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A Linear Parametrical Programming Approach for Studying and Solving Bilinear Programming Problem *

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Abstract. An approach for studying and solving a bilinear programming problem, based on linear parametrical programming, is proposed. Using duality principle for the considered problem we show that it can be transformed into a problem of determining the compatibility of a system of linear inequalities with a right-hand member that depends on parameters, admissible values of which are defined by another system of linear inequalities. Some properties of this auxiliary problem are obtained and a conical algorithm for its solving is proposed. We show that this algorithm can be used for finding the exact solution of bilinear programming problem as well as its approximate solution.

Mathematics subject classification: 65K05,68W25.
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1 Introduction and Problem Formulation

We consider the following bilinear programming problem (BPP) [1, 9]:
to minimize the object function

$$z = xCy + c'x + c''y \tag{1}$$

on subject

$$Ax \leq a, \quad x \geq 0; \tag{2}$$

$$By \leq b, \quad y \geq 0, \tag{3}$$

where C, A, B are matrices of size $n \times m_1,\ m_2 \times n,\ k \times m_1$, respectively, and c', $x \in R^n;\ c'', y \in R^{m_1};\ a \in R^{m_2},\ b \in R^k$. In order to simplify the notations we will omit transposition symbol for vectors.

This problem generalizes a large class of practical and theoretical combinatorial optimization problems [6, 9]. For example, a linear boolean programming problem, resource allocation problem, and determining Nash equilibria in bimatrix games, can be formulated as BPP (1)–(3).

It is easy to show that all local and global minima of the considered problem belong to basic solutions of systems (2), (3) and can be found in finite time. But it

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is well-known that BPP is NP-hard and therefore the elaboration of efficient polynomial-time algorithms for its solving looks to be unrealizable. Nevertheless in this paper we stress the attention to a general approach for studying and solving BPP, which is based on linear parametrical programming. The proposed approach allows us to elaborate such an exact algorithm that in the case of long time calculation it can be interrupted and an admissible solution, which is appropriate to an optimal one, can be obtained. Some classes of problems, for which the proposed approach can be used, are given.

2 Parametrical programming approach for studying and solving BPP

Let L be the size of BPP (1)-(3) with integer coefficients of the matrices C, A, B and vectors a, b, c', c''. So, $L = L_1 + L_2 + \log H + 2$, where

$$L_1 = \sum_{i=1}^{m_2}\sum_{j=1}^{n} \log(|a_{ij}|+1) + \sum_{i=1}^{m_2} \log(|a_i|+1) + \log m_2 n + 1;$$

$$L_2 = \sum_{i=1}^{k}\sum_{j=1}^{m_1} \log(|b_{ij}|+1) + \sum_{i=1}^{k} \log(|b_i|+1) + \log k m_1 + 1;$$

$$H = \max\{|c_{ij}|,\ |c'_i|,\ |c''_j|,\ i=\overline{1,n},\ j=\overline{1,m_1}\}.$$

In [7] the following lemma is proved.

Lemma 1. *If A, B, C and a, b, c', c'' are integer, then for nonempty and bounded solution sets of systems (2) and (3) the optimal value of the object function of BPP (1)–(3) is a quantity of the form t/r, where t and r are integer and $|t|, |r| \le 2^L$.*

On the basis of this lemma and results from [4, 5] we may conclude that if BPP (1)–(3) has solution then it can be solved by varying the parameter $h \in [-2^L, 2^L]$ in the problem of determining the compatibility of the system

$$\begin{cases} Ax \le a; \\ xCy + c'x + c''y \le h; \\ By \le b; \\ x \ge 0,\ y \ge 0. \end{cases} \tag{4}$$

If there exists an algorithm T for determining the compatibility of such a system then we can find the optimal value h^* of the object function and the solution of BPP (1)-(3) by using the dichotomy method on the segment $[-2^L, 2^L]$, checking at every step the compatibility of system (4) with $h = h_k$, where h_k is a current value of parameter h at the kth step of the method. On the basis of results from [4, 5, 7] we can conclude that using $3L + 2$ steps we obtain the optimal value h^* with the precision 2^{-2L-2}. As it is shown in [4,5] if an approximate solution for h^* is known with the precision 2^{-2L-2} then an exact solution can be found in polynomial time by using a special approximate procedure.

In the following we will reduce the problem of the compatibility of system (4) to the problem of finding the compatibility of the system of linear inequalities with a right-hand member depending on parameters. So, we prove the following theorems.

Theorem 2. *Let solution sets X and Y of systems (2) and (3) be nonempty. Then system (4) has no solution if and only if the following system of linear inequalities*

$$\begin{cases} -A^T u \le Cy + c'; \\ au < c''y - h; \\ u \ge 0 \end{cases} \tag{5}$$

is compatible with respect to u for every y satisfying (3).

