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# On topological torsion LCA groups with commutative ring of continuous endomorphisms

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**Abstract.** In this paper, we determine for some classes $\mathcal{S}$ of topological torsion LCA (locally compact abelian) groups the structure of those groups in $\mathcal{S}$ which have a commutative ring of continuous endomorphisms.

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## 1 Introduction

Given an LCA group $X$, let $E(X)$ denote the ring of all continuous endomorphisms of $X$. The very pleasant facts that, with respect to the compact-open topology, $E(X)$ is a complete Hausdorff topological ring and the evaluation map $(u, x) \to u(x)$ from $E(X) \times X$ to $X$ is continuous, where $E(X) \times X$ is taken with the product topology, provide a felicitous setting for the study of interconnections between the algebraic-topological properties of $X$ and those of $E(X)$. Similar problems for discrete $X$ constituted the subject of an enormous number of investigations.

The present paper is concerned with the following question:

For which LCA groups $X$ is the ring $E(X)$ commutative?

The prototype of this problem, corresponding to the case when $X$ is discrete, is listed in Fuchs' book [6] as problem 46(a), and was studied for the first time by T. Szele and J. Szendrei. In [14], they have completely solved the case of torsion groups and have obtained some partial results for the case of mixed groups. In the case of torsionfree groups a solution, due to L. C. A. van Leeuwen [9], has been obtained only for very special groups.

This paper is intended to be the first of several investigating the structure of LCA groups $X$ with a commutative ring $E(X)$. We begin our study by examining the case of topological torsion LCA groups, which represent a natural generalization of discrete torsion abelian groups within the class of all LCA groups. In contrast with the case of discrete torsion groups, this new situation is much more complicated and we do not settle it completely. Though we are unable to give a full description of topological torsion LCA groups $X$ having a commutative ring $E(X)$, we give such a description for certain important special cases of this kind of groups.

## 2 Notation

In the following, $\mathbb{P}$ is the set of prime numbers, $\mathbb{N}$ is the set of natural numbers (including zero), and $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.

For $p \in \mathbb{P}$, we denote by $\mathbb{Q}_p$ the group of $p$-adic numbers, by $\mathbb{Z}_p$ the group of $p$-adic integers (both with their usual topologies), by $\mathbb{Z}(p^\infty)$ the quasi-cyclic group corresponding to $p$ and by $\mathbb{Z}(p^n)$, where $n \in \mathbb{N}$, the cyclic group of order $p^n$ (all with the discrete topology).

We let $\mathcal{L}$ denote the class of locally compact abelian groups, and $\mathcal{L}_p$, where $p \in \mathbb{P}$, the subclass of $\mathcal{L}$ consisting of all topological $p$-primary groups.

Let $X$ be a group in $\mathcal{L}$. For any closed subgroup $C$ of $X$, $X/C$ will indicate the quotient group of $X$ by $C$, equipped with the quotient topology.

We let $1_X$, $c(X)$, $d(X)$, $k(X)$, $m(X)$, $t(X)$, and $X^*$ denote, respectively, the identity map on $X$, the connected component of $X$, the maximal divisible subgroup of $X$, the subgroup of compact elements of $X$, the smallest closed subgroup $K$ of $X$ such that the quotient group $X/K$ is torsionfree, the torsion subgroup of $X$, and the character group of $X$.

For $n \in \mathbb{N}$, we let

$$X[n] = \{x \in X \mid n \cdot x = 0\} \qquad \text{and} \qquad n \cdot X = \{n \cdot x \mid x \in X\}.$$

If $p \in \mathbb{P}$, $X_p$ stands the topological $p$-primary component of $X$, i. e.

$$X_p = \{x \in X \mid \lim_{n \to \infty} p^n x = 0\}.$$

If $X$ is a topological torsion group, we let

$$S(X) = \{p \in \mathbb{P} \mid X_p \neq 0\}.$$

For $a \in X$ and $S \subset X$, $o(a)$ is the order of $a$, $\langle a \rangle$ is the subgroup of $X$ generated by $a$, $\overline{S}$ is the closure of $S$ in $X$, and

$$A(X^*, S) = \{\gamma \in X^* \mid \gamma(x) = 0 \quad \text{for all} \quad x \in S\}.$$

For $u \in E(X)$, we let $u^*$ be the transpose of $u$, i.e. the endomorphism $u^* \in E(X^*)$ defined by the rule $u^*(\gamma) = \gamma \circ u$ for all $\gamma \in X^*$.

If $Y$ is another group in $\mathcal{L}$, then $H(X, Y)$ stands for the group of all continuous homomorphisms from $X$ into $Y$. For $h \in H(X, Y)$, we denote by $\text{im}(h)$ the image of $h$ and by $\ker(h)$ the kernel of $h$.

Also, we write $X = A \oplus B$ in case $X$ is a topological direct sum of its subgroups $A$ and $B$.

Let $(X_i)_{i \in I}$ be a collection of topological groups (rings) indexed by a set $I$. We write $\prod_{i \in I} X_i$ for the direct product of the family $(X_i)_{i \in I}$, taken with the product topology. In case each $X_i$ is a discrete abelian group, $\bigoplus_{i\in I} X_i$ denotes the external direct sum of the family $(X_i)_{i\in I}$, taken with the discrete topology. If each $X_i = X$ for some fixed $X$, then $\prod_{i\in I} X_i$ is denoted by $X^I$ and $\bigoplus_{i\in I} X_i$ by $X^{(I)}$.

