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The $GL(2,\mathbb{R})$-orbits of polynomial differential systems of degree four

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Abstract. In this paper we characterize the $GL(2,\mathbb{R})-$orbits of the differential systems $\dot{x_1} = P(x_1, x_2)$, $\dot{x_2} = Q(x_1, x_2)$, where P, Q are polynomials of degree four, with respects to their dimensions.

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1 Center-affine transformations

Consider the system

$$\dot{x_1} = \sum_{k=0}^{4} P_k(x_1, x_2) \equiv P(x_1, x_2), \quad \dot{x_2} = \sum_{k=0}^{4} Q_k(x_1, x_2) \equiv Q(x_1, x_2), \tag{1}$$

where

$$P_k(x_1, x_2) = \sum_{i+j=k} a_{ij} x_1^i x_2^j, \quad Q_k(x_1, x_2) = \sum_{i+j=k} b_{ij} x_1^i x_2^j.$$

Denote by E the space of the coefficients

$$a = (a_{00}, a_{10}, a_{01}, a_{20}, \ldots, a_{13}, a_{04}; b_{00}, b_{10}, b_{01}, b_{20}, \ldots, b_{13}, b_{04})$$

of system (1) and by $GL(2,\mathbb{R})$ the group of the center-affine transformations of the phase space Ox, $x = (x_1, x_2)$.

Applying in (1) the transformation $X = qx$, where $X = (X_1, X_2)$, $q \in GL(2,\mathbb{R})$, i.e.

$$q = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}, \alpha_{ij} \in \mathbb{R}, \Delta = det(q) \neq 0, \ q^{-1} = \frac{1}{\Delta} \begin{pmatrix} \alpha_{22} & -\alpha_{12} \\ -\alpha_{21} & \alpha_{11} \end{pmatrix},$$

we obtain the system

$$\dot{X}_1 = \sum_{i+j=0}^{4} a^*_{ij} X_1^i X_2^j, \quad \dot{X}_2 = \sum_{i+j=0}^{4} b^*_{ij} X_1^i X_2^j. \tag{2}$$

The coefficients a^* of (2) are expressed linearly by coefficients of system (1): $a^* = \Lambda_{(q)}(a)$, $det\Lambda_{(q)} \neq 0$. The set $\Lambda = \{\Lambda_{(q)} | q \in GL(2,\mathbb{R})\}$ forms a 4-parameter group with the operation of composition. Λ is called the representation of the $GL(2,\mathbb{R})$ group of the center-affine transformations of the phase space Ox in the space of coefficients E of system (1).

The set $O(a) = \{\Lambda_{(q)}(a) |\, q \in GL(2,\mathbb{R})\}$ is called a $GL(2,\mathbb{R})-$orbit of the point $a \in E$ or of the differential system (1) corresponding to this point.

Let

$$q_1^t = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix},\; q_2^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},\; q_3^t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},\; q_4^t = \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}$$

and $G_l = \{q_l^t |\, t \in \mathbb{R}\} \subset GL(2,\mathbb{R})$, $l = \overline{1,4}$. Denote $g_l^t = \Lambda_{(q_l^t)}$ and $a^{*l} = g_l^t(a) \in E$. Then $\Lambda_l = \{g_l^t\}$, $l = \overline{1,4}$, are representations in E of the subgroups G_l, respectively. Each of the pairs $(E, \{g_l^t\})$, $l = \overline{1,4}$, is a differential flow. They define in E the following differential system of linear equations

$$\frac{da}{dt} = \left(\frac{dg_l^t(a)}{dt}\right)\Big|_{t=0} = A^{(l)} \cdot a, \quad l = \overline{1,4}. \tag{3}$$

