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Minimum Cost Multicommodity Flows in Dynamic Networks and Algorithms for their Finding

Maria Fonoberova,* Dmitrii Lozovanu †

Abstract. We consider the minimum cost multicommodity flow problem in dynamic networks with time-varying capacities of arcs and transit times on arcs that depend on the sort of commodity entering them. We assume that cost functions, defined on arcs, are nonlinear and depend on time and flow, and the demand function also depends on time. Moreover, we study the problem in the case when transit time functions depend on time and flow. The modification of the time-expanded network method and new algorithms for solving the considered classes of problems are proposed.

Mathematics subject classification: 90B10, 90C35, 90C27.
Keywords and phrases: Network flows, dynamic networks, multicommodity flows, dynamic minimum cost flow problem.

1 Introduction and Problem Formulation

In this paper we study the dynamic version of the nonlinear minimum cost multicommodity network flow problem, which generalizes the classical static flow problem and extends some dynamic problems considered in [1,3,4]. We consider this problem on dynamic networks with time-varying capacities of arcs and transit times on arcs that depend on the sort of commodity entering them, what means that the transit time functions on the set of arcs for different commodities can be different. We assume that cost functions, defined on arcs, are nonlinear and depend on time and flow. Moreover, we assume that the demand function also depends on time. To solve the considered dynamic problem, we reduce it to the static one on a special time-expanded network, the structure of which differs from the standard one introduced by Ford and Fulkerson in [3]. We propose algorithms for solving the general minimum cost multicommodity flow problem and its variants with different forms of restrictions by parameters of network and time. We also consider dynamic networks with transit time functions that depend on flow and time and elaborate methods for solving problems on such networks.

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The minimum cost multicommodity dynamic flow problem asks to find the flow of a set of commodities through a network with given time horizon, satisfying all supplies and demands with minimum cost such that link capacities are not exceeded. We consider the discrete time model, in which all times are integral and bounded by horizon T. The time horizon is the time until which the flow can travel in the network and defines the makespan $\mathcal{T} = \{0, 1, \dots, T\}$ of time moments we consider. Time is measured in discrete steps, so that if one unit of flow of commodity k leaves node u at time t on arc $e = (u, v)$, one unit of flow arrives at node v at time $t + \tau_e^k$, where τ_e^k is the transit time on arc e for commodity k. Without loosing generality, we assume that no arcs enter sources or exit sinks. Accordingly the sources are nodes through which flow enters the network and the sinks are nodes through which flow leaves the network.

We consider a directed network $N = (V, E, K, w, u, \tau, d, \varphi)$ with set of vertices V, set of arcs E and set of commodities $K = \{1, 2, \dots, q\}$ that must be routed through the same network. A dynamic network N consists of capacity function w: $E \times K \times \mathcal{T} \to \mathbb{R}_+$, mutual capacity function u: $E \times \mathcal{T} \to \mathbb{R}_+$, transit time function τ: $E \times K \to \mathbb{R}_+$, demand function d: $V \times K \times \mathcal{T} \to \mathbb{R}$ and cost function φ: $E \times \mathbb{R}_+ \times \mathcal{T} \to \mathbb{R}_+$. So, $\tau_e = (\tau_e^1, \tau_e^2, \dots, \tau_e^q)$ is a vector, each component of which reflects the transit time on arc e for commodity $k \in K$. Such formulation of the problem extends models studied in [1, 2, 4, 5]. The demand function $d_v^k(t)$ satisfies the following conditions:

a) there exists $v \in V$ for every $k \in K$ with $d_v^k(0) < 0$;

b) if $d_v^k(t) < 0$ for a node $v \in V$ for commodity $k \in K$ then $d_v^k(t) = 0$, $t = 1, 2, \dots, T$;

c) $\displaystyle\sum_{t\in\mathcal{T}}\sum_{v\in V} d_v^k(t) = 0, \forall k \in K.$

Nodes $v \in V$ with $\sum_{t\in\mathcal{T}} d_v^k(t) < 0,\ k \in K$, are called sources for commodity k, nodes $v \in V$ with $\sum_{t\in\mathcal{T}} d_v^k(t) > 0,\ k \in K$, are called sinks for commodity k and nodes $v \in V$ with $\sum_{t\in\mathcal{T}} d_v^k(t) = 0,\ k \in K$, are called intermediate for commodity k.