Suppose, in addition, that for each $i \in I$ we are given an open subgroup (subring) $U_i$ of $X_i$. The local direct product of the family $(X_i)_{i\in I}$ with respect to $(U_i)_{i\in I}$ will be indicated by $\prod_{i\in I}(X_i; U_i)$. Recall that the group (ring) $\prod_{i\in I}(X_i; U_i)$ consists of all $(x_i)_{i\in I} \in \prod_{i\in I} X_i$ such that $x_i \in U_i$ for all but finitely many $i$ and is topologized by declaring all neighborhoods of zero in the topological group (ring) $\prod_{i\in I} U_i$ to be a fundamental system of neighborhoods of zero in $\prod_{i\in I}(X_i; U_i)$.

The symbol $\cong$ denotes topological group (ring) isomorphism.

## 3 Some technical lemmas

We collect here several facts which will be frequently used in the sequel.

**Lemma 3.1.** *For any $X \in \mathcal{L}$, the mapping $u \to u^*$ is a topological ring antiisomorphism from $E(X)$ onto $E(X^*)$.*

**Lemma 3.2.** *Let $X$ be a group in $\mathcal{L}$ such that $X = A \oplus B$ for some subgroups $A, B$ of $X$. If $\varepsilon_A \in E(X)$ is the canonical projection of $X$ onto $A$, then $E(A) \cong \varepsilon_A E(X)\varepsilon_A$, where $\varepsilon_A E(X)\varepsilon_A$ carries the induced topology.*

**Definition 3.3.** *A closed subgroup $C$ of a group $X \in \mathcal{L}$ is said to be topologically fully invariant in $X$ if $u(C) \subset C$ for all $u \in E(X)$.*

**Lemma 3.4.** *Let $(X_i)_{i\in I}$ be an indexed collection of groups in $\mathcal{L}$, and, for each $i \in I$, let $Y_i$ be a compact open subgroup of $X_i$. If $X = \prod_{i\in I}(X_i; Y_i)$ and if every subgroup*

$$X'_j = \{(x_i)_{i\in I} \in X \mid x_i = 0 \quad \text{for all} \quad i \neq j\}, j \in I,$$

*is topologically fully invariant in $X$, then $E(X)$ is topologically isomorphic with $\prod_{i\in I}(E(X_i); \Omega_{X_i}(Y_i, Y_i))$.*

**Proof.** See [11, (2.2)] □

The following lemma provides us with a tool of constructing noncommuting continuous endomorphisms.

**Lemma 3.5.** *Let $X$ be a group in $\mathcal{L}$ admitting a continuous endomorphism $w$ such that $\overline{\operatorname{im}(w)} = A \oplus B$ for some nonzero subgroups $A, B$ of $X$ with $w(A) \subset A$ and $w(B) \subset B$. If there exists $h \in H(A, B)$ satisfying $w(A) \not\subset \ker(h)$, then $E(X)$ fails to be commutative.*

**Proof.** Let $\pi_A : \overline{\operatorname{im}(w)} \to A$ and $\pi_B : \overline{\operatorname{im}(w)} \to B$ denote the canonical projections corresponding to the above decomposition of $\overline{\operatorname{im}(w)}$. If $\eta_A : A \to X$ and $\eta_B : B \to X$ are the canonical injections, define $u, v \in E(X)$ by setting $u = \eta_B \circ h \circ \pi_A \circ w$ and $v = \eta_A \circ \pi_A \circ w$. We cannot have

$$h \circ \pi_A \circ w \circ \eta_A \circ \pi_A \circ w = 0,$$

since otherwise it would follow that

$$(h \circ \pi_A \circ w \circ \eta_A \circ \pi_A)(\overline{\mathrm{im}(w)}) \subset \overline{(h \circ \pi_A \circ w \circ \eta_A \circ \pi_A \circ w)(X)} = \{0\}$$

[2, Ch. 1, §2, Theorem 1], which would imply

$$h(w(A)) = (h \circ \pi_A \circ w \circ \eta_A \circ \pi_A)(A) \subset (h \circ \pi_A \circ w \circ \eta_A \circ \pi_A)(\overline{\mathrm{im}(w)}) = \{0\},$$

a contradiction. Thus $h \circ \pi_A \circ w \circ \eta_A \circ \pi_A \circ w \neq 0$. It then follows that $uv \neq 0$, and since $vu = 0$, the proof is complete. □

## 4 Discrete and compact groups

As we have mentioned in Introduction, T. Szele and J. Szendrei characterized in [14] the major classes of discrete abelian groups with commutative endomorphism ring.

For torsion groups, the characterization of [14] may be paraphrased as follows:

**Theorem 4.1.** [14] *The endomorphism ring $E(X)$ of a discrete torsion group $X \in \mathcal{L}$ is commutative if and only if*

$$X \cong \bigoplus_{p \in S_1} \mathbb{Z}(p^\infty) \times \bigoplus_{p \in S_2} \mathbb{Z}(p^{n_p}),$$

*where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$ and $n_p \in \mathbb{N}_0$ for all $p \in S_2$.*

As a first application of this result, we obtain the description of compact totally disconnected groups in $\mathcal{L}$ with commutative ring of continuous endomorphisms.

**Corollary 4.2.** *The endomorphism ring $E(X)$ of a compact totally disconnected group $X \in \mathcal{L}$ is commutative if and only if*

$$X \cong \prod_{p \in S_1} \mathbb{Z}_p \times \prod_{p \in S_2} \mathbb{Z}(p^{n_p}),$$

*where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$ and $n_p \in \mathbb{N}_0$ for all $p \in S_2$.*

**Proof.** Since the rings $E(X)$ and $E(X^*)$ are topologically antiisomorphic, and since $X$ is discrete and torsion if and only if $X^*$ is compact and totally disconnected [8, (23.17) and (24.26)], the assertion follows from Theorem 4.1 by taking duals. □

## 5 Topological torsion groups

Theorem 4.1, due to T. Szele and J. Szendrei, gives a complete description of torsion discrete abelian groups $X$ whose ring $E(X)$ is commutative. In the present section, which contains our main results, we will be concerned with a natural generalization within $\mathcal{L}$ of this class of groups, namely, with the class of topological torsion groups.