Let

$$v_l = \sum_{i+j=0}^{4} \Big(A_{ij}^{(l)} \frac{\partial}{\partial a_{ij}} + B_{ij}^{(l)} \frac{\partial}{\partial b_{ij}}\Big), \quad l = \overline{1,4},$$

be the vector fields defined in E by systems (3). The coordinates of the vectors v_l, $l = \overline{1,4}$, are given by the formulas

$$\begin{aligned}
&A_{ij}^{(1)} = (1-i)a_{ij}, && B_{ij}^{(1)} = -ib_{ij};\\
&A_{i0}^{(2)} = b_{i0}, && A_{ij}^{(2)} = b_{ij} - (i+1)a_{i+1,j-1};\\
&B_{i0}^{(2)} = 0, && B_{ij}^{(2)} = -(i+1)b_{i+1,j-1}, \quad j \neq 0;\\
&A_{0j}^{(3)} = 0, && A_{ij}^{(3)} = -(j+1)a_{i-1,j+1};\\
&B_{0j}^{(3)} = a_{0j}, && B_{ij}^{(3)} = a_{ij} - (j+1)b_{i-1,j+1},\ i \neq 0;\\
&A_{ij}^{(4)} = -ja_{ij}, && B_{ij}^{(4)} = (1-j)b_{ij}.
\end{aligned}$$

If we denote by L_v the derivative with respect to the vector v and we set $w = [u,v]$, where $L_w = L_uL_v - L_vL_u$, it is easy to determine that the vector fields v_l, $l = \overline{1,4}$, generate a Lie algebra. The dimension of the orbit $O(a)$ is equal to the dimension of this algebra, i.e. with the rank of the matrix of dimension 4×30 [1, 2]:

$$M = \begin{pmatrix} A_{00}^{(1)} & A_{10}^{(1)} & A_{01}^{(1)} & A_{20}^{(1)} & \dots & A_{04}^{(1)} & B_{00}^{(1)} & \dots & B_{04}^{(1)} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ A_{00}^{(4)} & A_{10}^{(4)} & A_{01}^{(4)} & A_{20}^{(4)} & \dots & A_{04}^{(4)} & B_{00}^{(4)} & \dots & B_{04}^{(4)} \end{pmatrix}. \tag{4}$$

The purpose of this paper consists in the classification of systems (1) according to the dimensions of their $GL(2,\mathbb{R})-$orbits.

We notice that such classification was done for some particular cases of system (1) in [2–9].

From [10] follows

Lemma 1. *Let $O(a)$ be a $GL(2,\mathbb{R})-$orbit of the system (1). Then*

1) $dimO(a) = 0$ if and only if (1) has the form

$$\dot{x_1} = bx_1, \quad \dot{x_2} = bx_2, \quad b = const; \tag{5}$$

2) $dimO(a) \neq 1$, $\forall a \in E$.

By Lemma 1, $dimO(a) > 1$, i.e. $dimO(a)$ is equal to one of the numbers 2,3 or 4, if and only if

$$|P(x_1, x_2) - a_{10}x_1| + |Q(x_1, x_2) - a_{10}x_2| \not\equiv 0.$$

Therefore, if the right-hand sides of the system (1) have either at least one constant term a_{00}, b_{00} or one nonlinear term, then the dimension of the $GL(2,\mathbb{R})-$orbit is at least two.

For the linear system

$$\dot{x_1} = a_{10}x_1 + a_{01}x_2, \quad \dot{x_2} = b_{10}x_1 + b_{01}x_2 \tag{6}$$

the matrix (4) has the form:

$$M_1 = \begin{pmatrix} 0 & a_{01} & -b_{10} & 0 \\ b_{10} & b_{01} - a_{10} & 0 & -b_{10} \\ -a_{01} & 0 & a_{10} - b_{01} & a_{01} \\ 0 & -a_{01} & b_{10} & 0 \end{pmatrix}. \tag{7}$$

It is easy to determine that $rank\, M_1 \leq 2$. So, the linear system has the orbit's dimension equal to zero only if it has the form (5) and $dim\, O(a) = 2$ in other cases, i.e. when

$$\dot{x_1} = a_{10}x_1 + a_{01}x_2, \quad \dot{x_2} = b_{10}x_1 + b_{01}x_2, \; |a_{10} - b_{01}| + |a_{01}| + |b_{10}| \neq 0. \tag{8}$$

Applying in (1) the transformation of coordinates

$$x_1 \longrightarrow x_2, \quad x_2 \longrightarrow x_1, \tag{9}$$

we obtain

$$\dot{x_1} = Q(x_2, x_1), \quad \dot{x_2} = P(x_2, x_1). \tag{10}$$

Denote by v_l^*, $l = \overline{1,4}$, the vectorial fields associated to the differential system (10).

