BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(53), 2007, Pages 25–36
ISSN 1024–7696

Classification of $GL(2,\mathbb{R})$-orbit's dimensions for the differential equations' system with homogeneities of the 4th order

E. Naidenova, M.N. Popa, V. Orlov

Abstract. Center-affine invariant conditions for $GL(2,\mathbb{R})$-orbit's dimensions are defined for two-dimensional autonomous system of differential polynomial equations with homogeneities of the 4th order.

Mathematics subject classification: 34C14.
Keywords and phrases: Differential system, Lie algebra of the operators, $GL(2,\mathbb{R})$-orbit.

Consider two-dimensional differential system with homogeneities of the 4th order

$$\frac{dx^j}{dt} = a^j_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta \quad (j,\alpha,\beta,\gamma,\delta=\overline{1,2}), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta\gamma\delta}$ is symmetrical in lower indices in which the complete convolution holds.

Consider also the group of center-affine transformations $GL(2,\mathbb{R})$ given by the equalities

$$\bar{x}^1 = \alpha x^1 + \beta x^2, \quad \bar{x}^2 = \gamma x^1 + \delta x^2, \quad \Delta = det\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \neq 0.$$

Further will use the notations

$$a^1_{1111}=a,\ \ a^1_{1112}=b,\ \ a^1_{1122}=c,\ \ a^1_{1222}=d,\ \ a^1_{2222}=e,\ \ a^2_{1111}=f,\ \ a^2_{1112}=g,$$
$$a^2_{1122}=h,\ \ a^2_{1222}=k,\ \ a^2_{2222}=l,\ \ x^1=x,\ \ x^2=y. \tag{2}$$

According to [1] and taking into consideration (2) the representation operators of the group $GL(2,\mathbb{R})$ in the space of coefficients and variables of the system (1) will take the form

$$X_1 = x\frac{\partial}{\partial x} - 3a\frac{\partial}{\partial a} - 2b\frac{\partial}{\partial b} - c\frac{\partial}{\partial c} + e\frac{\partial}{\partial e} - 4f\frac{\partial}{\partial f} - 3g\frac{\partial}{\partial g} - 2h\frac{\partial}{\partial h} - k\frac{\partial}{\partial k};$$

$$X_2 = y\frac{\partial}{\partial x} + f\frac{\partial}{\partial a} + (g-a)\frac{\partial}{\partial b} + (h-2b)\frac{\partial}{\partial c} + (k-3c)\frac{\partial}{\partial d} + (l-4d)\frac{\partial}{\partial e} -$$

$$-f\frac{\partial}{\partial g}-2g\frac{\partial}{\partial h}-3h\frac{\partial}{\partial k}-4k\frac{\partial}{\partial l};$$

$$X_3=x\frac{\partial}{\partial y}-4b\frac{\partial}{\partial a}-3c\frac{\partial}{\partial b}-2d\frac{\partial}{\partial c}-e\frac{\partial}{\partial d}+(a-4g)\frac{\partial}{\partial f}+$$

$$+(b-3h)\frac{\partial}{\partial g}+(c-2k)\frac{\partial}{\partial h}+(d-l)\frac{\partial}{\partial k}+e\frac{\partial}{\partial l};$$

$$X_4=y\frac{\partial}{\partial y}-b\frac{\partial}{\partial b}-2c\frac{\partial}{\partial c}-3d\frac{\partial}{\partial d}-4e\frac{\partial}{\partial e}+f\frac{\partial}{\partial f}-h\frac{\partial}{\partial h}-2k\frac{\partial}{\partial k}-3l\frac{\partial}{\partial l}. \quad (3)$$

The operators (3) form a four-dimensional reductive Lie algebra [1].

Let $\tilde{a}=(a,b,...,l)\in E^{10}(\tilde{a})$, where $E^{10}(\tilde{a})$ is Euclidean space of the coefficients of the right-hand sides of the system (1). Denote by $\tilde{a}(q)$ the point from $E^{10}(\tilde{a})$ that corresponds to the system, obtained from the system (1) with coefficients $\tilde{a}$ by a transformation $q\in GL(2,\mathbb{R})$.

Definition 1. Call the set $O(\tilde{a})=\{\tilde{a}(q)|q\in GL(2,\mathbb{R})\}$ the $GL(2,\mathbb{R})$-orbit of the point $\tilde{a}$ for the system (1).

Definition 2. Call the set $M\subseteq E^{10}(\tilde{a})$ the $GL(2,\mathbb{R})$-invariant if for any point $\tilde{a}\in M$ its orbit $O(\tilde{a})\subseteq M$.