**Definition 5.1.** *A group $X \in \mathcal{L}$ is said to be a topological torsion group in case, for each $x \in X$, $\lim_{n\to\infty}(n!)x = 0$.*

Our first goal will be to describe the $p$-groups in $\mathcal{L}$ with commutative ring of continuous endomorphisms.

**Theorem 5.2.** *Let $p \in \mathbb{P}$, and let $X$ be a $p$-group in $\mathcal{L}$. The ring $E(X)$ is commutative if and only if $X$ is topologically isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^n)$ for some $n \in \mathbb{N}$.*

**Proof.** Let $E(X)$ be commutative. We first consider the case when $X$ is nonreduced. As is well known, $X$ contains then a closed subgroup $D$ topologically isomorphic with $\mathbb{Z}(p^\infty)$ [1, Proposition 4.22]. Let us fix an isomorphism $j : \mathbb{Z}(p^\infty) \to D$. Since the discrete divisible groups are splitting in the class of totally disconnected LCA groups [1, Proposition 6.21], we can write $X = D \oplus X_0$ for some closed subgroup $X_0$ of $X$. Assume by way of contradiction that $X_0 \neq \{0\}$, and let $U$ be a nonzero compact open subgroup of $X_0$. By the structure theorem for torsion compact groups [8, (25.9)], there is a topological isomorphism $\varphi$ from $U$ onto a group of the form $\prod_{i\in I} \mathbb{Z}(p^{m_i})$, where $I$ is a nonempty set and the $m_i$'s are nonzero natural numbers not exceeding a fixed $N \in \mathbb{N}$. Picking any $i_0 \in I$, let $\pi$ denote the canonical projection of $\prod_{i\in I} \mathbb{Z}(p^{m_i})$ onto $\mathbb{Z}(p^{m_{i_0}})$ and $\rho$ the canonical injection of $\mathbb{Z}(p^{m_{i_0}})$ into $\mathbb{Z}(p^\infty)$. Since $D$ is divisible and $U$ is open, $j \circ \rho \circ \pi \circ \varphi \in H(U, D)$ extends [8, (A.7)] to a nonzero homomorphism $h \in H(X_0, D)$ [3, Ch. III, §2, Proposition 23]. Then applying Lemma 3.5 to $w = 1_X$ and our $h \in H(X_0, D)$ leads to a contradiction. Consequently, we must have $X_0 = \{0\}$, and hence $X \cong \mathbb{Z}(p^\infty)$.

Next we dispose of the case when $X$ is reduced. Our first goal will be to prove that $X$ is of bounded order. Pick an arbitrary compact open subgroup $V$ of $X$. In view of the earlier mentioned structure theorem for torsion compact groups, we know that $V$ is of bounded order. Therefore, the desired fact that $X$ is of bounded order will follow if we show that $X/V$ is of bounded order.

It is not difficult to see that $X/V$ is reduced. Indeed, since $V$ is open, $X/V$ is a discrete $p$-group. If $X/V$ were nonreduced, we could write $X/V = D_1 \oplus G$, where $D_1 \cong \mathbb{Z}(p^\infty)$ and $G$ is a subgroup of $X/V$. Since $A(X^*; V) \cong (X/V)^*$ [8, (23.25)] it would then follow from [1, Corollary 6.10] and [8, (25.2)] that $A(X^*; V) = \Delta \oplus \Gamma$, where $\Delta \cong \mathbb{Z}_p$. Let $\psi \in H(A(X^*; V), \Delta)$ denote the canonical projection with kernel $\Gamma$ and choose any nonzero $\eta \in H(\Delta, \mathbb{Q}_p)$. Since $A(X^*; V)$ is open in $X^*$ (because $V$ is compact) and $\mathbb{Q}_p$ is divisible, $\eta \circ \psi$ extends to a nonzero $\chi \in H(X^*, \mathbb{Q}_p)$, and so the transpose map $\chi^*$ would be a nonzero member of $H(\mathbb{Q}_p, X)$, which would imply that $X$ is nonreduced, a contradiction. Consequently, $X/V$ must be reduced.

Having established this, we are ready to prove that $X/V$ is of bounded order. Let

$$n_V = \min\{n \in \mathbb{N} \mid p^n V = \{0\}\}.$$

It is easily seen that $p^{n_V}V^* = \{0\}$ as well. Since $V^* \cong X^*/A(X^*;V)$ [8, (24.5)], it then follows that

$$p^{n_V}X^* \subset A(X^*;V). \tag{5.1}$$

If $X/V$ were not of bounded order, it would follow that $X/V$ has cyclic direct summands of arbitrarily high orders [7, Chapter V, §27, Exercise 1]. Hence we could write $X/V = A \oplus B \oplus C \oplus F$, where $A \cong \mathbb{Z}(p^{n_A})$, $B \cong \mathbb{Z}(p^{n_B})$, $C \cong \mathbb{Z}(p^{n_C})$ and $n_C \geq n_B \geq n_A \geq 2n_V + 1$. By [1, Corollary 6.10] and [8, (23.25)], we then would obtain $A(X^*;V) = A_1 \oplus B_1 \oplus C_1 \oplus F_1$, where $A_1 \cong A$, $B_1 \cong B$ and $C_1 \cong C$. Letting $\alpha \in A_1$, $\beta \in B_1$ and $\gamma \in C_1$ be generators, define $f \in H(C_1, B_1)$ and $g \in H(B_1, A_1)$ by the rule $f(\gamma) = \beta$ and $g(\beta) = \alpha$. Further, letting $\xi \in H(A(X^*;V), B_1)$ and $\zeta \in H(A(X^*;V), C_1)$ be the canonical projections, $\sigma \in H(B_1, X^*)$ and $\tau \in H(A_1, X^*)$ the canonical injections, and taking account of (5.1), define $u, v \in E(X^*)$ by setting

$$u = \tau \circ g \circ \xi \circ p^{n_V} 1_{X^*} \qquad \text{and} \qquad v = \sigma \circ f \circ \zeta \circ p^{n_V} 1_{X^*}.$$

