see below) are different than those used in [10]. We also mention the papers [10, 11] where we investigated the system (S) with $n = 1, 2, \ldots, N$ $(N \geq 1)$ with some boundary conditions and initial data. Although the proposed problem appeared by discretization of $(\mathrm{S})_0$+$(\mathrm{BC})_0$+$(\mathrm{IC})_0$, our problem also covers some nonlinear differential systems in Hilbert spaces. For the basic concepts and results in the theory of monotone operators and nonlinear evolution equations of monotone type in Hilbert spaces we refer the reader to [1, 4, 8, 13].

We present the assumptions that we shall use in the sequel

(H1) The operators $A : D(A) \subset H \to H$, $B : D(B) \subset H \to H$ are maximal monotone, possibly multivalued, $0 \in A(0)$, $0 \in B(0)$.

(H2) The operator $\alpha : D(\alpha) \subset H \to H$ is maximal monotone, possibly multivalued, with $D(\alpha) \neq \emptyset$.

(H3) $D(A) \cap D(\alpha) \neq \emptyset$.

(H4) i) The operator α is bounded on bounded sets.

ii) $(\operatorname{int} D(\alpha)) \cap D(A) \neq \emptyset$.

(H5) The constant $h > 0$.

(H6) The constants $c_n > 0$, $d_n > 0$, $\forall\, n \geq 1$.

2 The results

We shall write our problem (S)+(BC)+(IC) as a Cauchy problem in a certain Hilbert space, and we shall apply the theory of nonlinear evolution equations of monotone type.

We consider the Hilbert *space* $X = l_h^2(H) \times l_h^2(H)$, where $l_h^2(H) = \{(u_n)_n \subset H,$ $\sum_{n=1}^{\infty} \|u_n\|^2 < \infty\}\ (= l^2(H))$, with the scalar product

$$< ((u_n)_n, (v_n)_n), ((\overline{u}_n)_n, (\overline{v}_n)_n) >_X = < (u_n)_n, (\overline{u}_n)_n >_{l_h^2(H)} +$$

$$< (v_n)_n, (\overline{v}_n)_n >_{l_h^2(H)} = \sum_{n=1}^{\infty} h < u_n, \overline{u}_n > + \sum_{n=1}^{\infty} h < v_n, \overline{v}_n > .$$

We define the *operator* $\mathcal{A} : D(\mathcal{A}) \subset X \to X$, with

$$D(\mathcal{A}) = \{((u_n)_n, (v_n)_n) \in X,\ \ u_1 \in D(\alpha)\},$$

$$\mathcal{A}((u_n)_n, (v_n)_n) = \left\{ \left(\left(\frac{v_n - v_{n-1}}{h} \right)_n, \left(\frac{u_{n+1} - u_n}{h} \right)_n \right),\ \text{ with }\ v_0 \in -\alpha(u_1) \right\},$$

and the *operator* $\mathcal{B} : D(\mathcal{B}) \subset X \to X$, with $D(\mathcal{B}) = \big\{((u_n)_n, (v_n)_n) \in X,$ $u_n \in D(A),\ \ v_n \in D(B),\ \ \forall\, n \geq 1,\ \ \{(c_n A(u_n))_n\} \subset l^2(H),\ \{(d_n B(v_n))_n\} \subset l^2(H)\big\}$, $\mathcal{B}((u_n)_n, (v_n)_n) = \big\{((c_n \gamma_n)_n, (d_n \delta_n)_n),\ \ \gamma_n \in A(u_n),\ \ \delta_n \in B(v_n),\ \ \forall\, n \geq 1\big\}$.

Theorem 1. *If the assumptions* (H2) *and* (H5) *hold, then the operator* $\mathcal{A}$ *is maximal monotone in* X.

Theorem 2. *If the assumptions* (H1), (H5) *and* (H6) *hold, then the operator* $\mathcal{B}$ *is maximal monotone in* X.

Theorem 3. *If the assumptions* (H1), (H2), (H3), [(H4)i) *or* (H4)ii)], (H5) *and* (H6) *hold, then the operator* $\mathcal{A} + \mathcal{B}$ *is maximal monotone.*

Using the operators $\mathcal{A}$ and $\mathcal{B}$ our problem (S)+(BC)+(IC) can be equivalently expressed as the following Cauchy problem in the space X

$$\text{(P)} \qquad \begin{cases} \dfrac{dU}{dt}(t) + \mathcal{A}(U(t)) + \mathcal{B}(U(t)) \ni F(t) \\ U(0) = U_0, \end{cases}$$

where $U = ((u_n)_n, (v_n)_n)$, $\quad U_0 = ((u_{n0})_n, (v_{n0})_n)$, $\quad F = ((f_n)_n, (g_n)_n)$.