Proof. $\Rightarrow$ Let us assume that system (4) has no solution. This means that for every $y \in Y$ the following system of linear inequalities

$$\begin{cases} Ax \le a, \\ x(Cy + c') \le h - c''y, \\ x \ge 0 \end{cases} \tag{6}$$

has no solution with respect to x. Then according to theorem 2.14 from [2] the incompatibility of system (6) involves the solvability with respect to u and t of the following system of linear inequalities

$$\begin{cases} A^T u + (Cy + c')t \ge 0; \\ au + (h - c''y)t < 0; \\ u \ge 0, \ t \ge 0, \end{cases} \tag{7}$$

for every $y \in Y$.

Note that for every fixed $y \in Y$ in obtained system (7) for an arbitrary solution (u^*, t^*) the condition $t^* > 0$ holds. Indeed, if $t^* = 0$, then it means that the system

$$\begin{cases} A^T u \ge 0; \\ au < 0, \ u \ge 0, \end{cases}$$

has solution, what, according to theorem 2.14 from [2], involves the incompatibility of initial system (2) that is contrary to the initial assumption. Consequently, $t^* > 0$.

Since $t > 0$ in (7) for every $y \in Y$, then, dividing every of inequalities of this system by t and denoting $z = (1/t)u$, we obtain the following system of linear inequalities

$$\begin{cases} -A^T z \le Cy + c'; \\ az < c''y - h; \\ z \ge 0, \end{cases}$$

which has solution with respect to z for every $y \in Y$.

$\Leftarrow$ Let system (5) have solution with respect to u for every $y \in Y$. Then the following system of linear inequalities

$$\begin{cases} A^T u + (Cy + c')t \geq 0; \\ au + (h - c''y)t < 0; \\ u \geq 0, \ t > 0, \end{cases}$$

is compatible with respect to u and t for every $y \in Y$. However this system is equivalent to system (7), as it was shown that for every solution (u, t) of system (7) the condition $t > 0$ holds. Again using theorem 2.14 from [2], we obtain from the solvability of system (7) with respect to u and t for every $y \in Y$ that system (6) is incompatible with respect to x for every $y \in Y$. This means that system (4) has no solution. $\square$

Theorem 3. *The minimal value of the object function in BPP (1)-(3) is equal to the maximal value h^* of the parameter h in the system*

$$\begin{cases} -A^T u \leq Cy + c'; \\ au \leq c''y - h; \\ u \geq 0 \end{cases} \tag{8}$$

for which it is compatible with respect to u for every $y \in Y$. An arbitrary point $y^* \in Y$, for which system (5) with $h = h^*$ and $y = y^*$ has no solution with respect to u, corresponds to one of optimal points for BPP (1)-(3).

Proof. Let h^* be a maximal value of parameter h, for which system (8) with $h = h^*$ has solution with respect to u for every $y \in Y$. Then system (5) with $h = h^*$ has solution with respect to u not for every $y \in Y$. From this on the basis of the previous theorem it results that system (4) with $h = h^*$ is compatible. Using the previous theorem we can see that if for every fixed $h < h^*$ system (5) has solution with respect to u for every $y \in Y$, then system (4) with $h < h^*$ has no solution. Consequently, the maximal value h^* of parameter h, for which system (8) has solution with respect to u for every $y \in Y$, is equal to the minimum value of the object function of BPP (1)-(3).

Now let us prove the second part of the theorem. Let $y^* \in E^{m_1}$ be an arbitrary point for which system (5) with $h = h^*$ and $y = y^*$ has no solution with respect to u. Then on the basis on the duality principle the following system

$$\begin{cases} Ax \leq a; \\ x(Cy^* + c') \leq h^* - c''y; \\ x \geq 0 \end{cases}$$

has solution with respect to x. Consequently, system (4) with $h = h^*$ is compatible and the point $y^* \in Y$ together with the certain $x^* \in X$ represents its solution, i.e. y^* is one of sought optimal points for BPP (1)-(3). $\square$

So, the problem of determining the compatibility of system (4) is equivalent to the problem of determining the compatibility of system (8) for every y satisfying (3). If an algorithm for solving this problem is elaborated, then we will obtain an algorithm for solving BPP (1)-(3).

3 Main properties of systems of linear inequalities with a right-hand member depending on parameters

The systems of linear inequalities with a right-hand member depending on parameters have been studied in [6–8].

3.1 Duality principle for parametrical systems of linear inequalities

Let the following system of linear inequalities be given

$$\begin{cases} \sum_{j=1}^{n} a_{ij}u_j \le \sum_{s=1}^{k} c_{is}y_s + c_{i0}, \ \ i=\overline{1,m}; \\ u_j \ge 0, \ \ j=\overline{1,p} \ (p \le n) \end{cases} \tag{9}$$

with the right-hand member depending on parameters $y_1, y_2, \dots, y_k$. We consider the problem of determining the compatibility of system (9) for every $y_1, y_2, \dots, y_k$ satisfying the following system

$$\begin{cases} \sum_{s=1}^{k} b_{is}y_s + b_{i0} \le 0, \ \ i=\overline{1,r}; \\ y_s \ge 0, \ \ s=\overline{1,q} \ (q \le k). \end{cases} \tag{10}$$

In [6, 7] the following theorem has been proved.