A multicommodity dynamic flow in N is a function x: $E \times \mathcal{T} \to \mathbb{R}_+$ that satisfies the following conditions:

$$\sum_{\substack{e\in E^+(v)\\ t-\tau_e^k\ge 0}} x_e^k(t - \tau_e^k) - \sum_{e\in E^-(v)} x_e^k(t) = d_v^k(t),\ \forall\, t \in \mathcal{T},\ \forall\, v \in V,\ \forall k \in K; \tag{1}$$

$$\sum_{k\in K} x_e^k(t) \le u_e(t),\ \forall\, t \in \mathcal{T},\ \forall e \in E; \tag{2}$$

$$0 \le x_e^k(t) \le w_e^k(t),\quad \forall\, t \in \mathcal{T},\ \forall\, e \in E,\ \forall k \in K; \tag{3}$$

$$x_e^k(t) = 0,\ \forall\, e \in E,\ t = \overline{T - \tau_e^k + 1, T},\ \forall k \in K, \tag{4}$$

where $E^-(v) = \{(v, z) \,|\, (v, z) \in E\}$, $E^+(v) = \{(z, v) \,|\, (z, v) \in E\}$.

Here the function x defines the value $x_e^k(t)$ of flow of commodity k entering arc e at time t. It is easy to observe that the flow of commodity k does not enter arc

e at time t if it will have to leave the arc after time T; this is ensured by condition (4). Capacity constraints (3) mean that at most $w_e^k(t)$ units of flow of commodity k can enter arc e at time t. Mutual capacity constraints (2) mean that at most $u_e(t)$ units of flow can enter arc e at time t. Conditions (1) represent flow conservation constraints.

To model transit costs, which may change over time, we define the cost function $\varphi_e(x_e^1(t), x_e^2(t), \dots, x_e^q(t), t)$ which indicates the cost of shipping flows over arc e entering arc e at time t. The total cost of the dynamic multicommodity flow x is defined as follows:

$$c(x) = \sum_{t \in \mathcal{T}} \sum_{e \in E} \varphi_e(x_e^1(t), x_e^2(t), \dots, x_e^q(t), t).$$

The object of the minimum cost multicommodity dynamic flow problem is to find a flow that minimizes this objective function.

It is important to notice that in many practical cases the cost functions are presented in the following form:

$$\varphi_e(x_e^1(t), x_e^2(t), \dots, x_e^q(t), t) = \sum_{k \in K} \varphi_e^k(x_e^k(t), t). \tag{5}$$

The case when $\tau_e^k = 0$, $\forall\, e \in E$, $\forall\, k \in K$ and $T = 0$ can be considered as the static minimum cost multicommodity flow problem.

2 The Main Results

We show that the minimum cost multicommodity flow problem on network N can be reduced to a static problem on an auxiliary network N^T, which we name the time-expanded network. The advantage of this approach is that it turns the problem of determining a minimum cost dynamic flow problem into a classical static minimum cost flow problem on the time-expanded network, which we regard as a static representation of the dynamic network.

2.1 Constructing the Time-Expanded Network for the General Case of the Problem

So, we study the general case of the considered minimum cost flow problem when transit times on an arc are different for different commodities. We define the time-expanded network $N^T = (V^T, E^T, K, d^T, w^T, u^T, \varphi^T)$ as follows:

1. $\overline{V}^T \colon = \{v(t) \,|\, v \in V,\ t \in \mathcal{T}\}$;
2. $\widetilde{V}^T \colon = \{e(v(t)) \,|\, v(t) \in \overline{V}^T,\ e \in E^-(v),\ t \in \mathcal{T} \setminus T\}$;
3. $V^T \colon = \overline{V}^T \cup \widetilde{V}^T$;

4. $\widetilde{E}^T\colon = \{\widetilde{e}(t) = (v(t), e(v(t)))\,|\, v(t) \in \overline{V}^T$ and corresponding $e(v(t)) \in \widetilde{V}^T,\ t \in \mathcal{T} \setminus T\}$;

5. $\overline{E}^T\colon = \{e^k(t) = (e(v(t)), z(t+\tau_e^k))\,|\, e(v(t)) \in \widetilde{V}^T,\ z(t+\tau_e^k) \in \overline{V}^T,\ e = (v,z) \in E,\ 0 \le t \le T - \tau_e^k,\ k \in K\}$;

6. $E^T\colon = \overline{E}^T \cup \widetilde{E}^T$;

7. ${d^k_{v(t)}}^T\colon = d^k_v(t)$ for $v(t) \in \overline{V}^T,\ k \in K$;
${d^k_{e(v(t))}}^T\colon = 0$ for $e(v(t)) \in \widetilde{V}^T,\ k \in K$;

8. ${w^l_{e^k(t)}}^T\colon = \begin{cases} w^k_e(t), & \text{if } l = k \text{ for } e^k(t) \in \overline{E}^T,\ k \in K; \\ 0, & \text{if } l \neq k \ \text{ for } e^k(t) \in \overline{E}^T,\ k \in K \end{cases}$
and ${w^l_{\widetilde{e}(t)}}^T = \infty$ for $\widetilde{e}(t) \in \widetilde{E}^T,\ l \in K$;

9. ${u_{\widetilde{e}(t)}}^T\colon = u_e(t)$ for $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$;
${u_{e^k(t)}}^T\colon = \infty$ for $e^k(t) \in \overline{E}^T,\ k \in K$;

10. $\varphi^T_{\widetilde{e}(t)}({x^1_{\widetilde{e}(t)}}^T, {x^2_{\widetilde{e}(t)}}^T, \dots, {x^q_{\widetilde{e}(t)}}^T)\colon = \varphi_e(x^1_e(t), x^2_e(t), \dots, x^q_e(t), t)$ for $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$;
$\varphi^T_{e^k(t)}({x^1_{e^k(t)}}^T, {x^2_{e^k(t)}}^T, \dots, {x^q_{e^k(t)}}^T)\colon = 0$ for $e^k(t) \in \overline{E}^T,\ k \in K$.