The main result for our problem (S)+(BC)+(IC)$\Leftrightarrow$(P) is

Theorem 4. *Assume that the assumptions* (H1), (H2), (H3), [(H4)i) *or* (H4)ii)], (H5) *and* (H6) *hold. If* $u_{10} \in D(A) \cap D(\alpha)$, $u_{n0} \in D(A)$, $\forall\, n \geq 2$, $v_{n0} \in D(\mathcal{B})$, $\forall\, n \geq 1$ *with* $(u_{n0})_n$, $(v_{n0})_n \in l^2(H)$, $\{(c_n A(u_{n0}))_n\}$, $\{(d_n B(v_{n0}))_n\} \subset l^2(H)$, *(that is* $U_0 \in D(\mathcal{A}) \cap D(\mathcal{B})$*), and* $(f_n)_n$, $(g_n)_n \in W^{1,1}(0,T; l^2(H))$, *then there exist unique functions* u_n, v_n, $n \geq 1$, $(u_n)_n$, $(v_n)_n \in W^{1,\infty}(0,T; l^2(H))$, $u_1(t) \in D(A) \cap D(\alpha)$, $u_n(t) \in D(A)$, $\forall\, n \geq 2$, $v_n(t) \in D(\mathcal{B})$, $\forall\, n \geq 1$, $\forall\, t \in [0,T]$, *that verify the system* (S) *for all* $t \in [0,T)$, *the boundary condition* (BC) *for all* $t \in [0,T)$ *and the initial data* (IC)*. Moreover* u_n, v_n, $n \geq 1$ *are everywhere differentiable from right in the topology of* H *and*

$$\frac{d^+ u_n}{dt} = \left(f_n - c_n A(u_n) - \frac{v_n - v_{n-1}}{h} \right)^0, \;\; n \geq 1,$$

$$\frac{d^+ v_n}{dt} = \left(g_n - d_n B(v_n) - \frac{u_{n+1} - u_n}{h} \right)^0, \;\; n \geq 1, \;\; t \in [0,T),$$

$$\textit{with} \;\; v_0(t) \in -\alpha(u_1(t)), \;\; \forall\, t \in [0,T).$$

Remark. If $U_0 \in \overline{D(\mathcal{A}) \cap D(\mathcal{B})}$ and $F \in L^1(0,T;X)$ then by [1, Corollary 2.2, Chapter III] the problem (P)$\Leftrightarrow$(S)+(BC)+(IC) has a unique weak solution $U \in C([0,T];X)$, that is there exist $(F_k)_k \subset W^{1,1}(0,T;\, X)$, $F_k \to F$, as $k \to \infty$, in $L^1(0,T;X)$ and $(U_k)_k \subset W^{1,\infty}(0,T;X)$, $U_k(0) = U_0$, $U_k \to U$ as $k \to \infty$ in $C([0,T];X)$, strong solutions for the problems

$$\frac{dU_k}{dt}(t) + (\mathcal{A} + \mathcal{B})(U_k(t)) \ni F_k(t), \;\; \text{for a.a. } t \in (0,T), \;\; k = 1, 2, \ldots$$

3 The proofs

The proof of Theorem 1. The operator $\mathcal{A}$ has $D(\mathcal{A}) \neq \emptyset$ and it is well defined in X; if $((u_n)_n, (v_n)_n) \in D(\mathcal{A})$ then $\mathcal{A}((u_n)_n, (v_n)_n) \in X$. The operator $\mathcal{A}$ is monotone;

indeed

$$< Z-\overline{Z}, U-\overline{U}>_X=< \left(\frac{v_n-v_{n-1}}{h}\right)_n - \left(\frac{\overline{v}_n-\overline{v}_{n-1}}{h}\right)_n ,$$

$$(u_n-\overline{u}_n)_n >_{l_h^2(H)} + < \left(\frac{u_{n+1}-u_n}{h}\right)_n - \left(\frac{\overline{u}_{n+1}-\overline{u}_n}{h}\right)_n ,$$

$$(v_n-\overline{v}_n)_n >_{l_h^2(H)} = \sum_{n=1}^{\infty} h < \frac{v_n-v_{n-1}-\overline{v}_n+\overline{v}_{n-1}}{h},$$

$$u_n-\overline{u}_n > + \sum_{n=1}^{\infty} h < \frac{u_{n+1}-u_n-\overline{u}_{n+1}+\overline{u}_n}{h},$$

$$v_n-\overline{v}_n >= - < v_0-\overline{v}_0, u_1-\overline{u}_1 > \geq 0, \quad \forall U=((u_n)_n,(v_n)_n),$$

$$\overline{U}=((\overline{u}_n)_n,(\overline{v}_n)_n)\in D(\mathcal{A}), \quad Z\in\mathcal{A}(U), \quad \overline{Z}\in\mathcal{A}(\overline{U}),$$

$$u_1\in D(\alpha), \quad \overline{u}_1\in D(\alpha), \quad v_0\in-\alpha(u_1), \quad \overline{v}_0\in-\alpha(\overline{u}_1).$$

To prove that $\mathcal{A}$ is maximal monotone, it is sufficient (and necessary) to show that for any λ (equivalently there exists a $\lambda>0$ such that) $R(I+\lambda\mathcal{A})=X$ (see [4, Proposition 2.2]). We consider $\lambda=h$ and we shall prove that for any $Y=((x_n)_n,(y_n)_n)\in X$, the equation

$$(I+h\mathcal{A})(U)\ni Y \tag{1}$$

has a solution $U=((u_n)_n,(v_n)_n)\in D(\mathcal{A})$.