Theorem 4. *System (9) is compatible with respect to $u_1, u_2, \dots, u_n$ for every $y_1, y_2, \dots, y_k$ satisfying (10) if and only if the following system*

$$\begin{cases} -\sum_{i=1}^{r} b_{is}v_i \le \sum_{i=1}^{m} c_{is}z_i, \ \ s=\overline{0,q}; \\ -\sum_{i=1}^{r} b_{is}v_i = \sum_{i=1}^{m} c_{is}z_i, \ \ s=\overline{q+1,k}; \\ v_i \ge 0, \ \ i=\overline{1,r} \end{cases}$$

is compatible with respect to $v_1, v_2, \dots, v_r$ for every $z_1, z_2, \dots, z_m$ satisfying the following system

$$\begin{cases} -\sum_{i=1}^{m} a_{ij}z_i \le 0, \ \ j=\overline{1,p}; \\ -\sum_{i=1}^{m} a_{ij}z_i = 0, \ \ j=\overline{p+1,n}; \\ z_i \ge 0, \ \ i=\overline{1,m}. \end{cases}$$

3.2 Two special cases of the parametrical problem

Note that if $r = 0$ and $q = k$ in system (10) then we obtain the problem of determining the compatibility of system (9) for every nonnegative values of parameters $y_1, y_2, \ldots, y_k$. It is easy to observe that in this case system (9) is compatible for every nonnegative values of parameters $y_1, y_2, \ldots, y_k$ if and only if each of the following $k+1$ systems ($s = \overline{0,k}$)

$$\begin{cases} \sum_{j=1}^{n} a_{ij} u_j \leq c_{is}, \ i = \overline{1,m}; \\ u_j \geq 0, \ j = \overline{1,p} \end{cases}$$

is compatible.

Another special case of the problem is the one when $n = 0$. This case can be reduced to the previous one using the duality problem for it.

In such a way, our problem can be solved in polynomial time for the mentioned above cases.

3.3 General approach for determining the compatibility property for parametrical systems

Let us assume that the solution sets UY and Y of systems (9) and (10) are bounded. Then it is easy to observe that the compatibility property of system (9) for all admissible values of parameters $y_1, y_2, \ldots, y_k$ satisfying (10) can be verified by checking the compatibility of system (9) for every basic solution of system (10). This fact follows from the geometrical interpretation of the problem. The set $\overline{Y} \subseteq R^k$ of vectors $y = (y_1, y_2, \ldots, y_k)$, for which system (9) is compatible, corresponds to the orthogonal projection on R^k of the set $UY \subseteq R^{n+k}$ of solutions of system (9) with respect to variables $u_1, u_2, \ldots, u_n$, $y_1, y_2, \ldots, y_k$. Therefore $Y \subseteq \overline{Y}$ if and only if system (9) is compatible with respect to $u_1, u_2, \ldots, u_n$ for every basic solution of system (10) (see Fig.1).

Another general approach which can be argued on the basis of the mentioned above geometrical interpretation is the following.

We find the system of linear inequalities

$$\sum_{j=1}^{r} c'_{ij} \ y_j + c'_{i0} \leq 0, \ i = \overline{1,m'}, \tag{11}$$

which determines the orthogonal projection $\overline{Y}$ of the set $UY \subseteq R^{n+k}$ on R^k; then we solve the problem from Section 3.2. System (11) can be found by using method of elimination of variables $u_1, u_2, \ldots, u_n$ from system (9). Such a method of elimination of variables can be found in [2]. Note that in final system (11) the number of inequalities m' can be too big. Therefore such an approach for solving our problem can be used only for a small class of problems.