The correspondence between flows in the dynamic network and the static time-expanded network is presented by the following lemma.

Lemma 1. *Let $x^T\colon E^T \to \mathbb{R}_+$ be a multicommodity flow in the static network N^T. Then the multicommodity flow $x\colon E \times \mathcal{T} \to \mathbb{R}_+$ in the dynamic network N can be defined in the following way. Let $e^k(t) = (e(v(t)), z(t + \tau_e^k)) \in \overline{E}^T$, $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$. Then the dynamic flow $x_e(t)$ on arc $e = (v,z)$ is determined as follows: $x^k_e(t) = {x^k_{e^k(t)}}^T = {x^k_{\widetilde{e}(t)}}^T,\ \forall k \in K,\ \forall t \in \mathcal{T}$.*

If $x\colon E \times \mathcal{T} \to \mathbb{R}_+$ is a multicommodity flow in the dynamic network N, then the multicommodity flow $x^T\colon E^T \to \mathbb{R}_+$ in the static network N^T can be determined as follows. Let $x_e(t)$ be a dynamic multicommodity flow on arc $e = (v,z) \in E$. Then the tuple $({x_{\widetilde{e}(t)}}^T, {\overline{x}_{\overline{e}(t)}}^T) = {x_{e(t)}}^T$ is a corresponding static multicommodity flow, where ${x_{\widetilde{e}(t)}}^T$ is a static multicommodity flow on additional arc $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$, at that ${x^k_{\widetilde{e}(t)}}^T = x^k_e(t),\ \forall k \in K;$ ${\overline{x}_{\overline{e}(t)}}^T = ({x_{e^1(t)}}^T, {x_{e^2(t)}}^T, \dots, {x_{e^q(t)}}^T)$ is a q dimension vector of static multicommodity flows on arcs $e^k(t) = (e(v(t)), z(t+\tau_e^k)) \in \overline{E}^T,\ k \in K$, at that ${x^k_{e^k(t)}}^T = x^k_e(t);$ ${x^l_{e^k(t)}}^T = 0,\ l \neq k$.

Proof. To prove the first part of the lemma we have to show that conditions (1)–(4) for defined above x in the dynamic network N are true. These conditions evidently result from the following definition of multicommodity flows in the static network N^T:

$$\sum_{e(t-\tau_e^k)\in E^+(v(t))} {x^k_{e(t-\tau_e^k)}}^T - \sum_{e(t)\in E^-(v(t))} {x^k_{e(t)}}^T = {d^k_{v(t)}}^T, \quad \forall\, t \in \mathcal{T},\ \forall\, v(t) \in V^T,\ \forall k \in K; \tag{6}$$

$$\sum_{k\in K} {x^k_{e(t)}}^T \le {u_{e(t)}}^T,\ \forall e(t) \in E^T,\ \forall\, t \in \mathcal{T}; \tag{7}$$

$$0 \le {x^k_{e(t)}}^T \le {w^k_{e(t)}}^T,\ \forall\, e(t) \in E^T,\ \ \forall\, t \in \mathcal{T},\ \forall k \in K; \tag{8}$$

$${x^k_{e(t)}}^T = 0,\ \forall\, e(t) \in E,\ t = \overline{T - \tau_e^k + 1, T},\ \forall k \in K. \tag{9}$$

In order to prove the second part of the lemma it is sufficient to show that conditions (6)–(9) hold. Correctness of these conditions results from the procedure of constructing the time-expanded network, correspondence between flows in static and dynamic networks and the satisfied conditions (1)–(4). □

The following theorem holds.

Theorem 2. *If ${x^*}^T$ is a static minimum cost multicommodity flow in the static network N^T, then the corresponding according to Lemma 1 dynamic multicommodity flow x^* in the dynamic network N is also a minimum cost one and vice-versa.*

