BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 1(53), 2007, Pages 101–106
ISSN 1024–7696

$GL(2, R)$-orbits in a competing species model

Raluca Mihaela Georgescu, Elena Naidenova

Abstract. A particular model with two parameters describing the dynamics of two competing species is analyzed from algebraic viewpoint involving the $GL(2, R)$-orbits.

Mathematics subject classification: 34C14.
Keywords and phrases: Differential system, parametric portrait, Lie algebra of the operators, $GL(2, \mathbb{R})$-orbit, first integral.

1 Introduction

In this paper we study a particular family of planar vector fields modelling the dynamics of two competing populations and corresponding to a couple of similar species of animals which compete with each other for a common food supply.

The competition between two species is modelled by the competitive Lotka-Volterra system

$$\begin{cases} \dot{x}_1 = x_1(r_1 - a_{11}x_1 - a_{12}x_2), \\ \dot{x}_2 = x_2(r_2 - a_{21}x_1 - a_{22}x_2), \end{cases} \quad (1)$$

where $x_1,\ x_2$ represent the number of the populations of the two species, $r_1,\ r_2$, represent the growth rate of the species, and $a_{ij} > 0,\ i, j = 1, 2$, represent the competitive impacts of species j on the growth of species i.

The model we study in this paper is proposed as an application by M. W. Hirsch, S. Smale and R. L. Devaney in [2] and has the form

$$\begin{cases} \dot{x}_1 = x_1(a - x_1 - ax_2), \\ \dot{x}_2 = x_2(b - bx_1 - x_2), \end{cases} \quad (2)$$

where $x_1,\ x_2$ represent the number of the populations of the two species, and a and b are positive parameters. The system (2) is a particular case of (1).

In [3] the equilibrium points are found and the phase portrait and the parameter portraits are determined. Herein we determine the $GL(2, R)$-orbits of the system (2) and construct the corresponding Lie algebra. Then we determine the first integrals of the system (2) for particular values of the parameters a and b.

2 $GL(2,R)$-orbits of the system (2)

In the tensorial form the system (2) reads

$$\dot{x}^j = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta, \quad (j,\alpha,\beta = 1,2). \tag{3}$$

The dimensions of the $GL(2,R)$-orbits of the system (3) are given by

Theorem 1. *[1]. The $GL(2,R)$-orbits of the system (3) has the dimension:*

4, *if* $K_5(K_9+\beta)\neq 0$,
or $K_5 \not\equiv 0,\ K_9+\beta\equiv 0,\ K_1\not\equiv 0,\ W_1\not\equiv 0$,
or $K_5 \not\equiv 0,\ K_9+\beta\equiv 0,\ K_1\equiv 0,\ K_2\not\equiv 0,\ W_2\not\equiv 0$,
or $K_5\equiv 0,\ K_1\not\equiv 0,\ I_4\neq 0$;

3, *if* $K_5 \not\equiv 0,\ K_9+\beta\equiv 0,\ K_1\not\equiv 0,\ W_1\equiv 0$,
or $K_5 \not\equiv 0,\ K_9+\beta\equiv 0,\ K_1\equiv 0,\ K_2\not\equiv 0, W_2=0$,
or $K_5 \not\equiv 0,\ K_9+\beta\equiv 0,\ K_1\equiv 0,\ K_2\equiv 0, K_7\not\equiv 0$,
or $K_5\equiv 0,\ K_1\not\equiv 0,\ I_4=0,\ K_2\not\equiv 0$;

2, *if* $K_5 \not\equiv 0,\ K_9+\beta\equiv 0,\ K_1\equiv 0,\ K_2\equiv 0,\ K_7\equiv 0$,
or $K_5\equiv 0,\ K_1\not\equiv 0,\ I_4=0,\ K_2\equiv 0$,
or $K_5\equiv 0,\ K_1\equiv 0,\ K_2\not\equiv 0$;