Remark 1. The equalities $\alpha v_1 + \beta v_2 + \gamma v_3 + \delta v_4 = 0$ and $\delta v_1^* + \gamma v_2^* + \beta v_3^* + \alpha v_4^* = 0$ $(\alpha, \beta, \gamma, \delta \in \mathbb{R})$ are equivalent.

Talking into consideration Remark 1, in order to determine the orbits of dimension two and three it is enough to examine the following two cases:

$$\alpha v_1 + v_4 = 0, \tag{11}$$

$$\alpha\, v_1 + v_2 + \gamma\, v_3 + \delta\, v_4 = 0. \tag{12}$$

2 The case $\alpha v_1 + v_4 = 0$

The equality (11) written in the coordinates of v_1 and v_4 represents a homogeneous linear algebraic system in coefficients a_{ij} of (1).

If $v_1 = v_4 = 0$, then (1) is of the form $\dot{x_1} = a_{10}x_1$, $\dot{x_2} = b_{01}x_2$ and it is a particular case of (6).

In the case when at least one of the vectors v_1 and v_4 is nonzero, this algebraic system has nontrivial solutions only for the following values of the parameter α :

$$\alpha = \pm 2;\ 3;\ 4;\ \pm 1;\ 0;\ \pm\frac{1}{2};\ \frac{1}{3};\ \frac{1}{4}.$$

According to Remark 1, it is enough to examine only the cases:

$$\alpha = 2;\ 3;\ 4;\ -1;\ -2;\ 0;\ 1.$$

To this values of α the following solutions correspond respectively:

1) $a_{ij} = 0,\ (i,j) \neq (1,0),\ (0,2),\ b_{ij} = 0,\ (i,j) \neq (0,1)$;

2) $a_{ij} = 0,\ (i,j) \neq (1,0),\ (0,3),\ b_{ij} = 0,\ (i,j) \neq (0,1)$;

3) $a_{ij} = 0,\ (i,j) \neq (1,0),\ (0,4),\ b_{ij} = 0,\ (i,j) \neq (0,1)$;

4) $a_{ij} = 0,\ (i,j) \neq (1,0),\ (2,1),\ b_{ij} = 0,\ (i,j) \neq (0,1), (1,2)$;

5) $a_{ij} = 0,\ (i,j) \neq (1,0),\ (2,2),\ b_{ij} = 0,\ (i,j) \neq (0,1), (1,3)$;

6) $a_{ij} = 0,\ j \neq 0,\ b_{ij} = 0,\ j \neq 1$;

6a) $a_{ij} = b_{ij} = 0,\ i + j \neq 1$.

Notice that in the case 6a) we obtain the linear system (6).

Denote

$i_1 = |a_{00}| + |b_{01} - a_{10}| + |b_{11} - a_{20}| + |b_{21} - a_{30}| + |a_{40}| + |b_{31}|$,

$i_2 = |b_{01} - a_{10}| + |a_{20}| + |a_{30}| + |a_{40}| + |b_{11}| + |b_{21}| + |b_{31}|$,

$i_3 = |a_{00}| + |a_{20}| + |a_{30}| + |a_{40}| + |b_{11}| + |b_{21}| + |b_{31}|$.

In order to separate the orbits of dimension two from those of dimension three, we will determine the conditions on the coefficients of system (1) such that in each of the cases 1) $-$ 6) all the minors of order three of matrix (4) should be equal to zero. We have respectively:

$1')$ $a_{02}(a_{10} - b_{01}) = 0$; $2')$ $a_{03}(a_{10} - b_{01}) = 0$;

$3')$ $a_{04}(a_{10} - b_{01}) = 0$; $4')$ $|a_{21}| + |b_{12}| = 0$;

$5')$ $|a_{22}| + |b_{13}| = 0$; $6')$ $i_1 \cdot i_2 \cdot i_3 = 0$.

