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Linear stability bounds in a convection problem for variable gravity field

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Abstract. A problem governing the convection-conduction in a horizontal layer bounded by rigid walls of a fluid heated from below for a linearly decreasing across the layer gravity field is reformulated as a variational problem. Stability bounds from the case of classical convection [1] and the case of convection in a linearly decreasing across the layer gravity field are compared. The new criterion, which yields good stability bounds for the stability limit, is shown by the numerical evaluations obtained in [2–4].

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1 Problem setting

Variations in the gravity field occur in, on and above the Earth's surface due to the fluid and atmosphere dynamics. In order to study the variable gravity effects on various convection problems and to compare them with the results obtained on a laboratory scale or deduced from the atmospheric models the mathematical model governing the conduction-convection must be investigated. In this paper we analyze the influence of a linearly decreasing gravity field on the stability bounds in a convection problem. The governing mathematical model is that given in [7]. This problem is quite unusual in the linear hydrodynamic stability theory due to the variable coefficients involved in the equations. It is a two-point eigenvalue problem, where the Rayleigh number is the eigenvalue which can be expressed by a functional defined on a Hilbert space of smooth functions satisfying some boundary conditions. The smallest eigenvalue, the only one of interest in applications, corresponds to the neutral stability in the case when the principle of exchange of stability holds. It can be computed as the minimum of that functional in the class H of admissible functions. This variational problem can be solved by means of a Fourier series technique and its solution is the smallest eigenvalue, called the linear stability limit. An alternative approach is to use isoperimetric and algebraic inequalities to provide bounds of this limit. Herein these two types of results are reported.

Consider a horizontal layer of a heat conducting viscous fluid situated between the planes $z = 0$ and $z = h$. For $t > 0$ the conduction and convective motion is governed by the conservation equations of momentum, mass and internal energy [7]

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot \mathrm{grad})\mathbf{v} = -\dfrac{1}{\rho}\mathrm{grad}p + \nu\Delta\mathbf{v} + \mathbf{g}(z)\alpha T, \\ \mathrm{div}\mathbf{v} = 0, \\ \dfrac{\partial T}{\partial t} + (\mathbf{v}\cdot \mathrm{grad})T = k\Delta T, \end{cases} \tag{1}$$

where ν is the coefficient of kinematic viscosity, ρ is the density, α the thermal expansion coefficient, k the thermal diffusivity, p the pressure, T the temperature, $\mathbf{v}$ the velocity and $\mathbf{g}(z) = gH(z)\mathbf{k}$ is the gravity, with g constant and $\mathbf{k}$ the unit vector in the z-direction. The boundary conditions at the rigid boundaries are [7]

$$\begin{cases} \mathbf{v} = 0, \text{ at } z = 0, h, \\ T = T_L, \text{ at } z = 0, \\ T = T_U, \text{ at } z = h, \text{ with } T_L > T_U. \end{cases} \tag{2}$$

The linear stability of the conduction stationary solution of equations (1) characterized by $\mathbf{v} = 0$, $T = -\beta z + T_L$, $\beta = \dfrac{T_L - T_U}{h}$, written in the nondimensional form, against normal mode perturbations is governed by the following two-point problem for the ordinary differential equations [5, 7]

$$\begin{cases} (D^2 - a^2)^2 W = RH(z)a^2\Theta, \\ (D^2 - a^2)\Theta = -RN(z)W, \end{cases} \tag{3}$$

$$W = DW = \Theta = 0 \quad \text{at } z = 0, 1. \tag{4}$$

Here $D = \dfrac{d}{dz}$, R^2 is the Rayleigh number and it represents the eigenvalue of the problem (3)–(4), a is the wavenumber and W and Θ are the amplitudes of the vertical velocity and pressure perturbation. They form the eigenfunction of the eigenvalue problem (3)–(4).