It is known from [1] that

$$dim_{\mathbb{R}}O(\tilde{a})=rankM_1, \quad (4)$$

where M_1 is the following matrix

$$M_1=\begin{pmatrix} 3a & 2b & c & 0 & -e & 4f & 3g & 2h & k & 0\\ -f & a-g & 2b-h & 3c-k & 4d-l & 0 & f & 2g & 3h & 4k\\ 4b & 3c & 2d & e & 0 & 4g-a & 3h-b & 2k-c & l-d & -e\\ 0 & b & 2c & 3d & 4e & -f & 0 & h & 2k & 3l \end{pmatrix}, \quad (5)$$

constructed on coordinate vectors of operators (3).

Will use the following notations for the matrix M_1: denote by Δ_{ijkl} the minor of the 4th order constructed on columns i, j, k, l, $(i,j,k,l=\overline{1,10})$; denote by Δ^{ijk}_{lmn} the minor of the 3rd order constructed on lines i, j, k, $(i,j,k=\overline{1,4})$ and columns l, m, n, $(l,m,n=\overline{1,10})$; and by Δ^{ij}_{kl} will be denoted the minor of the 2nd order constructed on lines i, j, $(i,j=\overline{1,4})$ and columns k, l, $(k,l=\overline{1,10})$.

For the system (1) two comitants of the first order with respect to its coefficients are known from [2]

$$F_3=(a+g)x^3+3(b+h)x^2y+3(c+k)xy^2+(d+l)y^3,$$

$$F_5=-fx^5+(a-4g)x^4y+(4b-6h)x^3y^2+(6c-4k)x^2y^3+(4d-l)xy^4+ey^5. \quad (6)$$

According to [3], write a transvectant of index k for binary forms f and φ as follows

$$(f,\varphi)^{(k)} = \frac{(r-k)!(\rho-k)!}{r!\rho!}\sum_{h=0}^{k}(-1)^h \mathrm{C}_k^h \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\frac{\partial^k \varphi}{\partial x^h \partial y^{k-h}}, \qquad (7)$$

where r and ρ are degrees of these forms with respect to x and y correspondingly.

According to [4], the transvectant (7) on two comitants of the system (1) is a comitant (invariant) of this system too.

Taking into consideration the above mentioned, the following comitants and invariants of the system (1) were constructed in [2]:

$$L_1 = (F_5,F_5)^{(2)}, \quad L_2 = (F_5,F_5)^{(4)}, \quad L_3 = (F_3,F_3)^{(2)},$$
$$L_4 = (F_3,F_5)^{(1)}, \quad L_5 = (F_3,F_5)^{(2)}, \quad L_6 = (F_3,F_5)^{(3)},$$
$$L_7 = (L_2,F_5)^{(2)}, \quad B_1 = (L_3,L_3)^{(2)}, \quad B_2 = (L_1,L_1)^{(6)},$$
$$B_3 = (L_1,L_4)^{(6)}, \quad B_4 = (L_3,L_6)^{(2)}, \quad B_5 = (L_5,L_5)^{(4)},$$
$$B_6 = ((L_3,L_5)^{(2)},L_3)^{(2)}, \quad B_7 = ((L_7,L_7)^{(2)},L_2)^{(2)},$$
$$B_8 = ((((F_3,L_4)^{(2)},F_3)^{(2)},L_5)^{(2)},L_5)^{(4)},$$
$$B_9 = (((L_7,L_7)^{(2)},L_7)^{(1)},L_7)^{(3)},$$
$$C_2 = (L_1,L_1)^{(2)}. \qquad (8)$$

Lemma 1. *For $F_5 \equiv 0$ the rang of matrix M_1 is equal to four if and only if $B_1 \neq 0$, where B_1 is from* (8).

Proof. Taking into consideration (6) from $F_5 \equiv 0$ we obtain

$$e = f = 0, \quad a = 4g, \quad b = \frac{3}{2}h, \quad k = \frac{3}{2}c, \quad l = 4d. \qquad (9)$$

As conditions (9) hold the matrix M_1 takes the form

$$M_1^{(1)} = \begin{pmatrix} 12g & 3h & c & 0 & 0 & 0 & 3g & 2h & \frac{3}{2}c & 0 \\ 0 & 3g & 2h & \frac{3}{2}c & 0 & 0 & 0 & 2g & 3h & 6c \\ 6h & 3c & 2d & 0 & 0 & 0 & \frac{3}{2}h & 2c & 3d & 0 \\ 0 & \frac{3}{2}h & 2c & 3d & 0 & 0 & 0 & h & 3c & 12d \end{pmatrix}. \qquad (10)$$

As conditions (9) hold the invariant B_1 takes the form

$$B_1 = -\frac{625}{8}(S^2 - 4TR), \qquad (11)$$

where

$$R = 2cg - h^2, \quad S = 4dg - ch, \quad T = 2dh - c^2. \qquad (12)$$

We note that all nonzero minors of the 4th order of matrix $M_1^{(1)}$ up to an constant factor coincide with $\Delta_{1234} = -\dfrac{108}{625}B_1$.