The cases $[1), 1'),\ a_{02} = 0]$; $[2), 2'),\ a_{03} = 0]$; $[3), 3'),\ a_{04} = 0]$; $[4), 4'), a_{21} = b_{12} = 0]$; $[5), 5'), a_{22} = b_{13} = 0]$ and $[6), 6'), i_3 = 0]$ lead us to a system of the form (6). Later on, assuming that $\alpha v_1 + v_4 = 0$, we have the following distribution by dimensions of orbits of the system (1) (the systems (5) and (6) are not included here):

dim O(a)=2

$$\dot{x_1} = a_{10}x_1 + a_{02}x_2^2, \quad \dot{x_2} = a_{10}x_2, \quad a_{02} \neq 0; \tag{13}$$

$$\dot{x_1} = a_{10}x_1 + a_{03}x_2^3, \quad \dot{x_2} = a_{10}x_2, \quad a_{03} \neq 0; \tag{14}$$

$$\dot{x_1} = a_{10}x_1 + a_{04}x_2^4, \quad \dot{x_2} = a_{10}x_2, \quad a_{04} \neq 0; \tag{15}$$

$$\dot{x_1} = x_1 \cdot F,\ \dot{x_2} = x_2 \cdot F,\ F = a_{10} + a_{20}x_1 + a_{30}x_1^2,\ |a_{20}| + |a_{30}| \neq 0; \tag{16}$$

$$\dot{x_1} = a_{00} + a_{10}x_1, \quad \dot{x_2} = a_{10}x_2, \quad a_{00} \neq 0. \tag{17}$$

dim O(a)=3

$$\dot{x_1} = a_{10}x_1 + a_{02}x_2^2, \quad \dot{x_2} = b_{01}x_2, \quad a_{02}(a_{10} - b_{01}) \neq 0; \tag{18}$$

$$\dot{x_1} = a_{10}x_1 + a_{03}x_2^3, \quad \dot{x_2} = b_{01}x_2, \quad a_{03}(a_{10} - b_{01}) \neq 0; \tag{19}$$

$$\dot{x_1} = a_{10}x_1 + a_{04}x_2^4, \quad \dot{x_2} = b_{01}x_2, \quad a_{04}(a_{10} - b_{01}) \neq 0; \tag{20}$$

$$\dot{x_1} = x_1(a_{10} + a_{21}x_1x_2), \quad \dot{x_2} = x_2(b_{01} + b_{12}x_1x_2),\ |a_{21}| + |b_{12}| \neq 0; \tag{21}$$

$$\dot{x_1} = x_1(a_{10} + a_{22}x_1x_2^2), \quad \dot{x_2} = x_2(b_{01} + b_{13}x_1x_2^2), \quad |a_{22}| + |b_{13}| \neq 0; \tag{22}$$

$$\begin{cases} \dot{x_1} = a_{00} + a_{10}x_1 + a_{20}x_1^2 + a_{30}x_1^3 + a_{40}x_1^4, \\ \dot{x_2} = x_2(b_{01} + b_{11}x_1 + b_{21}x_1^2 + b_{31}x_1^3),\ i_1 \cdot i_2 \cdot i_3 \neq 0. \end{cases} \tag{23}$$