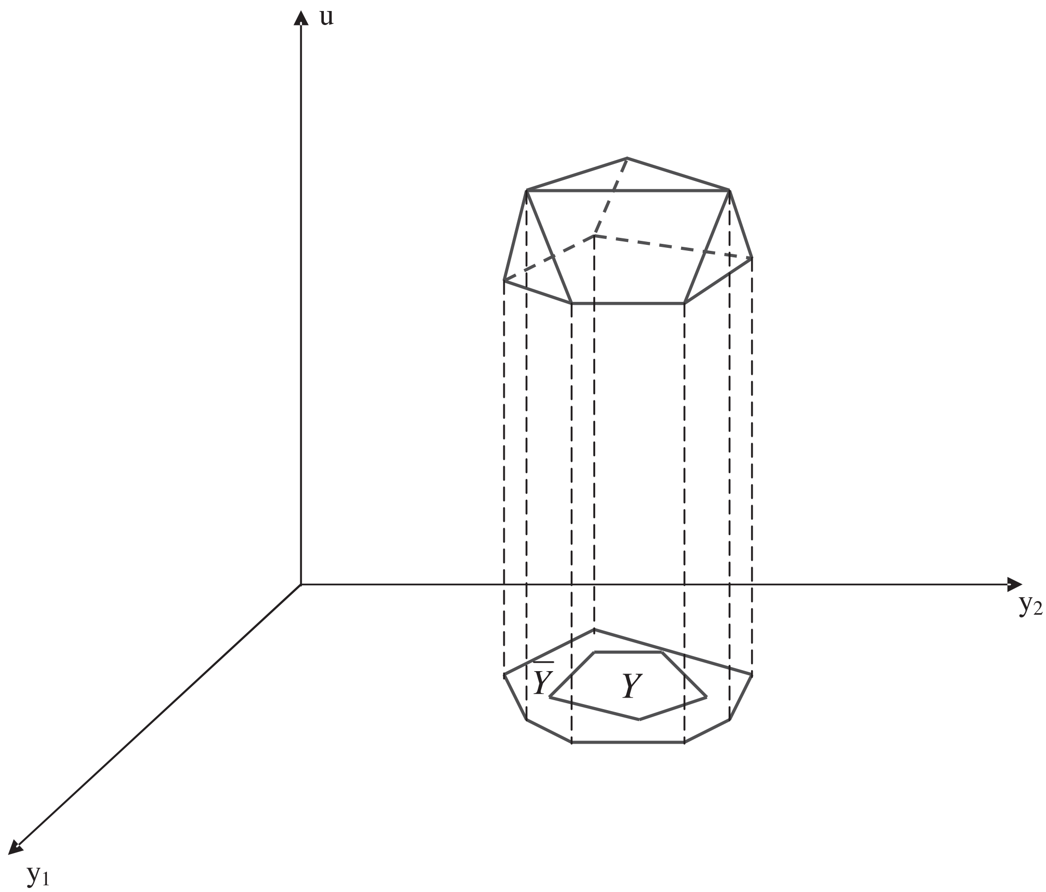

Fig.1

4 An algorithm for determining the compatibility of a parametrical problem

Let us assume that $h = h_k \in [M^1, M^2]$, where $M^1 \leq -2^L$, $M^2 \geq 2^L$. We propose an algorithm for determining the compatibility of system (8) with $h = h_k$ for every y satisfying (3). This algorithm works in the case when the solution sets of the considered systems are bounded. The case of the problem with unbounded solution sets can be easily reduced to the bounded one.

The proposed approach is based on the idea of conical algorithms from [3,9,10].

Algorithm 1.

Step 1. Choose an arbitrary basic solution y^0 of system (3). This solution corresponds to a solution of the system of linear equations

$$\sum_{j=1}^{m_1} b_{i_s j} y_j + b_{i_s 0} = 0, \ s = \overline{1, m_1}. \tag{12}$$

The matrix $\overline{B} = (b_{i_s j})_{m_1 \times m_1}$ of this system represents a submatrix of the matrix

$$B' = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m_1} \\ b_{21} & b_{22} & \dots & b_{2m_1} \\ \dots & \dots & \dots & \dots \\ b_{k1} & b_{k2} & \dots & b_{km} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

and the vector $(b_{i_1 0}, b_{i_2 0}, \dots, b_{i_{m_1} 0})^T$ is a "subvector" of $(b_1, b_2, \dots, b_k, 0, 0, \dots, 0)^T$.

If for $y = y^0$ system (8) is not compatible with respect to u then we fix $z = y^0$ and STOP.

If for $y = y^0$ system (8) is compatible with respect to u then we find the minimal cone Y^0, originating in y^0, which contains the polyhedral solution set Y of system (3) (see Fig.2).

It is easy to observe that the system of inequalities

$$\sum_{j=1}^{m_1} b_{i_s j} y_j + b_{i_s 0} \le 0, \ \ s = \overline{1, m_1},$$

which corresponds to system (12), determines in R^{m_1} the cone Y^0 with the following generating rays $y^s = y^0 + \overline{b}^s t, \ \ s = \overline{1, m_1}, \ \ t \ge 0$.

Here $\overline{b}^1, \overline{b}^2, \dots, \overline{b}^{m_1}$ represent directing vectors of respective rays originating in y^0. These directing vectors correspond to columns of the matrix $\overline{B}^{-1}$.