Hence, for $F_5 \equiv 0$, $B_1 \neq 0$ the rang of the matrix (5) is equal to four. Lemma 1 is proved.

Lemma 2. *For $F_5 \equiv 0$ the rang of matrix M_1 is equal to three if and only if holds*

$$B_1 = 0, \;\; L_3 \not\equiv 0, \tag{13}$$

where B_1, L_3 are from (8).

Proof. As conditions (9) hold, considering (1) and (2) the comitant L_3 takes the form

$$L_3 = \frac{25}{2}(Rx^2 + Sxy + Ty^2), \tag{14}$$

where R, S, T are from (12). Calculations yield that any nonzero third order minor of the matrix (10) up to a constant factor coincides with one of the following minors:

$$\Delta^{123}_{123} = -36hR + 18gS; \quad \Delta^{123}_{124} = -27cR; \quad \Delta^{123}_{134} = -9cS;$$

$$\Delta^{123}_{234} = -\frac{9}{2}cT; \quad \Delta^{124}_{123} = 36gR; \quad \Delta^{124}_{124} = 27gS;$$

$$\Delta^{124}_{134} = 36gT; \quad \Delta^{124}_{234} = -\frac{9}{4}(-3hT + 2dR);$$

$$\Delta^{134}_{123} = -72gT + 27hS; \quad \Delta^{134}_{124} = 54dR; \quad \Delta^{134}_{134} = 18dS;$$

$$\Delta^{134}_{234} = 9dT; \quad \Delta^{234}_{123} = -18hR; \quad \Delta^{234}_{124} = -\frac{27}{2}hS;$$

$$\Delta^{234}_{134} = -18hT; \quad \Delta^{234}_{234} = -9cT + \frac{9}{2}dS. \tag{15}$$

The necessity of the conditions (13) follows from Lemma 1, (14) and(15). It is evident that, according to (11) for $B_1 = 0$, $S^2 = 4TR$ holds and from $L_3 \not\equiv 0$ (see (14)) at least one of R, S and T will be nonzero. This fact with $c^2 + d^2 + g^2 + h^2 \neq 0$ ensure that at least one of minors (15) will be nonzero. Lemma 2 is proved.

Lemma 3. *For $F_5 \equiv 0$ the rang of matrix M_1 is equal to two if and only if*

$$L_3 \equiv 0, \;\; F_3 \not\equiv 0, \tag{16}$$

where F_3 is from (6) *and L_3 is from* (8).

Proof. According to (9) from (14) for $L_3 \equiv 0$ we obtain $T = R = S = 0$, where R, S, T are from (12). This implies $B_1 = 0$ and from Lemma 2 $rang M_1 < 3$. Will show that in this case $rang M_1 = 2$ if and only if

$$F_3 = 5gx^3 + \frac{15}{2}hx^2y + \frac{15}{2}cxy^2 + 5dy^3 \not\equiv 0. \tag{17}$$

And this is ensured by the existence of the following second order minors of the matrix $M_1^{(1)}$

$$\Delta_{12}^{12} = 36g^2, \quad \Delta_{12}^{34} = 9h^2, \quad \Delta_{34}^{12} = \frac{3}{2}c^2, \quad \Delta_{34}^{34} = 6d^2.$$

Lemma 3 is proved.

The next result is evident.

Lemma 4. *For $F_5 \equiv 0$ the rang of matrix M_1 is equal to zero if and only if $F_3 \equiv 0$, where F_3 is from* (6)*.*

From Lemmas 1–4 and equality (4) follows

Theorem 5. *For $F_5 \equiv 0$ the dimension of $GL(2,\mathbb{R})$-orbit of the system* (1) *is equal to*

4 for $B_1 \neq 0$;

3 for $B_1 = 0, \quad L_3 \not\equiv 0$;

2 for $L_3 \equiv 0, \quad F_3 \not\equiv 0$;