3 The case $\alpha\, v_1 + v_2 + \gamma\, v_3 + \delta\, v_4 = 0$.

In this section we will need the following notations:
$\alpha_i = (\delta - i\alpha)/(i+1), \delta_i = (\alpha - i\delta)/(i+1), i = \overline{1,4}, \nu_1 = \delta + 2\alpha, \nu_2 = \alpha + 2\delta;$
$j_1 = |a_{12} - 3a_{03}\alpha| + |\alpha + \delta| + |a_{00}| + |a_{11}| + |a_{02}| + |a_{13}| + |a_{04}|,$
$j_2 = |a_{00}| + |a_{01}| + |a_{02}| + |a_{03}| + |a_{04}|,$
$j_3 = |a_{11} - 2\alpha\, a_{02}| + |\alpha + \delta| + |a_{00}| + |a_{01}| + |a_{12}| + |a_{03}| + |a_{13}| + |a_{04}|,$
$j_4 = |a_{00}| + |a_{11}| + |a_{02}| + |a_{12}| + |a_{03}| + |a_{13}| + |a_{04}|,$
$j_5 = |a_{13} - 4\alpha\, a_{04}| + |\alpha + \delta| + |a_{00}| + |a_{01}| + |a_{11}| + |a_{02}| + |a_{12}| + |a_{03}|,$
$j_6 = |a_{01}| + |a_{03}| + |\gamma + \delta^2|,$
$j_7 = |a_{12} + 3\delta\, a_{03}| + |\gamma + \delta^2| + |a_{01}|,$
$j_8 = |a_{12}| + |a_{03}|,$
$j_9 = |a_{13}| + |a_{04}|,$
$j_{10} = |\alpha + \delta| + |a_{01}| + |a_{04}|,$
$j_{11} = |a_{13} + 4\delta\, a_{04}| + |\alpha + \delta| + |a_{01}|.$

The equality (12) holds if and only if at least one of the following seven series of conditions is realized:

7) $\gamma = \alpha_2 \cdot \delta_2,\ a_{20} = \delta_2^2 a_{02},\ a_{11} = 2\delta_2 a_{02},\ b_{10} = \alpha_2\delta_2 a_{01},\ b_{01} = a_{10} - 2\delta_1 a_{01},$ $b_{20} = -\delta_2^3 a_{02},\ b_{11} = -2\delta_2^2 a_{02},\ b_{02} = -\delta_2 a_{02},\ a_{ij} = b_{ij} = 0,\ i + j = 0, 3, 4;$

8) $\gamma = \alpha_3 \cdot \delta_3$, $a_{30} = a_{03}\delta_3^3$, $a_{21} = 3a_{03}\delta_3^2$, $a_{12} = 3a_{03}\delta_3$, $b_{10} = a_{01}\alpha_3\delta_3$, $b_{01} = a_{10} - 2\delta_1 a_{01}$, $b_{30} = -a_{03}\delta_3^4$, $b_{21} = -3a_{03}\delta_3^3$, $b_{12} = -3a_{03}\delta_3^2$, $b_{03} = -a_{03}\delta_3$, $a_{ij} = b_{ij} = 0$, $i + j = 0, 2, 4$;

9) $\gamma = \alpha_4 \cdot \delta_4$, $a_{40} = a_{04}\delta_4^4$, $a_{31} = 4a_{04}\delta_4^3$, $a_{22} = 6a_{04}\delta_4^2$, $a_{13} = 4a_{04}\delta_4$, $b_{10} = a_{01}\alpha_4\delta_4$, $b_{01} = a_{10} - 2\delta_1 a_{01}$, $b_{40} = -a_{04}\delta_4^5$, $b_{31} = -4a_{04}\delta_4^4$, $b_{22} = -6a_{04}\delta_4^3$, $b_{13} = -4a_{04}\delta_4^2$, $b_{04} = -a_{04}\delta_4$, $a_{ij} = b_{ij} = 0$, $i + j = 0, 2, 3$;

10) $\gamma = \alpha \cdot \delta$, $a_{20} = -\delta(a_{11} + \delta a_{02})$, $a_{30} = \delta^2(a_{12} + 2\delta a_{03})$, $a_{21} = -\delta(2a_{12} + 3a_{03}\delta)$, $a_{40} = -\delta^3(a_{13} + 3\delta a_{04})$, $a_{31} = \delta^2(3a_{13} + 8a_{04}\delta)$, $a_{22} = -3\delta(a_{13} + 2\delta a_{04})$, $b_{00} = -\alpha a_{00}$, $b_{10} = a_{01}\alpha\delta$, $b_{01} = a_{10} - 2\delta_1 a_{01}$, $b_{20} = -a_{02}\alpha\delta^2$, $b_{11} = \delta(4\delta_1 a_{02} - a_{11})$, $b_{02} = a_{11} - 3\delta_2 a_{02}$, $b_{30} = a_{03}\alpha\delta^3$, $b_{21} = \delta^2(a_{12} - 6a_{03}\delta_1)$, $b_{12} = \delta(9a_{03}\delta_2 - 2a_{12},)$ $b_{03} = a_{12} - 4\delta_3 a_{03}$, $b_{40} = -\alpha\delta^4 a_{04}$, $b_{31} = \delta^3(8a_{04}\delta_1 - a_{13})$, $b_{22} = 3\delta^2(a_{13} - 6a_{04}\delta_2)$, $b_{13} = \delta(16a_{04}\delta_3 - 3a_{13}), b_{04} = a_{13} - 5\delta_4 a_{04}$;