The equation (1) is equivalent to

$$\left\{\begin{array}{l} u_n+v_n-v_{n-1}=x_n \\ v_n+u_{n+1}-u_n=y_n, \ \ n=1,2,\ldots, \end{array}\right. \tag{2}$$
$$\text{with } \ v_0\in-\alpha(u_1).$$

We look for a solution for (2) in the form

$$\left\{\begin{array}{l} u_n=u_n^1+u_n^2 \\ v_n=v_n^1+v_n^2, \ \ n=1,2,\ldots, \end{array}\right.$$

where $((u_n^1)_n,(v_n^1)_n)$ is a solution to

$$\left\{\begin{array}{l} u_n^1+v_n^1-v_{n-1}^1=x_n \\ v_n^1+u_{n+1}^1-u_n^1=y_n, \ \ n=1,2,\ldots, \ \text{ in } \ H \end{array}\right. \tag{3}$$
$$\text{with } \ v_0^1=0,$$

and $((u_n^2)_n, (v_n^2)_n) = a((p_n)_n, (q_n)_n)$, where $a \in H$ will be determined below and $((p_n)_n, (q_n)_n) \in (l^2(I\!R))^2$ is solution of the system

$$\begin{cases} p_1 + q_1 = p \\ p_n + q_n - q_{n-1} = 0, \;\; n = 2, 3, \ldots \\ q_n + p_{n+1} - p_n = 0, \;\; n = 1, 2, \ldots, \;\; \text{with } \; p > 0, \;\; \text{in } \; I\!R. \end{cases} \tag{4}$$

The problem (3) has a solution. To prove this, we consider the *operator* $\mathcal{A}_0 : D(\mathcal{A}_0) = X \to X$, $\mathcal{A}_0((u_n)_n, (v_n)_n) = ((v_n - v_{n-1})_n, (u_{n+1} - u_n)_n)$, $v_0 = 0$. Then the problem (3) is equivalent to

$$U + \mathcal{A}_0(U) = Y. \tag{5}$$

The above equation (5) has solution, because the operator $\mathcal{A}_0$ is maximal monotone in X. Indeed, $\mathcal{A}_0$ is monotone

$$< \mathcal{A}_0(U) - \mathcal{A}_0(\overline{U}), U - \overline{U} >_X = \sum_{n=1}^{\infty} h < v_n - v_{n-1} - \overline{v}_n + \overline{v}_{n-1}, u_n - \overline{u}_n > +$$

$$+ \sum_{n=1}^{\infty} h < u_{n+1} - u_n - \overline{u}_{n+1} + \overline{u}_n, v_n - \overline{v}_n >= 0, \text{ where } v_0 = \overline{v}_0 = 0.$$

In addition, $\mathcal{A}_0$ is single-valued, everywhere defined and continuous. By [4, Proposition 2.4] we deduce that the operator $\mathcal{A}_0$ is maximal monotone and so the equation (5) ($\Leftrightarrow$ the problem (3)) has a (unique) solution.

Using the same argument used before, we deduce that the problem (4) (here $H = I\!R$) has a unique solution $((p_n)_n, (q_n)_n) \in l^2(I\!R) \times l^2(I\!R)$. We shall deduce in what follows the sequences $(p_n)_n$, $(q_n)_n$ by a direct computation (we shall need p_1, q_1).

We set $p_1 = r$. Then by (4) we have

$$\begin{aligned} &p_1 = r, \;\; q_1 = p - r, \\ &p_n = \Delta_{n-1} r - z_{n-2} p, \;\; n \geq 2, \\ &q_n = \Delta_{n-1} p - z_{n-1} r, \;\; n \geq 2, \end{aligned} \tag{6}$$

where $\Delta_0 = 1$, $\Delta_1 = 2$, $\Delta_2 = 5$, $\Delta_3 = 13$, $\Delta_4 = 34$, $\Delta_5 = 89, \ldots$, $z_0 = 1$, $z_1 = 3$, $z_2 = 8$, $z_3 = 21$, $z_4 = 55, \ldots$

The sequences $(\Delta_n)_n$, $(z_n)_n$ satisfy the recursive relations

$$\Delta_n = 3\Delta_{n-1} - \Delta_{n-2}, \;\; \Delta_0 = 1, \;\; \Delta_1 = 2,$$
$$z_n = 3z_{n-1} - z_{n-2}, \;\; z_0 = 1, \;\; z_1 = 3.$$