Then $(u \circ v)(\gamma) = u(p^{n_V}\beta) = p^{2n_V}\alpha \neq 0$ and $(v \circ u)(\gamma) = v(0) = 0$, so that $uv \neq vu$. This is a contradiction because, in view of Lemma 3.1, $E(X^*)$ must be commutative. In summary, $X/U$ is a group of bounded order, and hence so is $X$.

Finally, since in a group of bounded order every cyclic subgroup generated by an element of maximal order splits topologically [10, (3.8)], we can write $X = L \oplus M$, where $L \cong \mathbb{Z}(p^n)$ for some $n \in \mathbb{N}$ and $M$ is a subgroup of $X$. Again we must have $M = \{0\}$ since otherwise it would follow that $H(L, M) \neq \{0\}$, contradicting by Lemma 3.5 the commutativity of $E(X)$. Hence $X \cong \mathbb{Z}(p^n)$.

Since the converse is clear, the proof is complete. □

As a direct consequence of Theorem 5.2, we obtain the following result.

**Corollary 5.3.** *Let $X$ be a topological torsion group in $\mathcal{L}$ with torsion primary components. Then $E(X)$ is commutative if and only if*

$$X \cong \prod_{p \in S_1} (\mathbb{Z}(p^\infty); \mathbb{Z}(p^\infty)[p^{k_p}]) \times \prod_{p \in S_2} (\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{k_p}]),$$

*where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and $n_p, k_p \in \mathbb{N}$ for all $p \in S(X)$.*

**Proof.** By [4, Ch. III, §1, Th*é*or*è*me 1] we have

$$X \cong \prod_{p \in S(X)} (X_p; U_p),$$

where, for each $p \in S(X)$, $U_p$ is a compact open subgroup of $X_p$. Since the $X_p$'s are topologically fully invariant in $X$, it follows from Lemma 3.4 that

$$E(X) \cong \prod_{p \in S(X)} (E(X_p); \Omega(U_p, U_p)).$$

Consequently, the commutativity of $E(X)$ is equivalent to the commutativity of all the $E(X_p)$'s.

Now, since $X$ has torsion topological primary components, Theorem 5.2 shows that this last condition is equivalent to saying that, for each $p \in S(X)$, $X_p$ is topologically isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}$. It remains to put

$$S_1 = \{p \in S(X) \mid X_p \cong \mathbb{Z}(p^\infty)\} \qquad \text{and} \qquad S_2 = S(X) \setminus S_1. \qquad \square$$

To dualize the preceding corollary, a few definitions are in order.

**Definition 5.4.** *A group $X \in \mathcal{L}$ is said to be compact-by-bounded order in case $X$ admits a compact subgroup $K$ such that $X/K$ is of bounded order.*

**Definition 5.5.** *Let $X \in \mathcal{L}$. The subgroup $\bigcap_{n\in\mathbb{N}_0}\overline{nX}$ of $X$ is called the subgroup of elements of infinite topological height of $X$. If $\bigcap_{n\in\mathbb{N}_0}\overline{nX} = \{0\}$, $X$ is said to have no elements of infinite topological height.*

It is easy to see that if $X \in \mathcal{L}_p$ for some prime $p$, then $X$ is compact-by-bounded order if and only if $\overline{p^m X}$ is compact for some $m \in \mathbb{N}$, and $X$ has no elements of infinite topological height if and only if $\bigcap_{n\in\mathbb{N}} \overline{p^n X} = \{0\}$.

**Corollary 5.6.** *Let $X$ be a topological torsion group in $\mathcal{L}$ such that its primary components are compact-by-bounded order and have no elements of infinite topological height. Then $E(X)$ is commutative if and only if*

$$X \cong \prod_{p\in S_1} (\mathbb{Z}_p; p^{k_p}\mathbb{Z}_p) \times \prod_{p\in S_2} (\mathbb{Z}(p^{n_p}); p^{k_p}\mathbb{Z}(p^{n_p})),$$

*where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and $n_p, k_p \in \mathbb{N}_0$ for all $p \in S(X)$.*

**Proof.** Observing that a group $X \in \mathcal{L}$ is compact-by-bounded order and has no elements of infinite topological height if and only if $X^*$ is torsion, the assertion follows from [8, (23.33)], Lemma 3.1 and Corollary 5.3. $\square$

Specializing Theorem 5.2 to torsion groups, we arrive at the following corollary, which sharpens Theorem 4.1.