Step 2. For each $s = \overline{1, m_1}$, solve the following problem:

$$\textit{maximize } t$$

on subject

$$\begin{cases} By \le b; \\ y \ge 0; \\ y = y^0 + \overline{b}^s t, \ \ t \ge 0 \end{cases}$$

and find m_1 points $\overline{y}^1, \overline{y}^2, \dots, \overline{y}^{m_1}$, which correspond to m_1 basic solutions of system (3), i.e. $\overline{y}^1, \overline{y}^2, \dots, \overline{y}^{m_1}$ represent neighboring basic solutions for y^0. If system (8) is compatible with respect to u for each $y = \overline{y}^1, y = \overline{y}^2, \dots, y = \overline{y}^{m_1}$, then go to step 3; otherwise system (8) is not compatible for every y satisfying (3) and STOP.

Step 3. For each $s = \overline{1, m_1}$, solve the following problem:

$$\textit{maximize } t$$

on subject

$$\begin{cases} -A^T u \le Cy + c'; \\ au \le c'' y - h; \\ u \ge 0 \\ y = y^0 + \overline{b}^s t, \ \ t \ge 0 \end{cases}$$

and find m_1 solutions $t'_1, t'_2, \ldots, t'_{m_1}$. Then fix m_1 points $\widehat{y}^s = y^0 + \overline{b}^s t'_s,\ s = \overline{1, m_1}$. (On Fig.2 we can see $\widehat{y}^1$ and $\widehat{y}^2$.)

Step 4. Find the hyperplane Γ (see Fig.2), determined by the following equation $\sum_{j=1}^{m_1} a'_j y_j + a'_0 = 0$, which passes through the points $\widehat{y}^1, \widehat{y}^2, \ldots, \widehat{y}^{m_1}$.

Consider that the basic solution $y^0 = (y^0_1, y^0_2, \ldots, y^0_{m_1})$ satisfies the following condition $\sum_{j=1}^{m_1} a'_j y^0_j + a'_0 \leq 0$. Then add to system (3) the inequality $-\sum_{j=1}^{m_1} a'_j y_j - a'_0 \leq 0$. If after that the obtained system is not compatible, then conclude that system (8) is compatible for every y satisfying initial system (3) and STOP; otherwise change the initial system with the obtained one and go to step 1.

Note that this algorithm works well when the polyhedral set Y is a small part of the orthogonal projection $\overline{Y}$ (see Fig.3) or when the intersection of Y and $\overline{Y}$ is a small part of $\overline{Y}$ (see Fig.4). In the case when the polyhedral set Y is a big part of the orthogonal projection $\overline{Y}$ the algorithm may work too long (see Fig.5).