0 for $F_3 \equiv 0$,

where F_3 and F_5 are from (6)*, and B_1, L_3 are from* (8)*.*

Lemma 6. *For $F_3 \equiv 0$ the rang of matrix M_1 is equal to four if and only if*

$$3L_1L_2 + 105C_2 + 26F_5L_7 \not\equiv 0, \tag{18}$$

where F_3 and F_5 are from (6)*, and L_1, L_2, L_7, C_2 are from* (8)*.*

Proof. Taking into consideration (6) from $F_3 \equiv 0$ we obtain

$$a = -g, \ b = -h, \ c = -k, \ d = -l. \tag{19}$$

On the other hand, according to [5] such $GL(2,\mathbb{R})$-transformation exists that the comitant F_5 will take the form $F_5 = y\tilde{F}_4$, where $\tilde{F}_4$ is the polynomial of the forth order on variables, corresponding to the system (1) after the transformation. Due to this we obtain that $f = 0$. As this holds from conditions (19) we obtain that the matrix M_1 takes the form

$$M_1^{(2)} = \begin{pmatrix} -3g & -2h & -k & 0 & -e & 0 & 3g & 2h & k & 0 \\ 0 & -2g & -3h & -4k & -5l & 0 & 0 & 2g & 3h & 4k \\ -4h & -3k & -2l & e & 0 & 5g & 4h & 3k & 2l & -e \\ 0 & -h & -2k & -3l & 4e & 0 & 0 & h & 2k & 3l \end{pmatrix}. \tag{20}$$

Nonzero 4th order minors of the matrix (20) will coincide up to the numerical factor with one of the following:

$$\Delta_{1234} = 12eg^2k - 9egh^2 + 36g^2l^2 - 129ghkl + 72gk^3 + 72h^3l - 48h^2k^2;$$

$$\Delta_{1235} = -48eg^2l + 156eghk - 108eh^3 - 30ghl^2 + 90gk^2l - 60h^2kl;$$
$$\Delta_{1236} = 60g^3k - 45g^2h^2;$$
$$\Delta_{1245} = 24e^2g^2 + 39eghl + 144egk^2 - 144eh^2k + 135gkl^2 - 120h^2l^2;$$
$$\Delta_{1246} = 90g^3l - 60g^2hk; \quad \Delta_{1256} = -120eg^3 - 75g^2hl;$$
$$\Delta_{1345} = 36e^2gh + 126egkl + 36eh^2l - 96ehk^2 + 90gl^3 - 60hkl^2;$$
$$\Delta_{1346} = 135g^2hl - 120g^2k^2; \quad \Delta_{1356} = -180eg^2h - 150g^2kl;$$
$$\Delta_{1456} = -240eg^2k - 225g^2l^2;$$
$$\Delta_{2345} = -12e^2gk + 27e^2h^2 - 12egl^2 + 114ehkl - 72ek^3 + 60hl^3 - 45k^2l^2;$$
$$\Delta_{2346} = -30g^2kl + 90gh^2l - 60ghk^2; \quad \Delta_{2356} = 60eg^2k - 135egh^2 - 75ghkl;$$
$$\Delta_{2456} = 30eg^2l - 180eghk - 150ghl^2; \quad \Delta_{3456} = 45eghl - 120egk^2 - 75gkl^2. \quad (21)$$

Also holds the equality

$$3L_1L_2 + 105C_2 + 26F_5L_7 = -2\Delta_{1236}x^8 - 4\Delta_{1246}x^7y - 2(\Delta_{1256} + 3\Delta_{1346})x^6y^2 -$$
$$-4(\Delta_{1356} + 2\Delta_{2346})x^5y^3 + 2(-5\Delta_{1234} + \Delta_{15610} - 3\Delta_{2356})x^4y^4 -$$
$$-4(2\Delta_{1235} + \Delta_{2456})x^3y^5 - 2(3\Delta_{1245} + \Delta_{3456})x^2y^6 - 4\Delta_{1345}xy^7 - 2\Delta_{2345}y^8. \quad (22)$$

Let prove the necessity of condition (18). Assume the contrary, i.e. for $3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0$ there exists at least one nonzero 4th order minor of the matrix M_1. Taking into consideration (6), (8), (19) , (21) and (22), from $3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0$ we obtain the following series of conditions for coefficients of the system (1):

$$I. \quad g = h = k = 0; \quad (23)$$
$$II. \quad g = h = 0,\ e = -\frac{5l^2}{8k},\ k \neq 0; \quad (24)$$
$$III. \quad g = 0,\ l = \frac{2k^2}{3h},\ e = -\frac{10k^3}{27h^2},\ h \neq 0; \quad (25)$$
$$IV. \quad k = \frac{3h^2}{9g},\ l = \frac{h^3}{2g^2},\ e = -\frac{5h^4}{16g^3},\ g \neq 0. \quad (26)$$

With the aid of (21) one can verify that while any of the series of the conditions (23)–(26) holds, all the 4th order minors of the matrix (20) will be equal to zero. Thus obtained contradiction proves the necessity of condition (18).

The sufficiency of condition (18) is ensured by equality (22). Lemma 6 is proved.