**Corollary 5.7.** *The following are equivalent for a group $X \in \mathcal{L}$ :*

(i) *$X$ is discrete and torsion, and $E(X)$ is commutative.*

(ii) *$X$ is torsion, and $E(X)$ is commutative.*

(iii) *$X \cong \bigoplus_{p\in S_1} \mathbb{Z}(p^\infty) \times \bigoplus_{p\in S_2} \mathbb{Z}(p^{n_p})$, where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and $n_p \in \mathbb{N}_0$ for all $p \in S_2$.*

**Proof.** Clearly, (i) implies (ii), and (iii) implies (i). Assuming (ii), we deduce from Corollary 5.3 that

$$X \cong \prod_{p\in S_1} (\mathbb{Z}(p^\infty); \mathbb{Z}(p^\infty)[p^{k_p}]) \times \prod_{p\in S_2} (\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{k_p}]),$$

where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and $n_p, k_p \in \mathbb{N}$ for all $p \in S(X)$, so that in particular

$$\prod_{p\in S_1} \mathbb{Z}(p^\infty)[p^{k_p}] \times \prod_{p\in S_2} \mathbb{Z}(p^{n_p})[p^{k_p}] \qquad \Big(\cong \prod_{p\in S(X)} \mathbb{Z}(p^{k_p}),$$

since we may assume that $k_p \le n_p$ for all $p \in S_2\Big)$ has to be torsion. It then follows that $\{p \in S(X) \mid k_p \neq 0\}$ is finite, so

$$\prod_{p\in S_1} (\mathbb{Z}(p^\infty); \mathbb{Z}(p^\infty)[p^{k_p}]) \times \prod_{p\in S_2} (\mathbb{Z}(p^{n_p}); \mathbb{Z}(p^{n_p})[p^{k_p}])$$

is discrete by [8, (6.16)(d)], and hence (iii) holds. □

**Corollary 5.8.** *The following are equivalent for a group $X \in \mathcal{L}$ :*

(i) *$X$ is compact and totally disconnected, and $E(X)$ is commutative.*

(ii) *$X$ is a compact-by-bounded order topologically torsion group with no elements of infinite topological height, and $E(X)$ is commutative.*

(iii) *$X \cong \prod_{p\in S_1} \mathbb{Z}_p \times \prod_{p\in S_2} \mathbb{Z}(p^{n_p})$, where $S_1 \cup S_2 = S(X)$, $S_1 \cap S_2 = \varnothing$, and $n_p \in \mathbb{N}_0$ for all $p \in S_2$.*

We next show that Corollary 5.6 can be improved by dropping the assumption that the considered groups do not contain elements of infinite topological height.

**Theorem 5.9.** *Let $X$ be a topological torsion group in $\mathcal{L}$ with compact-by-bounded order topological primary components. The following are equivalent:*

(i) *$E(X)$ is commutative.*

(ii) *For each $p \in S(X)$, $X_p$ is topologically isomorphic with either $\mathbb{Z}_p$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0$.*

**Proof.** As we already mentioned in the proof of Corollary 5.3, for a topological torsion group $X \in \mathcal{L}$, the commutativity of $E(X)$ is equivalent to the commutativity of all the $E(X_p)$'s.

Pick any $p \in S(X)$, and assume that $E(X_p)$ is commutative. Since $X_p$ is compact-by-bounded order, there is a compact subgroup $K$ of $X_p$ such that $X_p/K$ is of bounded order. Hence $p^{n_0}(X_p/K) = \{0\}$ for some $n_0 \in \mathbb{N}$. It follows that $\overline{p^{n_0}X_p}$ is a closed subgroup of $K$, so that $\overline{p^{n_0}X_p}$ is compact, and hence $(\overline{p^{n_0}X_p})^*$ is a discrete $p$-group. But then $(\overline{p^{n_0}X_p})^*$ admits a direct summand isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^m)$ for some $m \in \mathbb{N}$ [7, Corollary 27.3]. Since every decomposition of $(\overline{p^{n_0}X_p})^*$ as a direct sum produces a decomposition into a topological direct sum of $\overline{p^{n_0}X_p}$ [1, Corollary 6.10], we can write $\overline{p^{n_0}X_p} = A \oplus B$, where $A$ is topologically isomorphic with either $\mathbb{Z}_p$ or $\mathbb{Z}(p^m)$. We must have $H(A, B) = \{0\}$, for otherwise we would obtain a contradiction by applying Lemma 3.5 with $\omega = p^{n_0} 1_{X_p}$ and any nonzero $h \in H(A, B)$.

Assume $A \cong \mathbb{Z}_p$. Since for every $x \in X_p$ there exists $f \in H(\mathbb{Z}_p, X_p)$ such that $x \in \operatorname{im}(f)$ [1, Lemma 2.10], the equality $H(A, B) = \{0\}$ can occur only if $B = \{0\}$. It follows that $\overline{p^{n_0} X_p} \cong \mathbb{Z}_p$, so that $\bigcap_{n\in\mathbb{N}} p^n X_p = \{0\}$, and hence $X_p \cong \mathbb{Z}_p$ by Corollary 5.6.

Now assume $A \cong \mathbb{Z}(p^m)$. Since $H(A, B) = \{0\}$, we must clearly have $t(B) = \{0\}$, so that $B \cong \mathbb{Z}_p^\nu$ for some cardinal number $\nu$ [8, 25.8]. But in view of Lemma 3.5 $H(B, A) = \{0\}$ too, which can only occur if $\nu = 0$. It follows that $(\overline{p^{n_0} X_p})^* \cong \mathbb{Z}(p^m)$, so $X_p$ is of bounded order, and hence, by Theorem 5.2, $X \cong \mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0$.

Since $E(\mathbb{Z}_p)$ and $E(\mathbb{Z}(p^{n_p}))$ are clearly commutative, the proof is complete. □

To dualize the preceding theorem, we need a new definition.

**Definition 5.10.** *A group $X \in \mathcal{L}$ is said to be bounded order-by-discrete in case $X$ contains an open subgroup of bounded order.*

The following extends Corollary 5.3.