11) $\alpha = -\delta$, $a_{30} = -\gamma(a_{12} + 2\delta a_{03})$, $a_{21} = -2a_{12}\delta - 4a_{03}\delta^2 - \gamma a_{03}$, $b_{10} = \gamma a_{01}$, $b_{01} = a_{10} + 2\delta a_{01}$, $b_{30} = -\gamma^2 a_{03}$, $b_{21} = -\gamma(a_{12} + 6\delta a_{03})$, $b_{12} = -2\delta a_{12} - 8\delta^2 a_{03} + \gamma a_{03}$, $b_{03} = a_{12} + 4\delta a_{03}$, $a_{ij} = b_{ij} = 0$, $i + j = 0, 2, 4$;

12) $\gamma = \nu_1 \cdot \nu_2$, $b_{10} = \nu_1\nu_2 a_{01}$ $b_{01} = a_{10} - 2\delta_1 a_{01}$, $a_{40} = \nu_1\nu_2^2(a_{13} + 3\delta a_{04})$, $a_{31} = -\nu_2(3\alpha a_{13} - 8\delta_1^2 a_{04})$, $a_{22} = -3(a_{04}\alpha^2 + a_{13}\delta + 2\alpha\delta a_{04} + 3\delta^2 a_{04})$, $b_{40} = \nu_1^2\nu_2^3 a_{04}$, $b_{31} = \nu_1\nu_2^2(a_{13} - 8\delta_1 a_{04})$, $b_{22} = -3\nu_2(a_{13}\alpha - a_{04}\alpha^2 + 6a_{04}\alpha\delta + a_{04}\delta^2)$, $b_{13} = -3\delta a_{13} + 4a_{04}\delta_1(\alpha + 5\delta)$, $b_{04} = a_{13} - 5a_{04}\delta_4$, $a_{ij} = b_{ij} = 0$, $i + j = 0, 2, 3$;

13) $a_{10} = \alpha b_{10}b_{01} - \delta b_{10}$, $b_{10} = \gamma a_{01}$, $a_{ij} = b_{ij} = 0$, $i + j = 0, 2, 3, 4$.

Notice that in conditions 13) we have a system of the form (6).

Equating to zero the minors of order three of the matrix (4) in each of the cases 7) − 12), we obtain respectively:

$7')$ $a_{01} \cdot a_{02} = 0$;
$8')$ $a_{01} \cdot a_{03} = 0$;
$9')$ $a_{01} \cdot a_{04} = 0$;
$10')$ $j_1 \cdot j_2 \cdot j_3 \cdot j_4 \cdot j_5 = 0$;
$11')$ $j_6 \cdot j_7 \cdot j_8 = 0$;
$12')$ $j_9 \cdot j_{10} \cdot j_{11} = 0$.