Lemma 7. *For $F_3 \equiv 0$ the rang of matrix M_1 is equal to three if and only if*

$$3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0,\ L_1 \not\equiv 0, \quad (27)$$

where F_3 and F_5 are from (6), and L_1, L_2, L_7, C_2 are from (8).

Proof. In proof we will use the $GL(2,\mathbb{R})$-transformation from the proof of Lemma 6, and therefore, the equality $f=0$ can be assumed. According to Lemma 6, as first condition from (27) holds for the coefficients of the system (1) besides (19) we obtain the values (23)–(26).

Consider the conditions (23). The matrix M_1 takes the form

$$M_1^{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5l & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2l & e & 0 & 0 & 0 & 0 & 2l & -e \\ 0 & 0 & 0 & -3l & 4e & 0 & 0 & 0 & 0 & 3l \end{pmatrix} \tag{28}$$

and the comitant L_1 takes the form

$$L_1 = -2l^2y^6. \tag{29}$$

Hence, it is evident that the condition $L_1 \not\equiv 0$ is sufficient and necessary.

Consider the conditions (24). The matrix M_1 takes the form (with $e = -\dfrac{5l^2}{8k}$)

$$M_1^{(4)} = \begin{pmatrix} 0 & 0 & -k & 0 & -e & 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & -4k & -5l & 0 & 0 & 0 & 0 & 4k \\ 0 & -3k & -2l & e & 0 & 0 & 0 & 3k & 2l & -e \\ 0 & 0 & -2k & -3l & 4e & 0 & 0 & 0 & 2k & 3l \end{pmatrix} \tag{30}$$

and the comitant L_1 takes the form $L_1 = -\frac{3}{4}(4kx+ly)^2y^4$. Since $k \neq 0$, considering (30) and (24) we obtain $L_1 \not\equiv 0$ and $\Delta_{234}^{123} = -12k^3 \neq 0$, i.e. $rang M_1^{(4)} = 3$.

Consider the conditions (25). The matrix M_1 takes the form (with the values of the parameters l and e from (25))

$$M_1^{(5)} = \begin{pmatrix} 0 & -2h & -k & 0 & -e & 0 & 0 & 2h & k & 0 \\ 0 & 0 & -3h & -4k & -5l & 0 & 0 & 0 & 3h & 4k \\ -4h & -3k & -2l & e & 0 & 0 & 4h & 3k & 2l & -e \\ 0 & -h & -2k & -3l & 4e & 0 & 0 & h & 2k & 3l \end{pmatrix} \tag{31}$$

and the comitant L_1 takes the form

$$L_1 = -\frac{4}{27h^2}(3hx+ky)^4y^2.$$

So, as $h \neq 0$ we get $L_1 \not\equiv 0$ as well as $\Delta_{123}^{123} = -24h^3 \neq 0$.

Consider the conditions (26).

The matrix M_1 takes the form $M_1^{(2)}$ (with the values of the parameters k, l and e from (26)), and the comitant L_1 takes the form

$$L_1 = -\frac{1}{32g^4}(2gx+hy)^6.$$

Since $g \neq 0$ we obtain $L_1 \not\equiv 0$ and $\Delta_{126}^{123} = 30g^3 \neq 0$, i.e. $rang M_1^{(2)} = 3$.

Lemma 7 is proved.

Lemma 8. *For $F_3 \equiv 0$ the rang of matrix M_1 is equal to two if and only if*

$$L_1 \equiv 0, \ \ F_5 \not\equiv 0, \tag{32}$$

where F_3 and F_5 are from (6)*, and L_1 is from* (8)*.*

Proof. As $C_2 = (L_1, L_1)^{(2)}$ (see (8)) it is evident that from $L_1 \equiv 0$ follows $C_2 \equiv 0$. Moreover, as L_1 is the Hessian of the comitant F_5 then, for $L_1 \equiv 0$ it follows $F_5 = (\alpha x + \beta y)^5$, $a, b \in \mathbb{R}$ (see [3]). Considering (8) it is easy to verify that the transvectant $L_7 = (((\alpha x+\beta y)^5, (\alpha x+\beta y)^5)^4, (\alpha x+\beta y)^5)^2 = 0$. Hence, the condition $L_1 \equiv 0$ implies $3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0$ and then from Lemma 7 follows the necessity of the conditions (32).