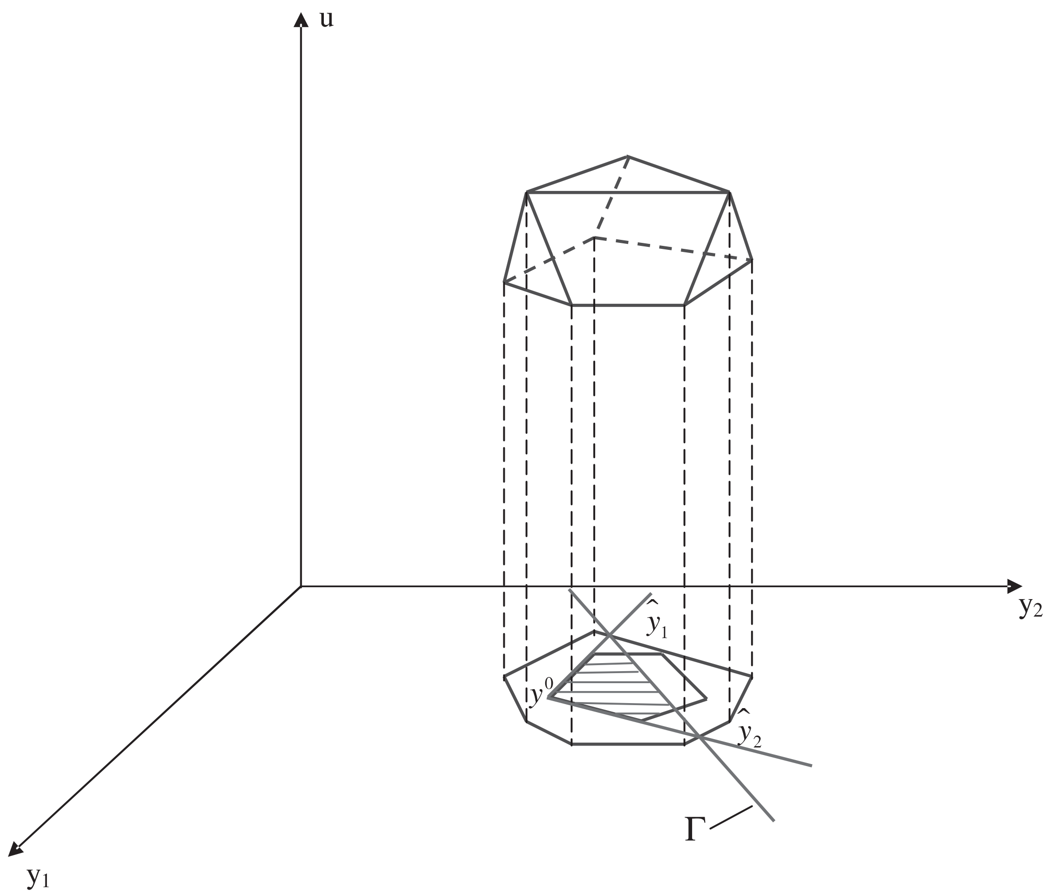

Fig.2

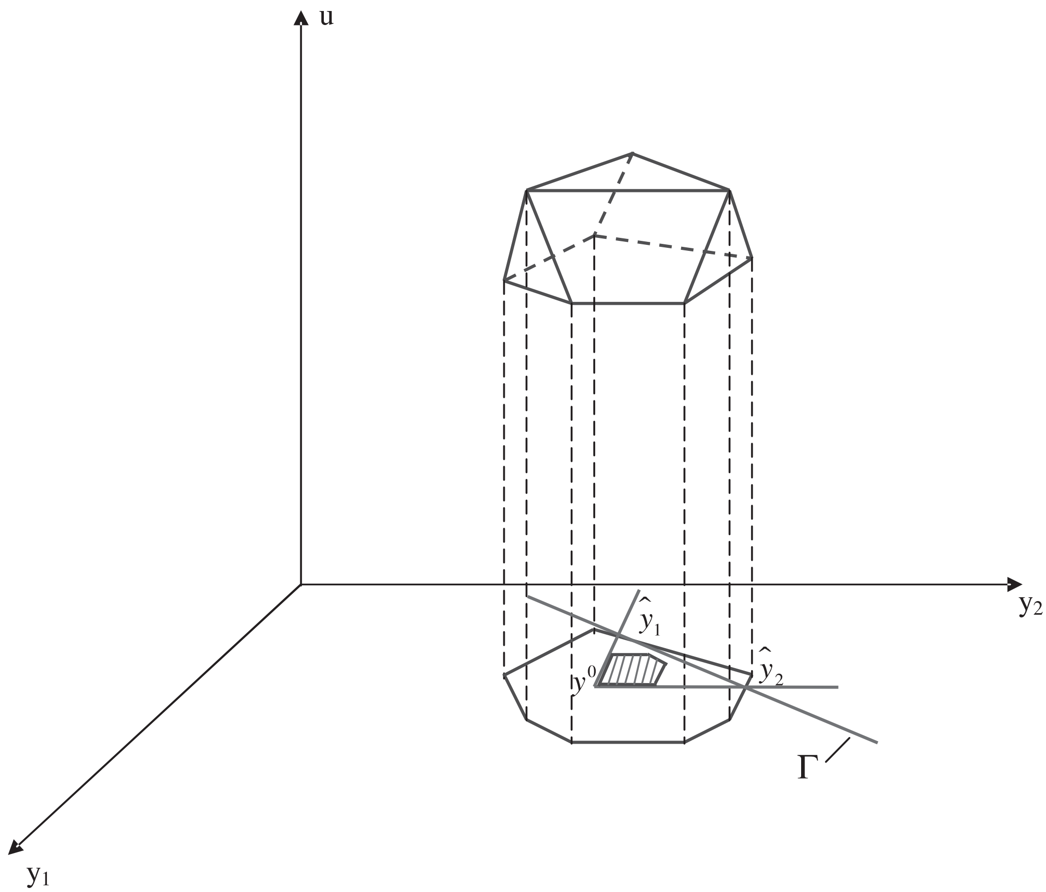

Fig.3

5 An algorithm for solving BPP

Using algorithm 1 for determining the compatibility of system (8) for every y satisfying (3) when $h = h_k$ is fixed, we can now propose an algorithm for solving BPP (1)-(3).

Algorithm 2.

Preliminary step (step 0). Fix an arbitrary basic solution $z = y^0$ of system (3) and put $M^1 = -2^L$, $M^2 = 2^L$, $h_0 = M^1$.

General step (step k). Find $\varepsilon = M^2 - M^1$. If $\varepsilon < \frac{1}{2^{2L+2}}$ then fix $y^k = z$ and STOP, otherwise find $h_k = \frac{M^1+M^2}{2}$. Then apply algorithm 1 with $h = h_k$ and determine if system (8) is compatible with respect to u for every y satisfying (3). If this condition is satisfied then change M^2 by $\frac{M^1+M^2}{2}$ and go to the next step; otherwise fix the basic solution $y^0 = z$ for which system (8) has no solution with respect to u, change M^1 by $\frac{M^1+M^2}{2}$ and go to the next step.