The relations [7), $7')$] − [12), $12')$] lead us to the following distribution of the $GL(2, \mathbb{R})$−orbits of the system (1) (the cases which lead us to the system (6) are not considered here):

dim O(a)=2

$$\dot{x_1} = a_{10}x_1 + F, \quad \dot{x_2} = a_{10}x_2 - \delta_2 \cdot F, \quad F = a_{02}(\delta_2 x_1 + x_2)^2 \not\equiv 0; \tag{24}$$

$$\dot{x_1} = a_{10}x_1 + F, \quad \dot{x_2} = a_{10}x_2 - \delta_3 \cdot F, \quad F = a_{03}(\delta_3 x_1 + x_2)^3 \not\equiv 0; \tag{25}$$

$$\dot{x_1} = a_{10}x_1 + F, \quad \dot{x_2} = a_{10}x_2 - \delta_4 \cdot F, \quad F = a_{04}(\delta_4 x_1 + x_2)^4 \not\equiv 0; \tag{26}$$

$$\begin{cases} \dot{x_1} = x_1 \cdot F, \quad \dot{x_2} = x_2 \cdot F, \\ F = a_{10} - a_{11}(\delta\, x_1 - x_2) + a_{12}(\delta\, x_1 - x_2)^2 - a_{13}(\delta\, x_1 - x_2)^3, \\ |a_{11}| + |a_{12}| + |a_{13}| \neq 0; \end{cases} \tag{27}$$

$$\dot{x_1} = a_{00} + a_{10}x_1, \quad \dot{x_2} = -\alpha\, a_{00} + a_{10}x_2, \quad a_{00} \neq 0. \tag{28}$$

dim O(a)=3

$$\begin{cases} \dot{x_1} = a_{10}x_1 + a_{01}x_2 + F, \\ \dot{x_2} = \alpha_2\delta_2 a_{01}x_1 + (a_{10} - 2\delta_1 a_{01})x_2 - \delta_2 \cdot F, \\ F = a_{02}(\delta_2 x_1 + x_2)^2, \quad a_{01} \cdot a_{02} \neq 0; \end{cases} \tag{29}$$

$$\begin{cases} \dot{x_1} = a_{10}x_1 + a_{01}x_2 + F, \\ \dot{x_2} = \alpha_3\delta_3 a_{01}x_1 + (a_{10} - 2\delta_1 a_{01})x_2 - \delta_3 \cdot F, \\ F = a_{03}(\delta_3 x_1 + x_2)^3, \quad a_{01} \cdot a_{03} \neq 0; \end{cases} \tag{30}$$

$$\begin{cases} \dot{x_1} = a_{10}x_1 + a_{01}x_2 + F, \\ \dot{x_2} = \alpha_4\delta_4 a_{01}x_1 + (a_{10} - 2\delta_1 a_{01})x_2 - \delta_4 \cdot F, \\ F = a_{04}(\delta_4 x_1 + x_2)^4, \quad a_{01} \cdot a_{04} \neq 0; \end{cases} \tag{31}$$

$$\begin{cases} \dot{x_1} = a_{00} + a_{10}x_1 + a_{01}x_2 - ((a_{11} + a_{02}\delta)x_1 + a_{02}x_2) \cdot F + \\ \qquad + ((a_{12} + 2a_{03}\delta)x_1 + a_{03}x_2) \cdot F^2 - \\ \qquad - ((a_{13} + 3a_{04}\delta)x_1 + a_{04}x_2) \cdot F^3, \\ \dot{x_2} = -\alpha a_{00} + \alpha\delta a_{01}x_1 + (a_{10} - 2\delta_1 a_{01})x_2 - \\ \qquad - (\alpha\,\delta a_{02}x_1 + (a_{11} - 3a_{02}\delta_2)x_2) \cdot F + \\ \qquad + (\alpha\,\delta a_{03}x_1 + (a_{12} - 4a_{03}\delta_3)x_2) \cdot F^2 - \\ \qquad - (\alpha\,\delta a_{04}x_1 + (a_{13} - 5a_{04}\delta_4)x_2) \cdot F^3, \\ F = \delta\, x_1 - x_2, \quad j_1 \cdot j_2 \cdot j_3 \cdot j_4 \cdot j_5 \neq 0; \end{cases} \tag{32}$$