Let prove the sufficiency. Assume $L_1 \equiv 0$, i.e. F_5 must be of the form $F_5 = (\alpha x + \beta y)^5$ (see above). On the other hand, as it was mentioned in the proof of Lemma 6, we assume $f = 0$ due to a $GL(2, \mathbb{R})$-transformation. Hence $\alpha = 0$ and considering (19) and (6) we obtain $g = h = k = l = 0$, $F_5 = ey^5$. In this case the matrix M_1 takes the form

$$M_1^{(6)} = \begin{pmatrix} 0 & 0 & 0 & 0 & -e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & -e \\ 0 & 0 & 0 & 0 & 4e & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{33}$$

It is evident that for $F_5 \equiv 0$ all 2nd order minors of the matrix $M_1^{(6)}$ will be equal to zero, and for $F_5 \not\equiv 0$ the 2nd order minor $\Delta_{45}^{13} = e^2$ will be nonzero.

Lemma 8 is proved.

With the aid of Lemmas 4-8 and equality (4) is proved

Theorem 9. *For $F_3 \equiv 0$ the dimension of $GL(2, \mathbb{R})$-orbit of the system* (1) *is equal to*

4 for $3L_1L_2 + 105C_2 + 26F_5L_7 \not\equiv 0$*;*
3 for $3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0, \quad L_1 \not\equiv 0$*;*
2 for $L_1 \equiv 0, \quad F_5 \not\equiv 0$*;*
0 for $F_5 \equiv 0$*,*

where F_3 and F_5 are from (6)*, and L_1, L_2, L_7, C_2 are from* (8)*.*

Lemma 10. *For $F_3F_5 \not\equiv 0$ the rang of matrix M_1 is equal to four if and only if*

$$12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 \not\equiv 0, \tag{34}$$

where F_3, F_5 are from (6) *and L_1, L_3, L_4 are from* (8)*.*

Proof. The sufficiency of the condition (34) follows from the equality

$$12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 = (\Delta_{1267} + \Delta_{1678})x^{12} + (4\Delta_{1268} + 2\Delta_{1367} + 2\Delta_{1679} + 4\Delta_{2678})x^{11}y +$$

$$+(\Delta_{1236} + 5\Delta_{1269} + 10\Delta_{1278} + 9\Delta_{1368} + \Delta_{1467} + \Delta_{16710} + 3\Delta_{1689} + 2\Delta_{2367} + 8\Delta_{2679} +$$