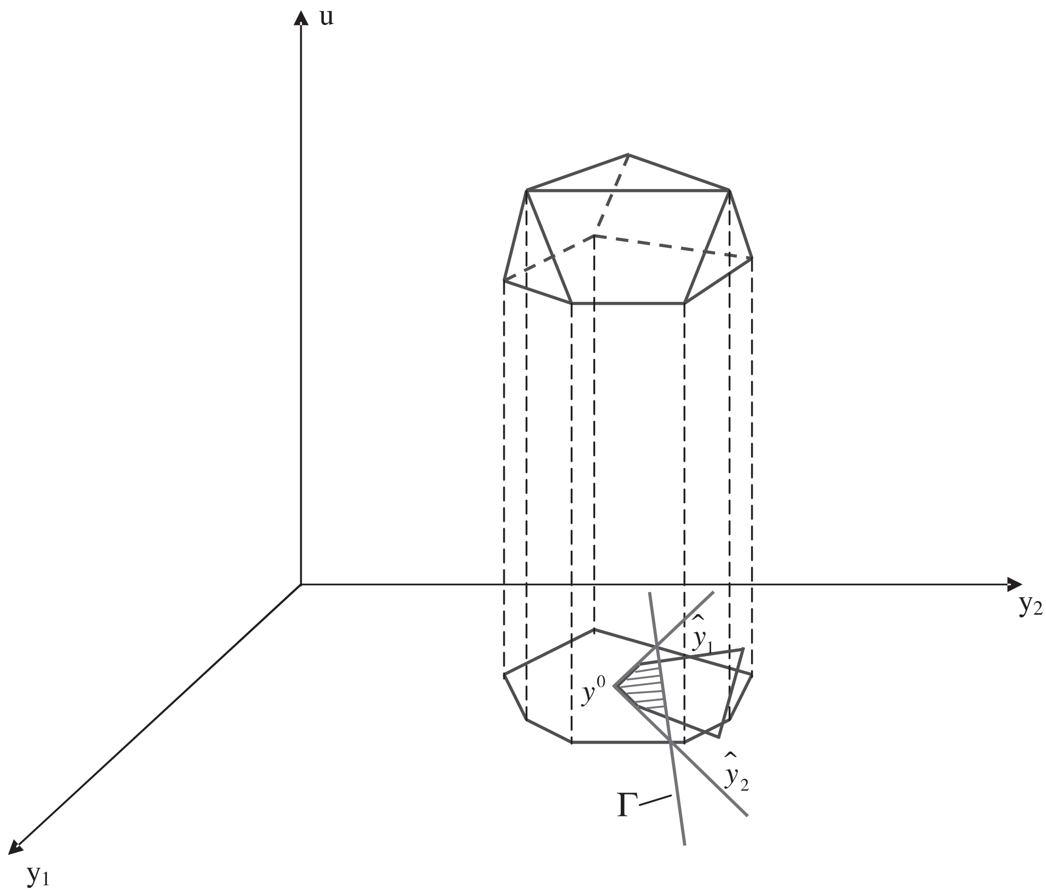

Fig.4

In general, this algorithm finds the exact solution of BPP (1)–(3). But if the process of calculation takes too much time then we can stop it and we obtain an admissible solution of BPP (1)-(3), which is appropriate to an optimal one.

Taking into account the geometrical interpretation of the auxiliary parametrical programming problem we may conclude that the proposed algorithm will work efficiently if BPP (1)-(3) has a global minimum with the corresponding value of the object function, which differs essentially from the values of local minima. Namely in this case for the auxiliary problem the set Y of the parametrical problem is a small part of the orthogonal projection $\overline{Y}$. In the case when BPP has many local minima with not essential deviations of the corresponding values of the object function the algorithm may work too much time.