Proof. Taking into account the correspondence between static and dynamic multicommodity flows on the basis of Lemma 1, we obtain that costs of multicommodity flow in the time-expanded network N^T and multicommodity flow in the dynamic network N are equal. Indeed, to solve the minimum cost multicommodity flow problem in the static time-expanded network N^T, we have to solve the following problem:

$$c^T(x) = \sum_{t\in\mathcal{T}} \sum_{e(t)\in E^T} \varphi^T_{e(t)}(x^1_{e(t)}, x^2_{e(t)}, \ldots, x^q_{e(t)}) \to \min$$

subject to (6)–(9). □

2.2 The Case of the Problem with Separable Cost Functions and without Mutual Capacity of Arcs

The minimum cost flow problem with separable cost functions (5) and without mutual capacity constraints for arcs allows us to simplify the procedure of constructing the time-expanded network. In this case we don't have to add a new set of vertexes $\widetilde{V}^T$ and a new set of arcs $\widetilde{E}^T$. In that way the time-expanded network N^T is defined as follows:

1. $V^T\!:\ = \{v(t) \,|\, v \in V,\ t \in \mathcal{T}\}$;
2. $E^T\!:\ = \{e^k(t) = (v(t), z(t + \tau_e^k)) \,|\, e = (v, z) \in E,\ 0 \le t \le T - \tau_e^k,\ k \in K\}$;

3. ${d^k_{v(t)}}^T\colon = d^k_v(t)$ for $v(t) \in V^T,\ k \in K$;

4. ${w^l_{e^k(t)}}^T\colon = \begin{cases} w^k_e(t), & \text{if } l = k \text{ for } e^k(t) \in E^T,\ k \in K; \\ 0, & \text{if } l \neq k \text{ for } e^k(t) \in E^T,\ k \in K; \end{cases}$

5. ${\varphi^l_{e^k(t)}}^T({x^l_{e^k(t)}}^T)\colon = \begin{cases} \varphi^k_e(x^k_e(t), t), & \text{if } l = k \text{ for } e^k(t) \in E^T,\ k \in K; \\ 0, & \text{if } l \neq k \text{ for } e^k(t) \in E^T,\ k \in K. \end{cases}$

The correspondence between flows in the dynamic network N and the static network N^T is defined as follows. Let $x^T\colon E^T \to \mathbb{R}_+$ be a multicommodity flow in the static network N^T. Then the following flow $x\colon E \times \mathcal{T} \to \mathbb{R}_+$, where $x^k_e(t) = {x^k_{e^k(t)}}^T$, $\forall e \in E,\ \forall k \in K,\ \forall t \in \mathcal{T}$, represents the multicommodity flow in the dynamic network N. If $x\colon E \times \mathcal{T} \to \mathbb{R}_+$ is a multicommodity flow in the dynamic network N, then the flow $x^T\colon E^T \to \mathbb{R}_+$, where ${x^k_{e^k(t)}}^T = x^k_e(t),\ {x^l_{e^k(t)}}^T = 0,\ \forall e^k(t) \in E^T,\ \forall k \in K,\ l \neq k$, represents the multicommodity flow in the static network N^T.

As above, it can be proved that if x^{*T} is a static minimum cost multicommodity flow in the static network N^T, then the corresponding dynamic multicommodity flow x^* in the dynamic network N is also a minimum cost flow and vice-versa.

2.3 The Case of the Problem with Common Transit Times on Arcs for Commodities

In the case of the minimum cost flow problem with common transit times for each commodity the time-expanded network also can be constructed in more simple way without adding a new set of vertexes $\widetilde{V}^T$ and a new set of arcs $\widetilde{E}^T$. Thus the time-expanded network N^T is defined as follows:

1. $V^T\colon = \{v(t) \,|\, v \in V,\ t \in \mathcal{T}\}$;

2. $E^T\colon = \{e(t) = (v(t), z(t + \tau_e)) \,|\, e = (v, z) \in E,\ 0 \leq t \leq T - \tau_e\}$;

3. ${d^k_{v(t)}}^T\colon = d^k_v(t)$ for $v(t) \in V^T,\ k \in K$;

4. $u^T_{e(t)}\colon = u_e(t)$ for $e(t) \in E^T$;

5. ${w^k_{e(t)}}^T\colon = w^k_e(t)$ for $e(t) \in E^T,\ k \in K$;

6. $\varphi^T_{e(t)}({x^1_{e(t)}}^T, {x^2_{e(t)}}^T, \ldots, {x^q_{e(t)}}^T)\colon = \varphi_e(x^1_e(t), x^2_e(t), \ldots, x^q_e(t), t)$ for $e(t) \in E^T$.

In this case the correspondence between flows in the dynamic network N and the static network N^T is defined in the following way. Let $x^T\colon E^T \to \mathbb{R}_+$ be a multicommodity flow in the static network N^T. Then the following flow $x\colon E \times \mathcal{T} \to$

$\mathbb{R}_+$, where $x_e^k(t) = {x_{e(t)}^k}^T$, $\forall e \in E$, $\forall k \in K$, $\forall t \in \mathcal{T}$, represents the multicommodity flow in the dynamic network N. If x: $E \times \mathcal{T} \to \mathbb{R}_+$ is a multicommodity flow in the dynamic network N, then the flow x^T: $E^T \to \mathbb{R}_+$, where ${x_{e(t)}^k}^T = x_e^k(t)$, $\forall e(t) \in E^T$, $\forall k \in K$, $\forall t \in \mathcal{T}$, represents the multicommodity flow in the static network N^T.