0, *if* $K_5\equiv 0,\ K_1\equiv 0,\ K_2\equiv 0$,

where $\beta = 27I_8 - I_9 - 18I_7$, $W_1 = K_1(2K_{11} - I_1K_5 - 2K_1K_2) + K_2K_6$, $W_2 = 3K_2K_7 - 2K_3K_5$, *the invariants* $I_1,\ I_4,\ I_7,\ I_8,\ I_9$ *and the comitants* $K_1,\ K_2,\ K_5,\ K_6,\ K_7,\ K_9,\ K_{11}$ *having the forms [6]:*

$$I_1 = a^\alpha_\alpha,\ I_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma}\varepsilon^{pq},\ I_7 = a^\alpha_{pr} a^\beta_{\alpha q} a^\delta_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},\ I_8 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\delta s} a^\delta_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},$$

$$I_9 = a^\alpha_{pr} a^\beta_{\beta q} a^\gamma_{\gamma s} a^\delta_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs},\ K_1 = a^\alpha_{\alpha\beta},\ K_2 = a^p_\alpha x^\alpha\varepsilon^{pq},\ K_3 = a^\alpha_\beta a^\beta_{\alpha\gamma}x^\gamma,$$

$$K_5 = a^p_{\alpha\beta}x^\alpha x^\beta x^p\varepsilon^{pq},\ K_6 = a^\alpha_{\alpha\beta}a^\beta_{\gamma\delta}x^\gamma x^\delta,\ K_7 = a^\alpha_{\beta\gamma}a^\beta_{\alpha\delta}x^\gamma x^\delta,\ K_9 = a^\alpha_{\alpha p}a^\beta_{\gamma q}a^\gamma_{\alpha\beta}x^\delta,$$

$$K_{11} = a^p_\alpha a^\alpha_{\beta\gamma}x^\beta x^\gamma x^q\varepsilon_{pq}.$$

For system (2) we have $a^1_1 = a$, $a^1_2 = 0$, $a^2_1 = 0$, $a^2_2 = b$, $a^1_{11} = -1$, $a^1_{12} = -a/2$, $a^1_{22} = 0$, $a^2_{11} = 0$, $a^2_{12} = -b/2$, $a^2_{22} = -1$. Therefore, in our particular case, we obtain

$$I_1 = a+b,\ I_4 = \frac{1}{4}(2+a)(a-b)(2+b),$$

$$I_7 = -a^2/4 + a/2 - 3a^2b/8 + ab/2 + b/2 - a^2b^2/4 - 3ab^2/8 - b^2/4,$$

$$I_8 = -a^2/4 + a/2 - a^2b/8 + b/2 - a^2b^2/4 - ab^2/8 - b^2/4,$$

$$I_9 = -(a+2)(b+2)(a+b+2ab-4)/8,\ K_1 = -(1+b/2)x - (1+a/2)y,$$

$$K_2 = (a-b)xy,\ K_3 = -(a+b^2/2)x - (b+a^2/2)y,$$

$$K_5 = (b-1)x^2y - (a-1)xy^2,\ K_6 = (1+b/2)x^2 + (a+ab+b)xy + (1+a/2)y^2,$$

$$K_7 = (1 + b^2/4)x^2 + (a + ab/2 + b)xy + (a^2/4 + 1)y^2,$$
$$K_9 = (1 - ab/2 - b/2)x - (1 - a/2 - ab/2)y,\ K_{11} = (b^2 - a)x^2y + (b - a^2)xy^2,$$
$$\beta = -2(b - 1)^2(a - 1)^2,\ W_1 = x^2y^2(a + b + 2ab - 4)(a - b)/2,$$
$$W_2 = (a + 2ab + b^3/4 + 3ab^2/4 - b^2 - 3b)x^3y + (5a^2b/2 - 5ab^2/2 - 2b + 2a)x^2y^2 -$$
$$-(3a^2b/4 - a^2 + b - 3a + a^3/4 + 2ab)xy^3,$$

whence, the theorem holds

Theorem 2. *$GL(2,R)$-orbits of the system (2) has the dimension:*
4, *if $a \neq 1$ or $b \neq 1$,*
2, *if $a = b = 1$.*

3 The parametric portrait and the phase portraits for the system (2)

The number and the nature of the equilibrium points of the system (2) are studied in [3]. Namely, from the biological viewpoint $(x, y > 0)$ and for $a, b \geq 0$, in the parametric portrait there are 12 strata (Fig. 1), i.e. there are 12 topological nonequivalent corresponding phase portraits (Fig. 2).