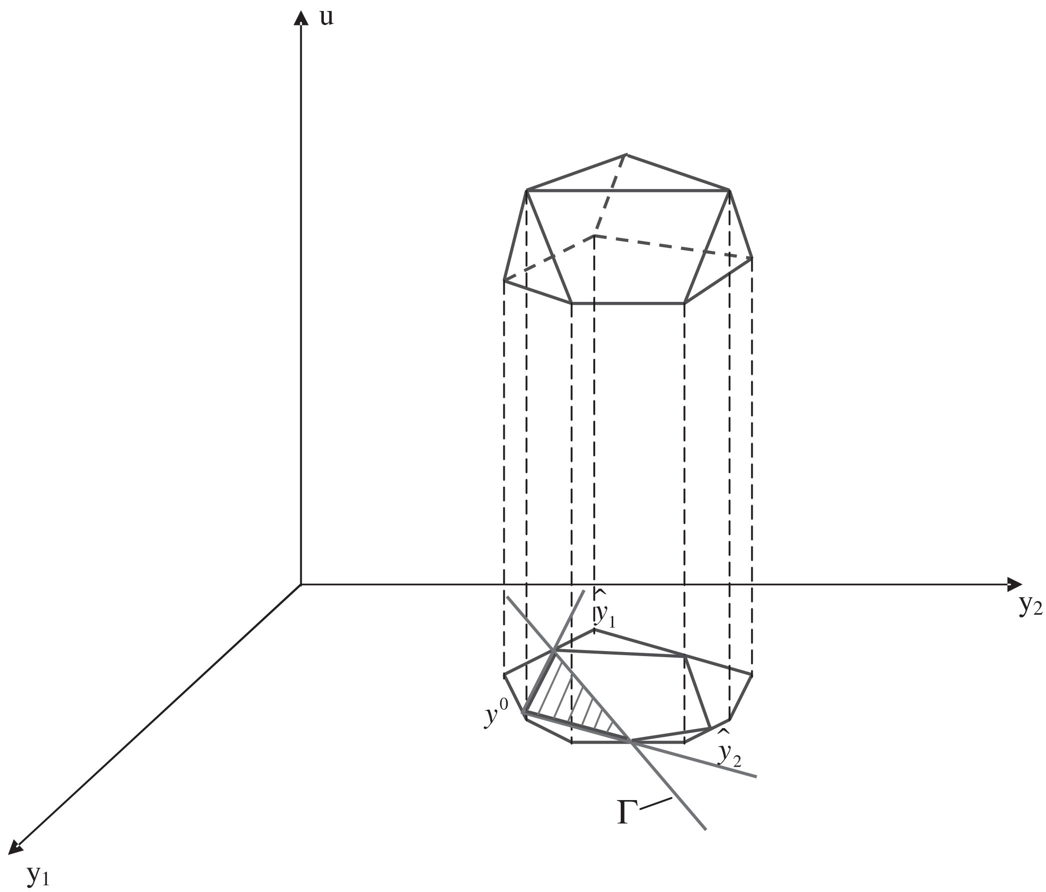

Fig.5

6 Applications

In this section we show that the proposed approach can be used for studying and solving the linear boolean programming problem and the resource allocation problem.

Let us consider the following linear boolean programming problem:
to minimize

$$z = \sum_{j=1}^{n} c_j x_j$$

on subject

$$\begin{cases} \sum_{j=1}^{n} a_{ij} x_j \leq a_i, \; i = \overline{1, m_2}; \\ x_j \in \{0, 1\}, \; j = \overline{1, n}. \end{cases}$$

It is easy to observe that if this problem has solution then it is equivalent to the following concave programming problem *to minimize*

$$z = \sum_{j=1}^{n} c_j x_j + M \sum_{j=1}^{n} \min\{x_j, 1 - x_j\}$$

on subject

$$\begin{cases} \sum\limits_{j=1}^{n} a_{ij} x_j \le a_i, \; i = \overline{1, m_2}; \\ 0 \le x_j \le 1, \; j = \overline{1, n}, \end{cases}$$

where $M > \sum_{j=1}^{n} |c_j|$.

In the following we represent this problem as BPP:
to minimize

$$z = \sum_{j=1}^{n} c_j x_j + M \sum_{j=1}^{n} (x_j y_j + (1 - x_j)(1 - y_j))$$

on subject

$$\begin{cases} \sum\limits_{j=1}^{n} a_{ij} x_j \le a_i, \; i = \overline{1, m_2}; \\ 0 \le x_j \le 1, \; j = \overline{1, n}; \\ 0 \le y_j \le 1, \; j = \overline{1, n}. \end{cases}$$

So, we obtain BPP (1)-(3), where

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

This means that BPP (1)-(3) is NP-hard even in such a case.

In [7] it is shown that the proposed approach can be used for studying and solving the following concave programming problem:
to minimize

$$z = \sum_{i=1}^{q} \min\{c^{il} x + c_0^{il}, \; l = \overline{1, r_i}\} \tag{13}$$

on subject

$$\begin{cases} Ax \le a; \\ x \ge 0, \end{cases} \tag{14}$$

where $x \in R^n$, $c^{il} \in R^n$, $c_0^{il} \in R^1$, A is an $m_2 \times n$-matrix, $a \in R^{m_2}$. This problem arises as an auxiliary one when solving a large class of resource allocation problems [7,9].

Problem (13)-(14) can be transformed into BPP:
to minimize

$$z = \sum_{i=1}^{q} \sum_{l=1}^{r_i} (c^{il} x + c_0^{il}) y_{il}$$

on subject

$$\begin{cases} Ax \le a, \quad x \ge 0; \\ \sum_{l=1}^{r_i} y_{il} = 1, \ i = \overline{1, q}; \\ y_{il} \ge 0, \ l = \overline{1, r_i}, \ i = \overline{1, q}. \end{cases}$$

In a more detailed form the algorithm for solving this problem is described in [7].

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