**Corollary 5.11.** *Let $X$ be a topological torsion group in $\mathcal{L}$ with bounded order-by-discrete topological primary components. The following are equivalent:*

(i) *$E(X)$ is commutative.*

(ii) *For each $p \in S(X)$, $X_p$ is topologically isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0$.*

Next we describe the nonreduced torsionfree topological $p$-primary groups $X \in \mathcal{L}$ with commutative ring $E(X)$.

**Theorem 5.12.** *Let $p \in \mathbb{P}$, and let $X$ be a nonreduced torsionfree group in $\mathcal{L}_p$. The ring $E(X)$ is commutative if and only if $X \cong \mathbb{Q}_p$.*

**Proof.** Assume $E(X)$ is commutative. Since $X$ is a nonreduced torsionfree topological $p$-primary group, it follows from [1, Theorem 4.23] that $X$ contains a closed subgroup $D$ topologically isomorphic with $\mathbb{Q}_p$. We shall show that $X$ must coincide with $D$. Suppose this is not the case. Then, taking into account that $\mathbb{Q}_p$ is splitting in the class of torsionfree LCA groups [1, Proposition 6.23], we can write $X = D \oplus Y$ for some nonzero subgroup $Y$ of $X$. Let $U$ be an arbitrary compact open subgroup of $Y$. Since $Y$ is torsionfree, it follows that $U \cong \mathbb{Z}_p^\nu$ for some cardinal number $\nu \geq 1$ [4, Ch. III, §1, Proposition 3]. Combining an arbitrary topological isomorphism from $U$ onto $\mathbb{Z}_p^\nu$ with a projection of $\mathbb{Z}_p^\nu$ onto $\mathbb{Z}_p$ and with an arbitrary continuous monomorphism from $\mathbb{Z}_p$ into $D$, we obtain a nonzero $h \in H(U, D)$. Since $D$ is divisible and $U$ is open in $Y$, $h$ extends to a nonzero homomorphism $h_0 \in H(Y, D)$, contradicting by Lemma 3.5 our assumption that $E(X)$ is commutative. Therefore we must have $X \cong \mathbb{Q}_p$.

The converse is clear. □

As a consequence we have the following two corollaries.

**Corollary 5.13.** *Let $X$ be a torsionfree topological torsion group in $\mathcal{L}$ such that, for each $p \in S(X)$, $X_p$ is nonreduced. The ring $E(X)$ is commutative if and only if $X \cong \prod_{p\in S(X)}(\mathbb{Q}_p;\mathbb{Z}_p)$.*

**Corollary 5.14.** *Let $X$ be a topological torsion densely divisible group in $\mathcal{L}$ such that, for each $p \in S(X)$, $m(X_p) \neq X_p$. The ring $E(X)$ is commutative if and only if $X \cong \prod_{p\in S(X)}(\mathbb{Q}_p;\mathbb{Z}_p)$.*

We now turn our attention to the case of topological torsion groups in $\mathcal{L}$ with mixed topological primary components. As we saw, the key argument used in proving Theorem 5.12 was the fact, due to L. C. Robertson [12], that, for each $p \in \mathbb{P}$, $\mathbb{Q}_p$ is splitting in the class of torsionfree LCA groups. In order to do with mixed nonreduced topological $p$-primary groups, we first extend Robertson's result to more general groups.

**Lemma 5.15.** *Let $X$ be a group in $\mathcal{L}$ satisfying $c(X) \subset m(X) \neq X$, and let $D$ be a closed subgroup of $X$ such that $D \cong \mathbb{Q}_p$ for some $p \in \mathbb{P}$ and $D \cap m(X) = \{0\}$. Then $D$ splits topologically from $X$.*

**Proof.** It is clear from the very definition of $m(X)$ that $m(X) \subset k(X)$. Since by hypothesis $c(X) \subset m(X)$, it follows that $c(X)$ is compact [5, Proposition 3.3.6], so that $m(X)/c(X)$ is closed in $X/c(X)$ [8, (5.18)]. Taking into account that $X/c(X)$ is totally disconnected [8, (7.3)], and

$$X/m(X) \cong (X/c(X))/(m(X)/c(X)) \quad [8, (5.35)],$$

we then deduce from [8, (7.11)] that $X/m(X)$ is totally disconnected as well. Let $\pi$ denote the canonical projection of $X$ onto $X/m(X)$. Fixing an arbitrary topological isomorphism $f$ from $\mathbb{Q}_p$ onto $D$, set $h = \pi \circ f$. Since $D \cap m(X) = \{0\}$, it follows that $\pi$ acts injectively on $D$, so that $h$ is injective too. Remembering that $X/m(X)$ is totally disconnected, we conclude from [1, Proposition 4.21] that $h(\mathbb{Q}_p)$ is a closed subgroup of $X/m(X)$ and that $h$ establishes a topological isomorphism from $\mathbb{Q}_p$ onto $h(\mathbb{Q}_p)$. Therefore, taking account of the above mentioned fact that $\mathbb{Q}_p$ is splitting in the class of torsionfree LCA groups, we can write

$$X/m(X) = h(\mathbb{Q}_p) \oplus \Gamma$$

for some closed subgroup $\Gamma$ of $X/m(X)$. Let $G = \pi^{-1}(\Gamma)$. We shall show that $X = D \oplus G$. Clearly, $G$ is a closed subgroup of $X$ containing $m(X)$, $\pi(G) = \Gamma$ and $\pi(D) = h(\mathbb{Q}_p)$. If there existed a nonzero $a \in D \cap G$, it would follow that