Using the characteristic equation $\lambda^2 - 3\lambda + 1 = 0$ with the solutions $\lambda_{1,2} = \dfrac{3 \pm \sqrt{5}}{2}$, we obtain for $(\Delta_n)_n$ and $(z_n)_n$ the formulas

$$\Delta_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n \frac{\sqrt{5}+1}{2} + \left(\frac{3-\sqrt{5}}{2}\right)^n \frac{\sqrt{5}-1}{2}\right], \quad n = 0, 1, 2, \ldots$$

$$z_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{3-\sqrt{5}}{2}\right)^{n+1}\right], \quad n = 0, 1, 2, \ldots \tag{7}$$

Then by (6) we obtain

$$p_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}\left(\frac{\sqrt{5}+1}{2}r - p\right) + \left(\frac{3-\sqrt{5}}{2}\right)^{n-1}\left(\frac{\sqrt{5}-1}{2}r + p\right)\right],$$

$$q_n = \frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^{n-1}\left(\frac{\sqrt{5}+1}{2}p - \frac{3+\sqrt{5}}{2}r\right) + \right.$$

$$\left. + \left(\frac{3-\sqrt{5}}{2}\right)^{n-1}\left(\frac{\sqrt{5}-1}{2}p + \frac{3-\sqrt{5}}{2}r\right)\right]. \tag{8}$$

The only bounded sequences $(p_n)_n$, $(q_n)_n$ which satisfy the relations (6) (of the form (8)) are that in which the coefficient of $\left(\dfrac{3+\sqrt{5}}{2}\right)^{n-1}$ in (8) is 0. Therefore we obtain the condition $\dfrac{\sqrt{5}+1}{2}r - p = 0 \;\Rightarrow\; r = \dfrac{\sqrt{5}-1}{2}p$. Then the condition $\dfrac{\sqrt{5}+1}{2}p - \dfrac{3+\sqrt{5}}{2}r = 0$ is also satified. In this way we found the sequences $(p_n)_n$, $(q_n)_n \in l^2(I\!R)$, solutions for (4)

$$p_n = \left(\frac{3-\sqrt{5}}{2}\right)^{n-1}\frac{\sqrt{5}-1}{2}p, \quad q_n = \left(\frac{3-\sqrt{5}}{2}\right)^n p, \quad \forall\, n \geq 1.$$

Evidently $u_n = u_n^1 + u_n^2 = u_n^1 + ap_n$, $n \geq 2$ and $v_n = v_n^1 + v_n^2 = v_n^1 + aq_n$, $n \geq 2$ verify the relations $(2)_1$ for $n = 2, 3, \ldots$ and $(2)_2$ for $n = 1, 2, \ldots$ We shall determine $a \in H$ such that

$$u_1 + v_1 - v_0 = x_1,\; v_0 \in -\alpha(u_1) \;\Leftrightarrow\; u_1^1 + u_1^2 + v_1^1 + v_1^2 - v_0 = x_1,\; v_0 \in -\alpha(u_1)$$
$$\Leftrightarrow\; u_1^1 + ap_1 + v_1^1 + aq_1 - v_0 = x_1,\; v_0 \in -\alpha(u_1^1 + ap_1) \;\Leftrightarrow\; ap_1 + aq_1 \in -\alpha(u_1^1 + ap) \;\Leftrightarrow$$
$$\frac{\sqrt{5}-1}{2}ap + \frac{3-\sqrt{5}}{2}ap \in -\alpha\left(u_1^1 + \frac{\sqrt{5}-1}{2}ap\right) \;\Leftrightarrow\; ap + \alpha\left(u_1^1 + \frac{\sqrt{5}-1}{2}ap\right) \ni 0, \tag{9}$$

where u_1^1 is the solution for (3).

We denote $z = \dfrac{\sqrt{5}-1}{2}ap$; then the equation (9) is equivalent to

$$\frac{\sqrt{5}+1}{2}z + \alpha(u_1^1 + z) \ni 0.$$

We obtain the equation

$$\Lambda_1(z) + \Lambda_2(z) \ni 0, \tag{10}$$

where $\Lambda_1 : H \to H$, $\Lambda_1(z) = \dfrac{\sqrt{5}+1}{2}z$ and $\Lambda_2 : D(\Lambda_2) \subset H \to H$, $D(\Lambda_2) = \{z \in H,\ u_1^1 + z \in D(\alpha)\}$, $\Lambda_2(z) = \alpha(u_1^1 + z)$. The operator Λ_1 is single-valued, everywhere defined, strongly monotone and continuous (so maximal monotone) and the operator Λ_2 is maximal monotone. Then by [1, Corollary 1.3, Chapter II] we deduce that the operator $\Lambda_1+\Lambda_2$ is strongly maximal monotone in H, so the equation (10) has a (unique) solution $z \in D(\Lambda_2)$. Then $a = \dfrac{\sqrt{5}+1}{2p}z$ verifies the relation (9). So we proved the existence of solution $U = ((u_n)_n, (v_n)_n) \in D(\mathcal{A})$ of the system (2) or equation (1). Therefore the operator $\mathcal{A}$ is maximal monotone in X. Q.E.D.