$$+6\Delta_{3678})x^{10}y^2+(4\Delta_{1237}+2\Delta_{1246}+2\Delta_{12610}+16\Delta_{1279}+12\Delta_{1369}+24\Delta_{1378}+6\Delta_{1468}+$$
$$+2\Delta_{16810}+4\Delta_{1789}+12\Delta_{2368}+4\Delta_{26710}+12\Delta_{2689}+12\Delta_{3679}+4\Delta_{4678})x^9y^3+(6\Delta_{1238}+$$
$$+8\Delta_{1247}+\Delta_{1256}+7\Delta_{12710}+14\Delta_{1289}+3\Delta_{1346}+5\Delta_{13610}+40\Delta_{1379}+9\Delta_{1469}+18\Delta_{1478}+$$
$$+\Delta_{1568}+\Delta_{16910}+3\Delta_{17810}+18\Delta_{2369}+36\Delta_{2378}+8\Delta_{2468}-\Delta_{2567}+8\Delta_{26810}+16\Delta_{2789}-$$
$$-2\Delta_{3467}+6\Delta_{36710}+18\Delta_{3689}+8\Delta_{4679}+\Delta_{5678})x^8y^4+(4\Delta_{1239}+12\Delta_{1248}+4\Delta_{1257}+$$
$$+8\Delta_{12810}+12\Delta_{1347}+2\Delta_{1356}+18\Delta_{13710}+36\Delta_{1389}+4\Delta_{14610}+32\Delta_{1479}+2\Delta_{1569}+$$
$$+4\Delta_{1578}+2\Delta_{17910}+4\Delta_{2346}+8\Delta_{23610}+64\Delta_{2379}+16\Delta_{2469}+32\Delta_{2478}+4\Delta_{26910}+$$
$$+12\Delta_{27810}+2\Delta_{3567}+12\Delta_{36810}+24\Delta_{3789}+4\Delta_{46710}+12\Delta_{4689}+2\Delta_{5679})x^7y^5+(\Delta_{12310}+$$
$$+8\Delta_{1249}+6\Delta_{1258}+3\Delta_{12910}+18\Delta_{1348}+8\Delta_{1357}+21\Delta_{13810}+\Delta_{1456}+15\Delta_{14710}+$$
$$+30\Delta_{1489}+\Delta_{15610}+8\Delta_{1579}+\Delta_{18910}+16\Delta_{2347}+3\Delta_{2356}+30\Delta_{23710}+60\Delta_{2389}+$$
$$+8\Delta_{24610}+64\Delta_{2479}+3\Delta_{2569}+6\Delta_{2578}+8\Delta_{27910}+6\Delta_{3469}+12\Delta_{3478}-3\Delta_{3568}+6\Delta_{36910}+$$
$$+18\Delta_{37810}-\Delta_{4567}+8\Delta_{46810}+16\Delta_{4789}+\Delta_{56710}+3\Delta_{5689})x^6y^6+(2\Delta_{12410}+4\Delta_{1259}+$$
$$+12\Delta_{1349}+12\Delta_{1358}+8\Delta_{13910}+4\Delta_{1457}+18\Delta_{14810}+4\Delta_{15710}+8\Delta_{1589}+24\Delta_{2348}+$$
$$+12\Delta_{2357}+36\Delta_{23810}+2\Delta_{2456}+32\Delta_{24710}+64\Delta_{2489}+2\Delta_{25610}+16\Delta_{2579}+4\Delta_{28910}+$$
$$+4\Delta_{34610}+32\Delta_{3479}+12\Delta_{37910}-2\Delta_{4568}+4\Delta_{46910}+12\Delta_{47810}+2\Delta_{56810}+4\Delta_{5789})x^5y^7+$$
$$+(\Delta_{12510}+3\Delta_{13410}+8\Delta_{1359}+6\Delta_{1458}+7\Delta_{14910}+5\Delta_{15810}+16\Delta_{2349}+18\Delta_{2358}+$$
$$+14\Delta_{23910}+8\Delta_{2457}+40\Delta_{24810}+9\Delta_{25710}+18\Delta_{2589}+\Delta_{3456}+18\Delta_{34710}+36\Delta_{3489}+$$
$$+\Delta_{35610}+8\Delta_{3579}+6\Delta_{38910}-\Delta_{4569}-2\Delta_{4578}+8\Delta_{47910}+\Delta_{56910}+3\Delta_{57810})x^4y^8+$$
$$+(2\Delta_{13510}+4\Delta_{1459}+2\Delta_{15910}+4\Delta_{23410}+12\Delta_{2359}+12\Delta_{2459}+16\Delta_{24910}+12\Delta_{25810}+$$
$$+4\Delta_{3457}+24\Delta_{34810}+6\Delta_{35710}+12\Delta_{3589}+4\Delta_{48910}+2\Delta_{57910})x^3y^9+(\Delta_{14510}+3\Delta_{23510}+$$
$$+8\Delta_{2459}+5\Delta_{25910}+6\Delta_{3458}+10\Delta_{34910}+9\Delta_{35810}+\Delta_{45710}+2\Delta_{4589}+$$
$$+\Delta_{58910})x^2y^{10}+(2\Delta_{24510}+4\Delta_{3459}+4\Delta_{35910}+2\Delta_{45810})xy^{11}+(\Delta_{34510}+\Delta_{45910})y^{12}, \quad (35)$$

where Δ_{ijkl}, $(1 \leq< j < k < l \leq 10)$ are the minors of the matrix M_1.

Let us prove the necessity of the condition (34). Assume the contrary, i.e. suppose that the condition

$$12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 \equiv 0 \quad (36)$$

is satisfied. We claim that in this case all minors Δ_{ijkl} $(1 \leq< j < k < l \leq 10)$ of the fourth degree vanish. Indeed, since the comitant $F_3 \not\equiv 0$ is a cubic binary form in x and y, via a center-affine transformation [3] it can be brought to one of the following 3 canonical forms (depending on its factorization over $\mathbb{C}$):

$$(i)\quad Ax(x^2 \pm y^2); \qquad (ii)\quad Ax^2y; \qquad (iii)\quad Ax^3,$$

where $A \neq 0$ due to $F_3 \not\equiv 0$. According to [5], these canonical forms can be used in order to construct the transformed system (1) via the same center-affine transformation. We shall consider each case separately.

(i) $F_3 = Ax(x^2 \pm y^2)$. Taking into consideration (6) we obtain the following values for the coefficients of the system (1):

$$a = A - g,\ b = -h,\ c = \pm\frac{1}{3}A - k,\ d = -l.$$

Then considering (36) we get the following relations:

$$b = d = e = f = h = l = 0, \quad a = 4g = \frac{4}{5}A, \quad k = \frac{3}{2}c = \pm\frac{1}{5}A.$$

However for these values of the coefficients of system (1) we obtain $F_5 \equiv 0$ and this contradicts to lemma's condition $F_3F_5 \not\equiv 0$.