$$\pi(a) \in \pi(D) \cap \pi(G) = h(\mathbb{Q}_p) \cap \Gamma = \{0\}.$$

This would imply that $a \in m(X)$, contradicting our assumption that $D \cap m(X) = \{0\}$. Thus we must have $D \cap G = \{0\}$. To see that also $X = D + G$, pick an arbitrary $x \in X$. Since

$$X/m(X) = h(\mathbb{Q}_p) \oplus \Gamma = \pi(D) \oplus \pi(G),$$

there exist $y \in D$ and $z \in G$ such that $\pi(x) = \pi(y) + \pi(z)$, so that $x = y + z + t$ for some $t \in m(X)$. But $z + t \in G$ because $m(X) \subset G$, and since $x \in X$ was chosen arbitrarily, this shows that $X = D + G$. Consequently, $X$ decomposes as an algebraic direct sum of $D$ and $G$. To conclude that the obtained decomposition is in fact topological, it remains to observe [1, Proposition 6.5] that $D$, being topologically isomorphic to $\mathbb{Q}_p$, is $\sigma$-compact. □

**Corollary 5.16.** *Let $X$ be a totally disconnected group in $\mathcal{L}$ having closed torsion subgroup. If $X$ contains a closed subgroup $D$ topologically isomorphic with $\mathbb{Q}_p$ for some $p \in \mathbb{P}$, then $D$ splits topologically from $X$.*

**Proof.** Since $c(X) = \{0\}$, $m(X) = t(X)$ and $\mathbb{Q}_p$ is torsionfree, the assertion follows from Lemma 5.15. □

We approach the description of mixed nonreduced topological $p$-primary groups $X \in \mathcal{L}$ with commutative ring $E(X)$ through two lemmas.

**Lemma 5.17.** *Let $p \in \mathbb{P}$, and let $X$ be a mixed group in $\mathcal{L}_p$. If $E(X)$ is commutative, then $t(X)$ is reduced.*

**Proof.** If $t(X)$ were nonreduced, $X$ would contain a copy $D$ of $\mathbb{Z}(p^\infty)$. Since $\mathbb{Z}(p^\infty)$ is splitting in the class of totally disconnected LCA groups, it would then follow that $X = D \oplus T$ for some nonzero (because $X \neq t(X)$) closed subgroup $T$ of $X$. Letting $U$ be an arbitrary nonzero compact open subgroup of $T$, then $U^*$ would be a nonzero discrete $p$-group, and so $U^*$ would admit by [7, Corollary 27.3] a direct summand isomorphic with either $\mathbb{Z}(p^\infty)$ or $\mathbb{Z}(p^n)$ for some $n \in \mathbb{N}$. We would then conclude from [8, (23.18)] that $U$ has a topological direct summand topologically isomorphic with either $\mathbb{Z}_p$ or $\mathbb{Z}(p^n)$, which would imply that $H(U, D) \neq \{0\}$. Extending the elements of $H(U, D)$, we would obtain that $H(T, D) \neq \{0\}$, so that by Lemma 3.5 $E(X)$ could not be commutative. □

**Lemma 5.18.** *Let $p \in \mathbb{P}$, and let $X$ be a group in $\mathcal{L}_p$ such that $d(X) \not\subset m(X)$. If $E(X)$ is commutative, then $d(X) \cong \mathbb{Q}_p$ and $X = d(X) \oplus m(X)$.*

**Proof.** Fix any $a \in d(X) \setminus m(X)$, and let $D$ denote the minimal divisible subgroup of $X$ containing $a$. It is clear from the definition of $m(X)$ that $t(X) \subset m(X)$, so $a \notin t(X)$, and hence $D$ is algebraically isomorphic to $\mathbb{Q}$. It follows that $\overline{D}$ is divisible, because every group in $\mathcal{L}$ containing a dense divisible subgroup of finite rank is itself divisible [1, (5.39)(e)]. We assert that $\overline{D} \cong \mathbb{Q}_p$. Indeed, since $X \in \mathcal{L}_p$, it follows from [1, Lemma 2.11] that $\overline{\langle a \rangle} \cong \mathbb{Z}_p$. Pick a topological isomorphism $\varphi$ from $\mathbb{Z}_p$ onto $\overline{\langle a \rangle}$. Since $\overline{D}$ is divisible, $\eta \circ \varphi$ extends to homomorphism $f \in H(\mathbb{Q}_p, \overline{D})$, where $\eta$ is the canonical injection of $\overline{\langle a \rangle}$ into $\overline{D}$. To show that $f$ is injective, pick any $x \in \ker(f)$. Since $\mathbb{Q}_p$ is the minimal divisible extension of $\mathbb{Z}_p$, we can find an $l \in \mathbb{N}$ such that $p^l x \in \mathbb{Z}_p$. It follows that $p^l x \in \ker(\eta \circ \varphi)$, so $p^l x = 0$, whence $x = 0$ because $\mathbb{Q}_p$ is torsionfree. Thus our claim is established. As every group in $\mathcal{L}_p$ is totally disconnected, it then follows from [1, Proposition 4.21] that $f(\mathbb{Q}_p)$ is closed in $\overline{D}$ and $f$ is a topological isomorphism from $\mathbb{Q}_p$ onto $f(\mathbb{Q}_p)$. But $\overline{\langle a \rangle} \subset f(\mathbb{Q}_p)$ and

$f(\mathbb{Q}_p)$ is divisible, so $D \subset f(\mathbb{Q}_p)$, and hence $f(\mathbb{Q}_p) = \overline{D}$, proving that $\overline{D} \cong \mathbb{Q}_p$. Next we show that $\overline{D} \cap m(X) = \{0\}$. Assume the contrary, and let $U = \overline{D} \cap m(X)$. Then $U$ is open in $\overline{D}$. Since $\lim_{n\to\infty} p^n a = 0$, there exists $k \in \mathbb{N}$ such that $p^k a \in U$, and so

$$a + m(X) \in t(X/m(X)).$$

As $X/m(X)$ is torsionfree, it follows that $a \in m(X)$, contradicting the choice of $a$. This proves that $\overline{D} \cap m(X) = \{0\}$.