As above, it can be proved that if x^{*T} is a static minimum cost multicommodity flow in the static network N^T, then the corresponding dynamic multicommodity flow x^* in the dynamic network N is also a minimum cost flow and vice-versa.

3 The Algorithm and Examples

On the basis of results from the previous section we can propose the following algorithm for solving the minimum cost multicommodity dynamic flow problem. In such a way, to solve the minimum cost multicommodity flow problem in N we have to build the time-expanded network N^T for the given dynamic network N, after what to solve the classical minimum cost multicommodity flow problem in the static network N^T and then to reconstruct according to Lemma 1 and Theorem 2 the solution of the static problem in N^T to the dynamic problem in N.

In the following we construct in different cases the time-expanded network N^T for the dynamic network N given on Fig. 1 with two commodities.

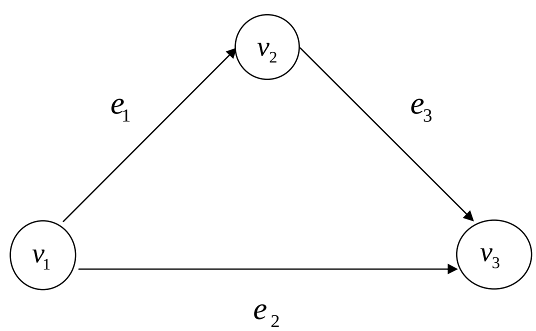

Figure 1. The dynamic network

The set of time moments we consider is $\mathcal{T} = \{0, 1, 2, 3\}$. The transit times on each arc for each commodity are defined in the following way: $\tau_{e_1}^1 = 2$, $\tau_{e_1}^2 = 1$, $\tau_{e_2}^1 = 1$, $\tau_{e_2}^2 = 3$, $\tau_{e_3}^1 = 1$, $\tau_{e_3}^2 = 2$. The mutual capacity, individual capacity, demand and cost functions are considered to be known.

The time-expanded network N^T for the dynamic network N in the general case is represented on Fig. 2. The time-expanded network N^T for the dynamic network N in the case of separable cost functions and without mutual capacity of arcs is represented on Fig. 3. The time-expanded network N^T for the dynamic network N in the case of common transit times for each commodity with $\tau_{e_1} = 1$, $\tau_{e_2} = 1$, $\tau_{e_3} = 2$ is represented on Fig. 4.

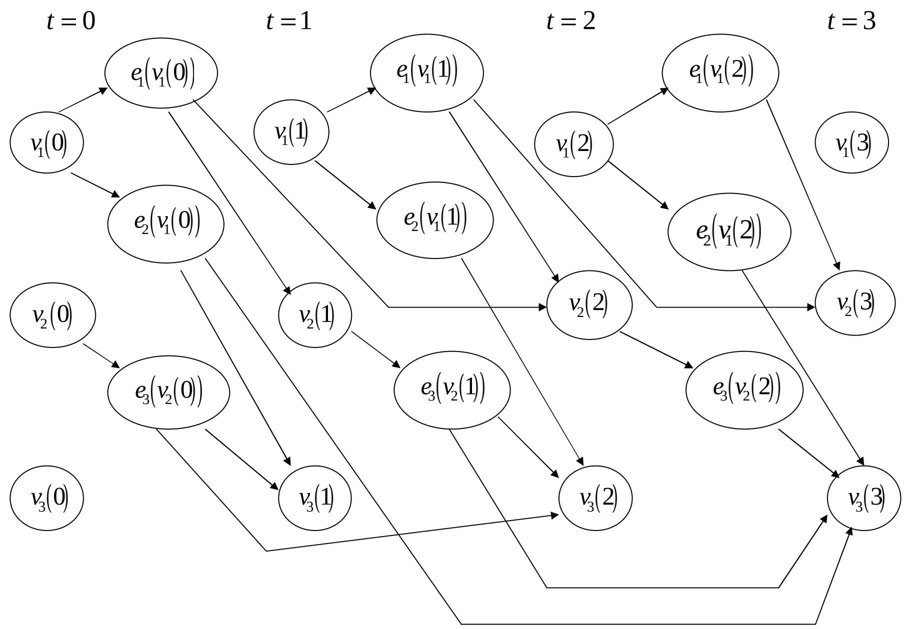

Figure 2. The time-expanded network

Remark 1. The proposed above approach can be used to solve some more general cases of the minimum cost dynamic multicommodity flow problem such as the problem when only a part of the flow is dumped into the considered network at the time 0, when flow storage at nodes is allowed and when the cost functions also depend on the flow at the nodes. The same reasoning to solve the minimum cost flow problem in the dynamic networks and its generalization can be held in the case when, instead of the condition (3) in the definition of the multicommodity dynamic flow, the following condition takes place: $w_e'^k(t) \leq x_e^k(t) \leq w_e''^k(t)$, $\forall\, t \in \mathcal{T}$, $\forall\, e \in E$, $\forall\, k \in K$, where $w_e'^k(t)$ and $w_e''^k(t)$ are lower and upper bounds of the capacity of the arc e respectively.