The proof of Theorem 2. We suppose without loss of generality (for an easy writing) that A and B are single-valued. By (H1), $D(\mathcal{B}) \neq \emptyset$. Because $\mathcal{B}$ is defined by a standard product construction, this operator is evidently monotone. Moreover $\mathcal{B}$ is maximal monotone, that is $\forall\, \lambda > 0\ \ R(I+\lambda\mathcal{B}) = X \ \Leftrightarrow\ \forall\, Y = ((x_n)_n, (y_n)_n) \in X\ \exists U = ((u_n)_n, (v_n)_n) \in D(\mathcal{B})$ such that $U + \lambda\mathcal{B}(U) = Y$. The last relation is equivalent to

$$((u_n)_n, (v_n)_n) + \lambda((c_n A(u_n))_n, (d_n B(v_n))_n) = ((x_n)_n, (y_n)_n) \ \Leftrightarrow$$

$$\begin{cases} (u_n)_n + \lambda(c_n A(u_n))_n = (x_n)_n \\ (v_n)_n + \lambda(d_n B(v_n))_n = (y_n)_n \end{cases} \Rightarrow \begin{cases} u_n + \lambda c_n A(u_n) = x_n \\ v_n + \lambda d_n B(v_n) = y_n, \ \ n \geq 1 \end{cases} \Rightarrow$$

$$u_n = (I + \lambda c_n A)^{-1}(x_n) = J^A_{\lambda c_n}(x_n),\ \ v_n = (I + \lambda d_n B)^{-1}(y_n) = J^B_{\lambda d_n}(y_n),\ \ \forall\, n \geq 1.$$

Because $A(0) = 0$ we have $J^A_\mu(0) = 0$, $\forall\, \mu > 0$ and

$$\|J^A_\mu(x) - J^A_\mu(0)\| \leq \|x\| \ \Rightarrow\ \|J^A_\mu(x)\| \leq \|x\|,\ \ \forall\, x \in H,\ \ \forall\, \mu > 0.$$

Similarly by $B(0) = 0$ we deduce $J^B_\mu(0) = 0$, $\forall \mu > 0$ and $\|J^B_\mu(x)\| \leq \|x\|$, $\forall x \in H$, $\forall \mu > 0$. With this remark we have

$$\sum_{n=1}^\infty \|J^A_{\lambda c_n}(x_n)\|^2 \leq \sum_{n=1}^\infty \|x_n\|^2 < \infty,\ \ \sum_{n=1}^\infty \|J^B_{\lambda d_n}(y_n)\|^2 \leq \sum_{n=1}^\infty \|y_n\|^2 < \infty,$$

so $U = ((u_n)_n, (v_n)_n) \in D(\mathcal{B})$. Q.E.D.

The proof of Theorem 3. The operator $\mathcal{A} + \mathcal{B} : D(\mathcal{A}) \cap D(\mathcal{B}) \subset X \to X$ has $D(\mathcal{A}) \cap D(\mathcal{B}) = \{((u_n)_n, (v_n)_n),\ u_1 \in D(A) \cap D(\alpha),\ u_n \in D(A),\ \forall\, n \geq 2,\ v_n \in D(\mathcal{B}),\ \forall\, n \geq 1,$ with $\{(c_n A(u_n))_n\},\ \{(d_n B(v_n))_n\} \subset l^2(H)\} \neq \emptyset$, by (H1), (H3).

First, we suppose (H4)i) holds. The operator $\mathcal{A} + \mathcal{B}$ is monotone ($\mathcal{A}$, $\mathcal{B}$ are monotone). To prove that $\mathcal{A} + \mathcal{B}$ is maximal monotone, we shall show that for any $F_0 = ((f_n^0)_n, (g_n^0)_n) \in X$ the equation

$$U + \mathcal{A}(U) + \mathcal{B}(U) \ni F_0 \tag{11}$$

has at least a solution $U \in D(\mathcal{A}) \cap D(\mathcal{B})$.

For let $F_0 \in X$ be given. The equation (11) is equivalent to

$$\begin{cases} u_n + \dfrac{v_n - v_{n-1}}{h} + c_n A(u_n) \ni f_n^0 \\ v_n + \dfrac{u_{n+1} - u_n}{h} + d_n B(v_n) \ni g_n^0, \quad n = 1, 2, \ldots, \end{cases} \tag{12}$$

$$\text{with } \ v_0 \in -\alpha(u_1). \tag{13}$$

We consider the following approximate problem

$$\begin{cases} U^\lambda + \mathcal{A}(U^\lambda) + \mathcal{B}_\lambda(U^\lambda) \ni F_0 \\ U^\lambda \in D(\mathcal{A}), \quad \lambda > 0, \end{cases} \tag{14}$$

where $\mathcal{B}_\lambda((u_n)_n, (v_n)_n) = ((c_n A_\lambda(u_n))_n, (d_n B_\lambda(v_n))_n)$ with A_λ, B_λ the Yosida approximations of A, respectively B, ($A_\lambda = \frac{1}{\lambda}(I - J_\lambda^A)$, $B_\lambda = \frac{1}{\lambda}(I - J_\lambda^B)$).