Now, according to Lemma 5.15, we can write $X = D \oplus G$ for some closed subgroup $G$ of $X$. Since, in view of Lemma 3.5, $H(D, G)$ and $H(G, D)$ cannot be nonzero groups, we must have $d(G) = \{0\}$ and $m(G) = G$, so that $D = d(X)$ and $G = m(X)$. □

**Theorem 5.19.** *Let $p \in \mathbb{P}$, and let $X \in \mathcal{L}_p$ be a nonreduced mixed group having closed torsion subgroup. The ring $E(X)$ is commutative if and only if $X$ is topologically isomorphic to $\mathbb{Q}_p \times \mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}_0$.*

**Proof.** Assume $E(X)$ is commutative. Since $t(X)$ is closed in $X$, we clearly have $m(X) = t(X)$. But $t(X)$ is reduced by Lemma 5.17, so that $d(X) \not\subset m(X)$. It then follows from Lemma 5.18 that $X = D \oplus t(X)$, where $D \cong \mathbb{Q}_p$. As, by Lemma 3.2, $E(t(X))$ is also commutative, we deduce from Theorem 5.2 that $t(X) \cong \mathbb{Z}(p^{n_p})$ for some $n \in \mathbb{N}_0$.

Assume the converse. We have $X = d(X) \oplus t(X)$, where $d(X) \cong \mathbb{Q}_p$ and $t(X) \cong \mathbb{Z}(p^{n_p})$. Since $d(X)$ is torsionfree and $t(X)$ is reduced, it follows that $d(X)$ and $t(X)$ are topologically fully invariant subgroups of $X$, so that $E(X) \cong E(\mathbb{Q}_p) \times E(\mathbb{Z}(p^{n_p}))$. □

We prove now

**Theorem 5.20.** *Let $X$ be a topological torsion group in $\mathcal{L}$ such that its topological primary components have closed torsion subgroup and compact-by-bounded order quotient modulo the subgroup of elements of infinite topological height. The following are equivalent:*

(i) *$E(X)$ is commutative.*

(ii) *For each $p \in S(X)$, $X_p$ is topologically isomorphic with one of the groups $\mathbb{Z}(p^\infty)$, $\mathbb{Z}(p^{n_p})$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ or $\mathbb{Q}_p \times \mathbb{Z}(p^{n_p})$, where $n_p \in \mathbb{N}$.*

**Proof.** As we know, $E(X)$ is commutative if and only if all the $E(X_p)$'s have this property.

Pick any $p \in S(X)$, and assume that $E(X_p)$ is commutative. If $X_p = t(X_p)$, we deduce from Theorem 5.2 that either $X \cong \mathbb{Z}(p^\infty)$ or $X \cong \mathbb{Z}(p^{n_p})$ for some $n_p \in \mathbb{N}$. Let us suppose further that $X_p \neq t(X_p)$. Since

$$\bigcap_{n\in\mathbb{N}} \overline{p^n X_p^*} = A(X_p^*, t(X_p)) \quad [8, (24.24)]$$

and $t(X_p)$ is closed in $X_p$, it then follows from [8, (23.24)(b)] that $\bigcap_{n\in\mathbb{N}} \overline{p^n X_p^*} \neq \{0\}$. But

$$\big(X_p/t(X_p)\big)^* \cong \bigcap_{n\in\mathbb{N}} \overline{p^n X_p^*} \quad [8,\ (23.25)]$$

and since $X_p/t(X_p)$ is torsionfree, we conclude by a theorem of Robertson [13, Theorem 5.2] that $\bigcap_{n\in\mathbb{N}} \overline{p^n X_p^*}$ is densely divisible, so that $X_p^*$ is nonreduced. Now, if $X_p^* = t(X_p^*)$, we use Theorem 5.2 again to deduce that $X_p^* \cong \mathbb{Z}(p^\infty)$, whence $X_p \cong \mathbb{Z}_p$. Further, if $t(X_p^*) = \{0\}$, we conclude from Theorem 5.12 that $X_p^* \cong \mathbb{Q}_p$, so $X_p \cong \mathbb{Q}_p$ because $\mathbb{Q}_p$ is self-dual. Thus, it only remains to consider the case when $X_p^*$ is mixed. As $X_p/\bigcap_{n\in\mathbb{N}} \overline{p^n X_p}$ is compact-by-bounded order, it is then easily seen that $t(X_p^*)$ is closed in $X_p^*$, so that by Theorem 5.19 $X_p^* \cong \mathbb{Q}_p \times \mathbb{Z}(p^{m_p})$ for some $m_p \in \mathbb{N}$, whence $X_p \cong \mathbb{Q}_p \times \mathbb{Z}(p^{m_p})$.

On the other hand, it is clear that the groups $\mathbb{Z}(p^\infty)$, $\mathbb{Z}(p^{n_p})$, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{Q}_p \times \mathbb{Z}(p^{m_p})$ have commutative ring of continuous endomorphisms. □

**Remark.** Observe that by dualizing Theorem 5.20 we would obtain nothing new because the class $\mathcal{S}$ of topological torsion groups in $\mathcal{L}$ whose topological primary components have closed torsion subgroup and compact-by-bounded order quotient by the subgroup of elements of infinite topological height is self-dual, i.e. if $X \in \mathcal{S}$, then $X^* \in \mathcal{S}$ too.

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