(ii) $F_3 = Ax^2y$. Considering (6) in this case we have

$$a = -g,\ b = \frac{1}{3}A - h,\ c = -k,\ d = -l.$$

Then from the identity (36) we calculate

$$a = c = d = e = f = g = k = l = 0, \quad b = \frac{1}{3}A - h. \tag{37}$$

In this case the matrix M_1 takes the form

$$M_1^{(7)} = \begin{pmatrix} 0 & \frac{2}{3}(A-3h) & 0 & 0 & 0 & 0 & 0 & 2h & 0 & 0 \\ 0 & 0 & \frac{2}{3}A - 3h & 0 & 0 & 0 & 0 & 0 & 3h & 0 \\ \frac{4}{3}(A-3h) & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3}A + 4h & 0 & 0 & 0 \\ 0 & \frac{1}{3}(A-3h) & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 \end{pmatrix}$$

and $F_5 = \frac{2}{3}(2A - 15h)x^3y^2$. It is easy to observe that all 4^{th} order minors of the matrix $M_1^{(7)}$ are equal to zero.

(iii) $F_3 = Ax^3$. In the same manner as above in this case we obtain

$$a = A - g,\ b = -h,\ c = -k,\ d = -l.$$

and then from (36) we get

$$b = c = d = e = h = k = l = 0, \quad a = A - g. \tag{38}$$

For these values of the coefficients of system (1) the matrix M_1 takes the form

$$M_1^{(8)} = \begin{pmatrix} 3(A-g) & 0 & 0 & 0 & 0 & 4f & 3g & 0 & 0 & 0 \\ -f & A - 2g & 0 & 0 & 0 & 0 & f & 2g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -A + 5g & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -f & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $F_5 = -\big[fx + (5g - A)y\big]x^4$. And we observe again that all 4^{th} order minors of the matrix $M_1^{(8)}$ are equal to zero. As all possible cases are considered our claim is proved. This has completed the proof of Lemma 10.

Lemma 11. *For $F_3F_5 \not\equiv 0$ the rang of matrix M_1 is equal to three if and only if*

$$12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 \equiv 0, \tag{39}$$

where F_3, F_5 are from (6) *and L_1, L_3, L_4 are from* (8)*.*

Proof. The necessity follows from Lemma 10.

Let prove the sufficiency. Assume $12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 \equiv 0$. Since $F_3F_5 \not\equiv 0$ from the proof of Lemma 10 it follows that we need to consider only two series of relations among the coefficients: (37) and (38).

If the relations (37) hold then the sufficiency is ensured by the equality

$$F_3F_5^2 = 3(\Delta_{123}^{123} + 4\Delta_{129}^{123} + 9\Delta_{138}^{123} - 6\Delta_{378}^{123})x^8y^5.$$

In the case when (38) holds then the sufficiency is ensured by the equality

$$3F_3F_5^2 = \Delta_{167}^{124}x^{13} + 2(\Delta_{126}^{124} + 3\Delta_{168}^{124} + 4\Delta_{267}^{124})x^{12}y - (\Delta_{126}^{123} + 3\Delta_{168}^{123} + 4\Delta_{267}^{123})x^{11}y^2.$$

Lemma 11 is proved.

From Theorems 5, 9 and Lemmas 10–11 follows

Theorem 12. *The dimension of $GL(2,\mathbb{R})$-orbit of the system* (1) *is equal to*

4 for $F_5 \equiv 0,\ B_1 \not\equiv 0,$ *or*
$F_3 \equiv 0,\ 3L_1L_2 + 105C_2 + 26F_5L_7 \not\equiv 0,$ *or*
$F_3F_5(12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2) \not\equiv 0$*;*

3 for $F_5 + B_1 \equiv 0,\ L_3 \not\equiv 0,$ *or*
$F_3 + 3L_1L_2 + 105C_2 + 26F_5L_7 \equiv 0,\ L_1 \not\equiv 0,$ *or*
$F_3F_5 \not\equiv 0,\ 12L_4^2 - 3L_3F_5^2 + 6L_1F_3^2 \equiv 0$*;*

2 for $F_5 + L_3 \equiv 0,\ F_3 \not\equiv 0,$ *or*
$F_3 \equiv L_1 \equiv 0,\ F_5 \not\equiv 0$*;*

0 for $F_3 \equiv F_5 \equiv 0,$

where F_3 and F_5 are from (6)*, and B_1, L_1, L_2, L_3, L_4, L_7, C_2 are from* (8)*.*

The authors tender thanks to Professor N.I. Vulpe for effective discussion of the results of the paper.

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E. Naidenova, M.N. Popa
Institute of Mathematics and Computer Sciences
Academy of Sciences of Moldova
str. Academiei, 5
MD-2028 Chisinau
Moldova
E-mail: *hstarus@gmail.com, popam@math.md*

Received July 14, 2006

V. Orlov
Technical University of Moldova
str. Stefan cel Mare, 168
MD-2004 Chisinau, Moldova