Because A_λ, B_λ are everywhere defined ($D(A_\lambda) = D(B_\lambda) = H$), single-valued, monotone, continuous, we deduce that $\mathcal{B}_\lambda$ is also everywhere defined in X, single-valued, monotone and continuous, $\forall\, \lambda > 0$. As $\mathcal{A}$ is maximal monotone operator (Theorem 1), then it follows that $\mathcal{A} + \mathcal{B}_\lambda$ is maximal monotone, $\forall\, \lambda > 0$. Therefore, for any $\lambda > 0$ the problem (14) has a solution $U^\lambda = ((u_n^\lambda)_n, (v_n^\lambda)_n) \in D(\mathcal{A})$. The problem (15) is equivalent to

$$\begin{cases} u_n^\lambda + \dfrac{v_n^\lambda - v_{n-1}^\lambda}{h} + c_n A_\lambda(u_n^\lambda) \ni f_n^0 \\ v_n^\lambda + \dfrac{u_{n+1}^\lambda - u_n^\lambda}{h} + d_n B_\lambda(v_n^\lambda) \ni g_n^0, \quad n = 1, 2, \ldots, \end{cases} \tag{15}$$

$$v_0^\lambda \in -\alpha(u_1^\lambda), \quad (u_1^\lambda \in D(\alpha)). \tag{16}$$

Let $U^0=((u_n^0)_n,(v_n^0)_n)\in D(\mathcal{A})$, $u_1^0\in D(\alpha)$. We denote

$$F_\lambda=((f_n^\lambda)_n,(g_n^\lambda)_n):=U^0+\mathcal{A}(U^0)+\mathcal{B}_\lambda(U^0),\ \ \lambda>0. \tag{17}$$

The set $\{\mathcal{B}_\lambda(U^0);\ \ \lambda>0\}$ is bounded in the space X; indeed

$$\|\mathcal{B}_\lambda(U^0)\|_X^2=\sum_{n=1}^\infty h(c_n^2\|A_\lambda(u_n^0)\|^2+d_n^2\|B_\lambda(v_n^0)\|^2)\leq$$

$$\leq\sum_{n=1}^\infty h(c_n^2\|A^0(u_n^0)\|^2+d_n^2\|B^0(v_n^0)\|^2)=\|\mathcal{B}^0(U^0)\|_X^2,\ \ \forall\,\lambda>0,$$

(where A^0 is the minimal section of A, that is $A^0(x)\in A(x)$, $\|A^0(x)\|=\inf\{\|y\|,\ y\in A(x)\}$, $\forall\,x\in D(A)$).

We deduce by the above inequality and (17) that $\|F_\lambda\|_X\leq$ const., $\forall\,\lambda>0$, (const. is a positive constant independent of λ).

Using (14) and (17) (we substract them and we multiply the obtained relation by $U^\lambda-U^0$ in X), we get

$$\|U^\lambda-U^0\|_X\leq\|F_0-F_\lambda\|_X\ \Rightarrow\ \|U^\lambda\|_X\leq\|U^0\|_X+\|F_0\|_X+\|F_\lambda\|_X\leq\text{const.},\ \forall\lambda>0.$$

We deduce that $\displaystyle\sum_{n=1}^\infty h(\|u_n^\lambda\|^2+\|v_n^\lambda\|^2)\leq$ const. Because $\{u_1^\lambda;\ \lambda>0\}$ is bounded in H, by (H4)i) we deduce that $\{v_0^\lambda;\ \lambda>0\}$ is also bounded in H. So we obtain that $\{\mathcal{A}(U^\lambda);\ \lambda>0\}$ is bounded in X. By (14) we get $\{\mathcal{B}_\lambda(U^\lambda);\ \lambda>0\}$ is bounded in X, $\|\mathcal{B}_\lambda(U^\lambda)\|_X\leq$ const., $\forall\,\lambda>0$, so

$$\sum_{n=1}^\infty h(\|c_nA_\lambda(u_n^\lambda)\|^2+\|d_nB_\lambda(v_n^\lambda)\|^2)\leq\text{const.},\ \ \forall\,\lambda>0. \tag{18}$$