Remark 2. The maximum multicommodity dynamic flow problem also can be solved by reduction to a static problem in an auxiliary time-expanded network N^T, which is defined as above but without demand and cost functions.

4 Determining the Minimum Cost Flows in Dynamic Networks with Transit Time Functions that Depend on Flow and Time

In the problems studied in the previous sections the transit time functions are assumed to be constant at every moment of time for each arc of the network. A more general class of dynamic multicommodity flow problems can be obtained if the transit time functions τ_e^k, $\forall e \in E$, $\forall k \in K$, depend on flows and on time. From the practical point of view we can state that the transit time function possesses the

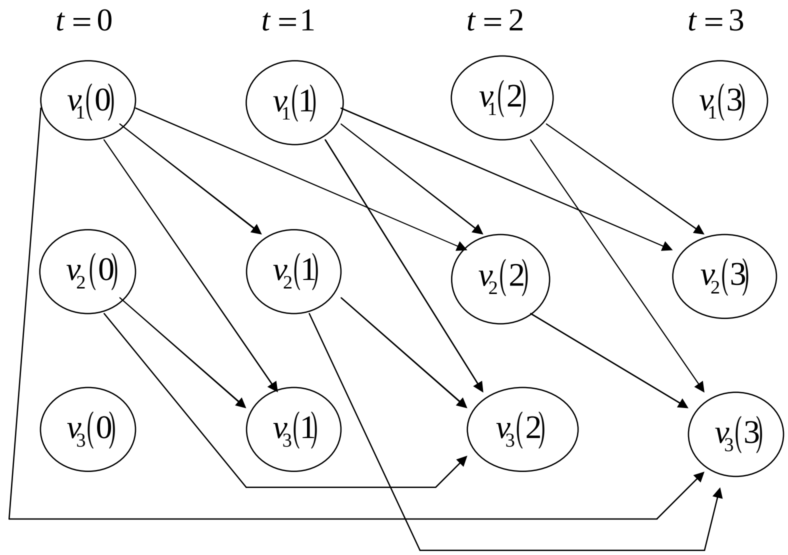

Figure 3. The time-expanded network (case of separable cost functions and without mutual capacity of arcs)

property of being a non-negative and non-decreasing function. So, we will assume that the transit time function is a non-decreasing non-negative step function. First we will describe the method for solving the minimum cost single-commodity flow problem in dynamic networks with transit time functions that depend on flow and time. Then the dynamic multicommodity flow problem with transit time functions that depend on flows and time can be solved by using the similar approach extended to the multicommodity case of the problem. The detailed elaboration of the time-expanded network method for such class of the problem can be obtained for the case of separable transit-time functions $\tau_e^k(x_e^1, x_e^2, \dots, x_e^q, t) = \sum_{p=1}^{q} \tau_e^k(x_e^p, t)$, $\forall e \in E$, $\forall t \in \mathcal{T}$, $\forall k \in K$, where the functions $\tau_e^k(x_e^p, t)$ satisfy the conditions described below.

4.1 The Minimum Cost Dynamic Flow Problem with Transit Time Functions that Depend on Flow and Time

Let us formulate the minimum cost single-commodity flow problem in dynamic networks with transit time functions that depend on flow and time. Let be given a directed network $N = (V, E, u', u'', \tau, d, \varphi)$ with set of vertices V and set of arcs E, lower and upper capacity functions u', u'': $E \times \mathcal{T} \to \mathbb{R}_+$, transit time function τ: $E \times \mathcal{T} \times \mathbb{R}_+ \to \mathbb{R}_+$, demand function d: $V \times \mathcal{T} \to \mathbb{R}$ and cost function φ: $E \times \mathbb{R}_+ \times \mathcal{T} \to \mathbb{R}_+$. As above, we consider the discrete time model, in which all times are integral and bounded by a time horizon T, which defines the set $\mathcal{T} = \{0, 1, \dots, T\}$ of time moments we consider. We suppose that all flow is dumped into the network at