$$\begin{cases} \dot{x_1} = a_{10}x_1 + a_{01}x_2 - ((a_{12} + 2a_{03})x_1 + a_{03}x_2) \cdot F, \\ \dot{x_2} = \gamma\, a_{01}x_1 + (a_{10} - 2a_{01}\delta_1)x_2 + (a_{03}\gamma x_1 + (a_{12} + 4a_{03})x_2) \cdot F, \\ F = \gamma\, x_1^2 + 2\delta\, x_1x_2 - x_2^2, \quad j_6 \cdot j_7 \cdot j_8 \neq 0; \end{cases} \tag{33}$$

$$\begin{cases} \dot{x_1} = a_{10}x_1 + a_{01}x_2 + ((a_{13} + 3\delta\, a_{04})x_1 + a_{04}x_2) \cdot F, \\ \dot{x_2} = \nu_1\,\nu_2\, a_{01}x_1 + (a_{10} - 2\delta_1 a_{01})x_2 + \\ \qquad + (\nu_1\,\nu_2\, a_{04}x_1 + (a_{13} - 5\delta_4\, a_{04})x_2) \cdot F, \\ F = (\nu_1\, x_1 + x_2)(\nu_2\, x_1 - x_2)^2, \quad j_9 \cdot j_{10} \cdot j_{11} \neq 0. \end{cases} \tag{34}$$

Remark 2. It is easy to see that the systems (13) − (15), (17) are particular cases of the systems (24) − (26), (28) respectively. The (16) by substitution (9) can be reduced to a system of the form (27).

The results obtained above are gathered in the following theorem:

Theorem. *Up to a transformation (9), the dimension of the $GL(2,\mathbb{R})-$orbit of the system (1) is equal to*

0 *if it has the form (5);*
2 *if it has one of the forms (8), (24) − (28);*
3 *if it has one of the forms (18) − (23), (29) − (34);*
4 *in other cases.*

References

[1] Ovsyanikov L.V. *Group analysis of differential equations.* Moscow, Nauka, 1978 (English transl. by Academic press, 1982.)

[2] Popa M.N. *Applications of algebras to differential systems.* Academy of Sciences of Moldova, Chişinău, 2001 (in Russian).

[3] Braicov A.V., Popa M.N. *The* $GL(2,\mathbb{R})-$*orbits of differential system with homogeneites second order.* The Internationals Conference "Differential and Integral Equations", Odessa, September 12–14, 2000, p. 31.

[4] Boularas D., Braicov A.V., Popa M.N. *Invariant conditions for dimensions of* $GL(2,\mathbb{R})-$*orbits for quadratic differential system.* Bul. Acad. Sci. Rep. Moldova, Math., 2000, N 2(33), p. 31–38.

[5] Boularas D., Braicov A.V., Popa M.N. *The* $GL(2,\mathbb{R})-$*orbits of differential system with cubic homogeneites.* Bul. Acad. Sci. Rep. Moldova, Math., 2001, N 1(35), p. 81–82.

[6] Naidenova E.V., Popa M.N. *On a classification of Orbits for Cubic Differential Systems.* Abstracts of "16th International Symposium on Nonlinear Acoustics", section "Modern group analysis" (MOGRAN-9), August 19–23, 2002, Moscow, p. 274.

[7] Naidenova E.V., Popa M.N. $GL(2,\mathbb{R})-$*orbits for one cubic system.* Abstracts of "11th Conference on Applied and Industrial Mathematics", May 29–31, 2003, Oradea, Romania, p. 57.

[8] Starus E.V. *Invariant conditions for the dimensions of the* $GL(2,\mathbb{R})-$*orbits for one differential cubic system*, Bul. Acad. Sci. Rep. Moldova, Math., 2003, N 3(43), p. 58–70.

[9] Starus E.V. *The classification of the* $GL(2,\mathbb{R})-$*orbit's dimensions for the system* $s(0,2)$ *and a factorsystem* $s(0,1,2)/GL(2,\mathbb{R})$. Bul. Acad. Sci. Rep. Moldova, Math., 2004, N 1(44), p. 120–123.

[10] Păşcanu A., Şubă A. $GL(2,\mathbb{R})-$*orbits of the polynomial systems of differential equation.* Bul. Acad. Sci. Rep. Moldova, Math., 2004, N 3(46), p. 25–40.

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