We shall prove in what follows that the sets $\{(u_n^\lambda)_n;\ \ \lambda>0\}$, $\{(v_n^\lambda)_n;\ \ \lambda>0\}$ are Cauchy sequences (in $l^2(H)$). For this, let $U^\lambda=((u_n^\lambda)_n,(v_n^\lambda)_n)$, $U^\mu=((u_n^\mu)_n,(v_n^\mu)_n)$, λ, $\mu>0$, be solutions for (14), $u_1^\lambda\in D(\alpha)$, $v_0^\lambda\in-\alpha(u_1^\lambda)$, $u_1^\mu\in D(\alpha)$, $v_0^\mu\in-\alpha(u_1^\mu)$. Then by (14) we have $U^\lambda+Z^\lambda+\mathcal{B}_\lambda(U^\lambda)=F_0$, $U^\mu+Z^\mu+\mathcal{B}_\mu(U^\mu)=F_0$, $Z^\lambda\in\mathcal{A}(U^\lambda)$, $Z^\mu\in\mathcal{A}(U^\mu)$ and $U^\lambda-U^\mu+Z^\lambda-Z^\mu+\mathcal{B}_\lambda(U^\lambda)-\mathcal{B}_\mu(U^\mu)=0$.

We multiply the above relation by $U^\lambda-U^\mu$ in X and after some computations we obtain

$$\sum_{n=1}^\infty(\|u_n^\lambda-u_n^\mu\|^2+\|v_n^\lambda-v_n^\mu\|^2)\leq-\sum_{n=1}^\infty\{c_n[<A_\lambda(u_n^\lambda)-A_\mu(u_n^\mu),J_\lambda^A(u_n^\lambda)-J_\mu^A(u_n^\mu)>+$$

$$+<A_\lambda(u_n^\lambda)-A_\mu(u_n^\mu),\lambda A_\lambda(u_n^\lambda)-\mu A_\mu(u_n^\mu)>]+d_n[<B_\lambda(v_n^\lambda)-B_\mu(v_n^\mu),J_\lambda^B(v_n^\lambda)-$$

$$-J^B_\mu(v^\mu_n) > + < B_\lambda(v^\lambda_n) - B_\mu(v^\mu_n), \lambda B_\lambda(v^\lambda_n) - \mu B_\mu(v^\mu_n) >]\}.$$

Because $A_\lambda(u^\lambda_n) \in A(J^A_\lambda(u^\lambda_n))$, $B_\lambda(v^\lambda_n) \in B(J^B_\lambda(v^\lambda_n))$, $\forall\, \lambda > 0$,$\forall\, n \geq 1$, by the above inequality we deduce

$$\sum_{n=1}^{\infty}(\|u^\lambda_n - u^\mu_n\|^2 + \|v^\lambda_n - v^\mu_n\|^2) \leq \sum_{n=1}^{\infty}[c_n(\lambda\|A_\lambda(u^\lambda_n)\|^2 + \frac{\lambda}{2}\|A_\mu(u^\mu_n)\|^2 +$$

$$+\frac{\lambda}{2}\|A_\lambda(u^\lambda_n)\|^2 + \frac{\mu}{2}\|A_\lambda(u^\lambda_n)\|^2 + \frac{\mu}{2}\|A_\mu(u^\mu_n)\|^2 + \mu\|A_\mu(u^\mu_n)\|^2)+$$

$$+d_n(\lambda\|B_\lambda(v^\lambda_n)\|^2 + \frac{\lambda}{2}\|B_\mu(v^\mu_n)\|^2 + \frac{\lambda}{2}\|B_\lambda(v^\lambda_n)\|^2 + \frac{\mu}{2}\|B_\lambda(v^\lambda_n)\|^2 + \frac{\mu}{2}\|B_\mu(v^\mu_n)\|^2 +$$

$$+\mu\|B_\mu(v^\mu_n)\|^2)] \leq \text{const.}(\lambda+\mu),\ \ \forall\, \lambda,\, \mu > 0,\ \ (\text{by}\ \ (18)).$$

Therefore

$$\sum_{n=1}^{\infty}(\|u^\lambda_n - u^\mu_n\|^2 + \|v^\lambda_n - v^\mu_n\|^2) \leq \text{const.}(\lambda+\mu),\ \ \forall\, \lambda,\, \mu > 0.$$

We deduce that $\{(u^\lambda_n)_n;\ \lambda > 0\}$ and $\{(v^\lambda_n)_n;\ \lambda > 0\}$ are Cauchy sequences in $l^2(H)$. Then, there exist $\lim\limits_{\lambda\to 0}(u^\lambda_n)_n = (u_n)_n$, $\lim\limits_{\lambda\to 0}(v^\lambda_n)_n = (v_n)_n$, in $l^2(H)$, (evidently $u^\lambda_n \to u_n$, $v^\lambda_n \to v_n$ as $\lambda \to 0$, in H, $\forall\, n \geq 1$).