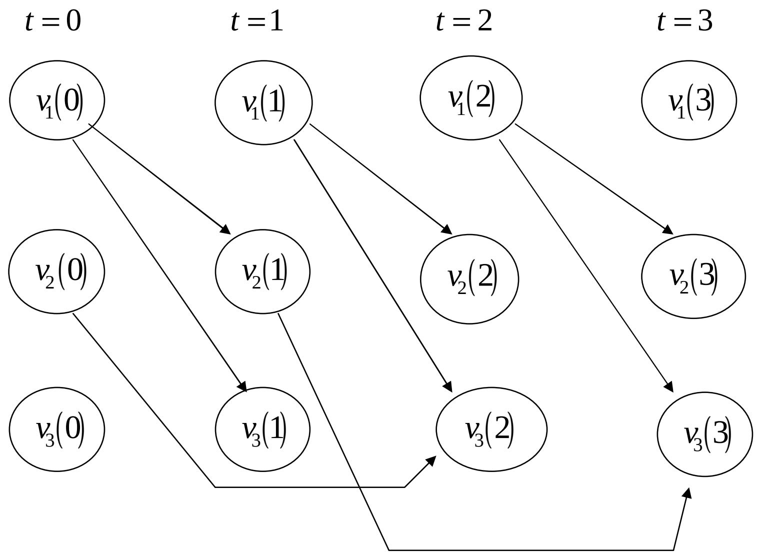

Figure 4. The time-expanded network (case of common transit times for each commodity)

time 0 and the supply is equal to the demand, i.e. $\sum_{t\in\mathcal{T}}\sum_{v\in V} d_v(t) = 0$. Without losing generality, we assume that no arcs enter sources or exit sinks.

A dynamic flow in N is represented by a function x: $E\times\mathcal{T}\to\mathbb{R}_+$, which defines the value $x_e(t)$ of flow entering arc e at time t. Since we require that all arcs must be empty after time horizon T, the following implication must hold for all $e\in E$ and $t\in\mathcal{T}$: if $x_e(t) > 0$, then $t+\tau_e(x_e(t),t)\le T$. The dynamic flow x must satisfy the flow conservation constraints, which mean that at any time moment $t\in\mathcal{T}$ for every vertex $v\in V$ the difference between the total amount of flow that leaves node v and the total amount of flow that enters node v, is equal to $d_v(t)$. The dynamic flow x is called feasible if it satisfies the following capacity constraints: $u'_e(t)\le x_e(t)\le u''_e(t)$, $\forall\, t\in\mathcal{T}$, $\forall\, e\in E$.

The total cost of the dynamic flow x is defined as follows:

$$F(x) = \sum_{t\in\mathcal{T}}\sum_{e\in E}\varphi_e(x_e(t),t).$$

The object of the minimum cost dynamic flow problem is to find a feasible flow that minimizes this objective function.

4.2 The Method for Solving the Problem

We propose an approach for solving the formulated problem, which is based on reduction of this problem to a static one on a special auxiliary time-expanded network N^T. We define the network $N^T = (V^T, E^T, d^T, u'^T, u''^T, \varphi^T)$ as follows:

1. $\overline{V}^T\colon\ = \{v(t)\,|\,v \in V,\ t \in \mathcal{T}\}$;

2. $\widetilde{V}^T\colon\ = \{e(v(t))\,|\,v(t) \in \overline{V}^T,\ e \in E^-(v),\ t \in \mathcal{T} \setminus T\}$;

3. $V^T\colon\ = \overline{V}^T \cup \widetilde{V}^T$;

4. $\widetilde{E}^T\colon\ = \{\widetilde{e}(t) = (v(t), e(v(t)))\,|\,v(t) \in \overline{V}^T$ and corresponding $e(v(t)) \in \widetilde{V}^T,\ t \in \mathcal{T} \setminus T\}$;

5. $\overline{E}^T\colon\ = \{e^p(t) = (e(v(t)), z(t+\tau_e^p))\,|\,e(v(t)) \in \widetilde{V}^T,\ z(t+\tau_e^p) \in \overline{V}^T,\ e = (v,z) \in E,\ 0 \le t \le T - \tau_e^p,\ p \in P\}$;

6. $E^T\colon\ = \overline{E}^T \cup \widetilde{E}^T$;

7. $d^T_{v(t)}\colon\ = d_v(t)$ for $v(t) \in \overline{V}^T,\ k \in K$;
   $d^T_{e(v(t))}\colon\ = 0$ for $e(v(t)) \in \widetilde{V}^T$;

8. ${u'}_{\widetilde{e}(t)}^T\colon\ = u'_e(t)$ for $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$;
   ${u''}_{\widetilde{e}(t)}^T\colon\ = u''_e(t)$ for $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$;
   ${u'}_{e^p(t)}^T\colon\ = \overline{x_e^{p-1}}(t)$ for $e^p(t) \in \overline{E}^T,\ p \in P$, where $\overline{x_e^0}(t) = 0$;
   ${u''}_{e^p(t)}^T\colon\ = \overline{x_e^p}(t)$ for $e^p(t) \in \overline{E}^T,\ p \in P$;

9. $\varphi^T_{\widetilde{e}(t)}({x_{\widetilde{e}(t)}}^T)\colon\ = \varphi_e(x_e(t), t)$ for $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$;
   $\varphi^T_{e^p(t)}({x_{e^p(t)}}^T)\colon\ = \varepsilon_p$ for $e^p(t) \in \overline{E}^T,\ p \in P$, where $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{|P|}$ are small numbers.