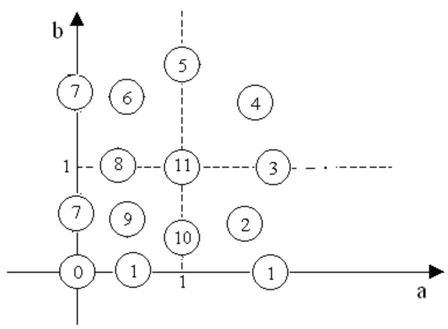
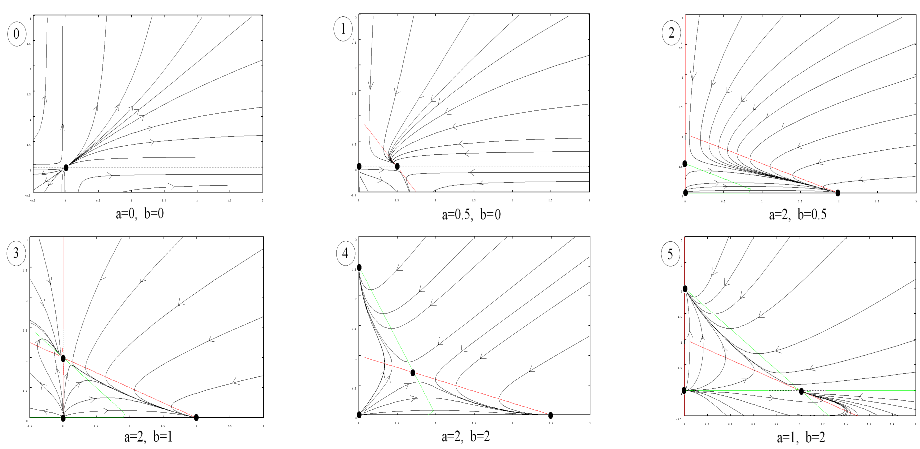


Fig. 1 The parametric portrait



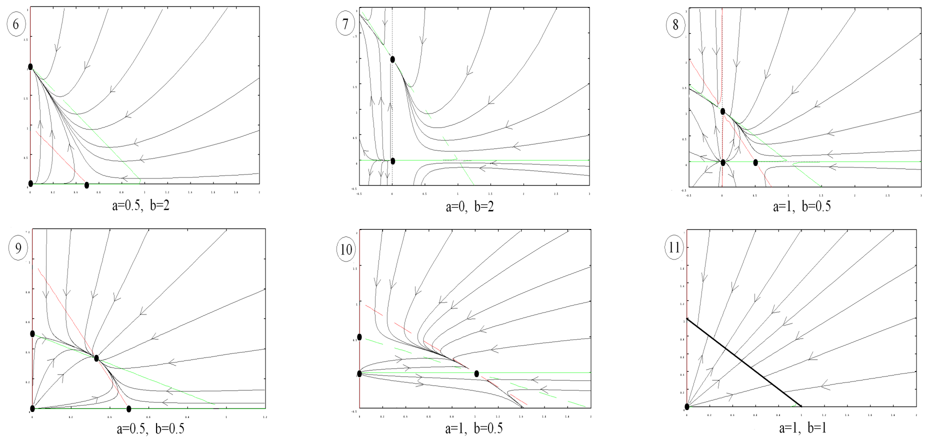


Fig. 2. Phase portraits for (2)

Remark 1. The case 11 in Fig.2 corresponds to the orbit of dimension 2, and the others to the orbit of dimension 4.

4 Lie algebras and some first integrals of the system (2)

To complete our algebraic investigation of the system (2), we attempted to construct Lie algebras for each system from Section 3. We supposed that system (2) admits the Lie algebra corresponding to the linear group of transformations having as generator the operator [5]:

$$X = \xi_1 \frac{\partial}{\partial x} + \xi_2 \frac{\partial}{\partial y}, \tag{4}$$

where

$$\xi_1 = Ax + By + C, \quad \xi_2 = Dx + Ey + F.$$

It was found that the only system which admits such an algebra correspond to the orbit of dimension 2. One can verify using CSA Mathematica or Maple that the system (2) on the orbit with dimension 4 does not admit such algebra. Similarly we have found that this systems does not admit Lie algebra having as generator the operator (4) with coefficient vectors as follows:

1. $\xi_1 = A_1x^2 + A_2xy + A_3y^2 + A_4x + A_5y + A_6,$

$$\xi_2 = B_1x^2 + B_2xy + B_3y^2 + B_4x + B_5y + B_6;$$

2. $\xi_1 = A_1x^3 + A_2x^2y + A_3xy^2 + A_4y^3 + A_5x + A_6y + A_7,$

$$\xi_2 = B_1x^3 + B_2x^2y + B_3xy^2 + B_4y^3 + B_5x + B_6y + B_7;$$

3. $\xi_1 = A_1x^4 + A_2x^3y + A_3x^2y^2 + A_4xy^3 + A_5y^4 + A_6x + A_7y,$

$$\xi_1 = B_1x^4 + B_2x^3y + B_3x^2y^2 + B_4xy^3 + B_5y^4 + B_6x + B_7y;$$

$$4.\ \ \xi_1 = \frac{A_1x + B_1y + C_1}{D_1x + E_1y + F_1}, \quad \xi_2 = \frac{A_2x + B_2y + C_2}{D_2x + E_2y + F_2}.$$

Assume that $x^1 = x,\ x^2 = y$. So, following [3] we have considered the cases:

1. The system (2) for $a = b = 1$ corresponds to the case 11 (Fig.2). As this system is on the orbits with dimension 2, it admits the one-dimensional Lie algebra with operator $X = -x\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. By means of this operator the first integral $\mathcal{F}_1 \equiv \frac{y}{x} = C_1$ was found.

2. The system (2) for $a = b = 0$ corresponds to the case 0 (Fig.2). It admits the first integral $\mathcal{F}_2 \equiv -\frac{1}{y} + \frac{1}{x} + C_2 = 0$.

3. The system (2) for $a \neq 0,\ b = 0$ corresponds to the case 1 (Fig.2). It admits the first integral

$$\mathcal{F}_3 \equiv \left(y^a \left(-\frac{a}{y}\right)^a \Gamma\left(-a, -\frac{a}{y}\right) ax - y^a \left(-\frac{a}{y}\right)^a \Gamma\left(-a\right) ax + e^{\frac{a}{y}} ay^a + C_3\, ax - \right.$$

$$\left. -e^{\frac{a}{y}} y^a x \right) a^{-1} x^{-1} = 0.$$

4. The system (2) for $b \neq 0,\ a = 0$ corresponds to the case 7 (Fig. 2). It admits the first integral

$$\mathcal{F}_4 \equiv \left(x^b \left(-\frac{b}{x}\right)^b \Gamma\left(-b\right) by - x^b \left(-\frac{b}{x}\right)^b \Gamma\left(-b, -\frac{b}{x}\right) by - be^{\frac{b}{x}} x^b + C_4\, by + \right.$$

$$\left. +e^{\frac{b}{x}} x^b y \right) b^{-1} y^{-1} = 0.$$

Remark 2. The integrals $\mathcal{F}_1$ - $\mathcal{F}_4$ can not be expressed by center-affine invariants and comitants of the system (2). Moreover, integrals $\mathcal{F}_3$, $\mathcal{F}_4$ contain Gamma-functions.

Remark 3. For the system with $ab \neq 0,\ a \neq 1$ and $b \neq 1$ the authors were not able to find the first integral.

Throughout the paper the Computer Algebra Systems Maple 9.5 and Mathematica 5 were widely used.

References

[1] Boularas D., Braicov A., Vulpe N. *The $GL(2,\mathbb{R})$-orbits of differential system with quadratic nonlinearities*. Proc. of the 2-nd Conference of the Mathematical Society of the Rep. of Moldova, Chisinau, 2004, p. 65–67.

[2] Hirsch M.W., Smale S., Devaney R.L. *Differential equations, dynamical systems, and an introduction to chaos*. Academic Elsevier, New York-Amsderdam, 2004, p. 246–254.

[3] Georgescu R.-M. *Bifurcation in a two competing species model*. Theodor Angheluta seminar, 2006 (in press).

[4] Ovsyannikov L.V. *Group analysis of differential equations*. Moscow, Nauka, 1978 (in Russian, English transl. by Academic New York, 1982).

[5] Popa M.N. *Application of algebraic methods to differental equations*. Flower Power, Pitesti, 2004 (in Romanian).

[6] Sibirsky K.S. *Introduction to algebraic theory of invariants of differential equations*, Chisinau, Stiinta, 1982 (in Russian, published in English in 1988).

Raluca Mihaela Georgescu
Department of Mathematics, University of Pitesti
Romania
E-mail: *gemiral@yahoo.com*

Received January 22, 2007

Elena Naidenova
Institute of Mathematics and Computer Science
Academy of Sciences of Moldova
E-mail: *hstarus@gmail.com*