Because $u^\lambda_1 \to u_1$, as $\lambda \to 0$, in H, $\{v^\lambda_0;\ \lambda > 0\}$ is bounded in H, so on a subsequence $v^\lambda_0 \rightharpoonup v_0$, as $\lambda \to 0$ ($v_0 \in H$), $v^\lambda_0 \in -\alpha(u^\lambda_1)$ and α is demiclosed, we deduce that $u_1 \in D(\alpha)$ and $v_0 \in -\alpha(u_1)$, that is (13).

Then, by $u^\lambda_n \to u_n$ and $v^\lambda_n \to v_n$ as $\lambda \to 0$, we deduce that $J^A_\lambda u^\lambda_n \to u_n$, $J^B_\lambda v^\lambda_n \to v_n$, as $\lambda \to 0$, $\forall\, n \geq 1$. Because $\{A_\lambda(u^\lambda_n);\ \lambda > 0\}$, $\{B_\lambda(v^\lambda_n);\ \lambda > 0\}$, $n \geq 1$ are bounded (by (18), for any n fixed we have $\|c_n A_\lambda(u^\lambda_n)\| \leq$ const., $\forall\, \lambda > 0$, so $\|A_\lambda(u^\lambda_n)\| \leq \frac{1}{c_n}$const., $\forall\, \lambda > 0$) and A, B are demiclosed, we deduce that $u_n \in D(A)$ and $A_\lambda(u^\lambda_n) \rightharpoonup p_n$, as $\lambda \to 0$ ($p_n \in H$), $p_n \in A(u_n)$, $v_n \in D(B)$ and $B_\lambda(v^\lambda_n) \rightharpoonup q_n$, as $\lambda \to 0$ ($q_n \in H$), $q_n \in B(v_n)$, $\forall\, n \geq 1$ (eventually on some sequences). By passing to $\lambda \to 0$ in (15), (16) we deduce that $U = ((u_n)_n, (v_n)_n)$ is a solution for (12) and (13). By (12) we obtain $(c_n A(u_n))_n$, $(d_n B(v_n))_n \in l^2(H)$, so $U \in D(\mathcal{A}) \cap D(\mathcal{B})$ and $\mathcal{A} + \mathcal{B}$ is maximal monotone in X.

If (H4)ii) holds, by Theorem 1 and Theorem 2 we have that $\mathcal{A}$ and $\mathcal{B}$ are maximal monotone with $\operatorname{int} D(\mathcal{A}) = \{((u_n)_n, (v_n)_n) \in X,\ u_1 \in \operatorname{int} D(\alpha)\}$ and so $\operatorname{int} D(\mathcal{A}) \cap D(\mathcal{B}) \neq \emptyset$. Therefore, by [4, Corollaire 2.7] we deduce that $\mathcal{A} + \mathcal{B}$ is maximal monotone. Q.E.D.

The proof of Theorem 4. By Theorem 3, the operator $\mathcal{A}+\mathcal{B}:D(\mathcal{A})\cap D(\mathcal{B})\subset X\to X$ is maximal monotone in X. Using [1, Theorem 2.2, Corollary 2.1, Chapter III] we deduce that for $U_0=((u_{n0})_n,(v_{n0})_n)\in D(\mathcal{A})\cap D(\mathcal{B})$ and $F=((f_n)_n,(g_n)_n)\in W^{1,1}(0,T;X)$, the problem (P)$\Leftrightarrow$(S)+(BC)+(IC) has a unique strong solution $U=((u_n)_n,(v_n)_n)\in W^{1,\infty}(0,T;X)$, $U(t)\in D(\mathcal{A})\cap D(\mathcal{B})$, $\forall\, t\in[0,T)$. By considering the equation $(\mathrm{P})_1$ in the interval $[0,T+\varepsilon]$, with $\varepsilon>0$ (by extending correspondingly the functions f_n, g_n, $n\geq 1$) we obtain $U(T)\in D(\mathcal{A})\cap D(\mathcal{B})$. The solution U is everywhere differentiable from right and $\dfrac{d^+U}{dt}(t)=(F(t)-\mathcal{A}(U(t))-\mathcal{B}(U(t)))^0$, $\forall\, t\in[0,T)$, that is the relations from the conclusion of the theorem are verified. In addition we have

$$\left\|\frac{d^+U}{dt}(t)\right\|_X\leq\left\|(F(0)-\mathcal{A}(U_0)-\mathcal{B}(U_0))^0\right\|_X+\int_0^t\left\|\frac{dF}{ds}(s)\right\|_X ds,\ \ \forall\, t\in[0,T).$$

If U and V are the solutions of (P) corresponding to (U_0,F), $(V_0,G)\in(D(\mathcal{A})\cap D(\mathcal{B}))\times W^{1,1}(0,T;X)$, then

$$\|U(t)-V(t)\|_X\leq\|U_0-V_0\|_X+\int_0^t\|F(s)-G(s)\|_X ds,\ \ \forall\, t\in[0,T].$$

Q.E.D.

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Received August 21, 2006