To make the notations more clear we construct a part of the time-expanded network for the fixed moment of time t for the given arc $e = (v, z)$ with the transit time function presented by Fig. 5.

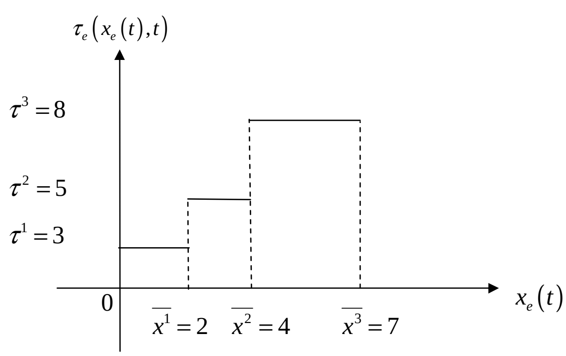

Figure 5. The transit time function

The constructed part of the time-expanded network is given on Fig. 6, where lower and upper capacities are written above each arc and the cost is written below each arc.

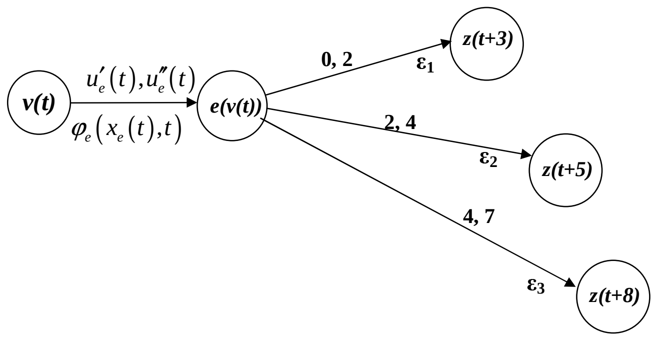

Figure 6. The constructed part of the time-expanded network

The following lemma gives the correspondence between flows in the dynamic network and the time-expanded network.

Lemma 3. *Let $x^T\colon E^T \to \mathbb{R}_+$ be a flow in the static network N^T. Then the flow $x\colon E \times \mathcal{T} \to \mathbb{R}_+$ in the dynamic network N can be defined in the following way. Let $e^p(t) = (e(v(t)), z(t + \tau_e^p)) \in \overline{E}^T$, $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$. Then the dynamic flow $x_e(t)$ on arc $e = (v, z)$ is determined as follows: $x_e(t) = x^T_{\widetilde{e}(t)} = x^T_{e^p(t)}$, $\forall t \in \mathcal{T}$, where $p \in P$ is such that $x^T_{\widetilde{e}(t)} \in (\overline{x_e^{p-1}}(t), \overline{x_e^p}(t)]$.*

If $x\colon E \times \mathcal{T} \to \mathbb{R}_+$ is a flow in the dynamic network N, then the flow $x^T\colon E^T \to \mathbb{R}_+$ in the static network N^T can be determined as follows. Let $x_e(t)$ be a dynamic flow on arc $e = (v, z) \in E$. Then the tuple $({x_{\widetilde{e}(t)}}^T, {\overline{x}_{\overline{e}(t)}}^T) = {x_{e(t)}}^T$ is a corresponding static flow, where ${x_{\widetilde{e}(t)}}^T$ is a static flow on additional arc $\widetilde{e}(t) = (v(t), e(v(t))) \in \widetilde{E}^T$, at that $x^T_{\widetilde{e}(t)} = x_e(t)$; ${\overline{x}_{\overline{e}(t)}}^T = ({x_{e^1(t)}}^T, {x_{e^2(t)}}^T, \dots, {x_{e^{|P|}(t)}}^T)$ is a $|P|$ dimensional vector of static flows on arcs $e^p(t) = (e(v(t)), z(t + \tau_e^p)) \in \overline{E}^T$, $p \in P$, at that $x^T_{e^p(t)} = x_e(t)$ if $x_e(t) \in (\overline{x_e^{p-1}}(t), \overline{x_e^p}(t)]$ or $x^T_{e^p(t)} = 0$ otherwise.

The proof of this lemma is similar to the proof of Lemma 1.

The following theorem holds.

Theorem 4. *If x^{*T} is a static minimum cost flow in the static network N^T, then the corresponding according to Lemma 3 dynamic flow x^* in the dynamic network N is also a minimum cost flow and vice-versa.*

In such a way, the minimum cost multicommodity flow problem in the dynamic network can be solved by static flow computations in the corresponding time-expanded network. To solve the minimum cost flow problem in dynamic networks with transit time functions that depend on flow and time we construct the time-expanded network, then solve the static minimum cost flow problem and reconstruct the solution of the static problem to the dynamic problem.

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Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
Academiei str. 5, MD-2028 Chişinău
Moldova

E-mail: *fonoberov@math.md*, *lozovanu@math.md*

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