In the sequel we consider that $H(z) = 1 - \varepsilon z$, $N(z) \equiv 1$ and $k = 1$, so $\mathbf{g}(z) = g(1 - \varepsilon z)\mathbf{k}$.

2 Stability criteria

For $\varepsilon \in [0, 1]$, the principle of exchange of stability holds [5] for the eigenvalue problem

$$\begin{cases} (D^2 - a^2)^2 W = R(1 - \varepsilon z)a^2\Theta, \\ (D^2 - a^2)\Theta = -RW, \end{cases} \tag{5}$$

with the boundary conditions (4). Eliminating W between the equations (5) we obtain the following six-order ordinary differential equation

$$(D^2 - a^2)^3\Theta = -R^2 a^2 (1 - \varepsilon z)\Theta,$$

and the boundary conditions, written in Θ only, are

$$\Theta = (D^2 - a^2)\Theta = D(D^2 - a^2)\Theta = 0, \quad \text{at} \quad z = 0, 1.$$

The problem (5)–(4) possesses a non-trivial solution only for particular values of R. So we have an eigenvalue problem for R. For a given a we must determine the lowest value of R. This minimum value with respect to a is the critical Rayleigh number at which the instability sets in. It corresponds to the most unstable mode.

Introduce the new function

$$\Psi = (D^2 - a^2)\Theta. \tag{6}$$

In this way, we have

$$(D^2 - a^2)^2\Psi = -R^2 a^2 (1 - \varepsilon z)\Theta, \tag{7}$$

with the boundary conditions

$$\Theta = \Psi = D\Psi = 0 \ \text{ at } \ z = 0, 1. \tag{8}$$

Some practical criteria can be derived for the hydrodynamic stability problem using the following three isoperimetric inequalities due to Joseph [6]

$$I_1^2 \geq \lambda_1^2 I_0^2, \quad I_2^2 \geq \lambda_2^2 I_1^2, \quad I_3^2 \geq \lambda_3^2 I_0^2, \tag{9}$$

where $\lambda_1 = \pi$, $\lambda_2 = 2\pi$, $\lambda_3 = (4.73)^2$ and $I_i^2(\Phi) = \displaystyle\int_0^1 (D^i\Phi)^2$. These isoperimetric inequalities are valid in the Hilbert space H_1 of real-valued four times continuously differentiable functions Φ on $[0, 1]$ satisfying the boundary conditions

$$\Phi(0) = \Phi(1) = D\Phi(0) = D\Phi(1) = 0.$$

Here, the functions Ψ and Θ are both indefinitely differentiable functions on the Hilbert space $L^2(0, 1)$. The unknown function Ψ satisfies the necessary boundary conditions so that the isoperimetric inequalities are valid for Ψ. Denote by H_2 the Hilbert subspace of $L^2(0, 1)$ consisting of real-valued four times continuously differentiable functions Φ on $[0, 1]$ satisfying the boundary conditions $\Phi(0) = \Phi(1) = 0$.

Multiplying (7) by Ψ, integrating the result over $[0, 1]$ and taking into account the boundary conditions (8) we obtain

$$I_2^2(\Psi) + 2a^2 I_1^2(\Psi) + a^4 I_0^4(\Psi) = -R^2 a^2 \int_0^1 (1 - \varepsilon z)\Theta\Psi. \tag{10}$$

Taking into account the isoperimetric inequalities (9), it is proved that the following stability criterion holds.

Proposition 1 [2]. *For $A \equiv a(a-\varepsilon) > 0$, a stability bound in the two-point problem (5), (4) is $R^2 \geq B(\pi^2 + a^2)^2/(a^2(\pi^2 + A))$, where $B = 4.73^4 + 2a^2\pi^2 + a^4$.*

Proposition 2 [2]. *For $A<0$ the stability bound in the two-point problem (5), (4) is given by $R^2 \geq B(\pi^2+a^2)/(a^2(1+\varepsilon))$.*

Let us recall that, for $\varepsilon = 0$, (5) becomes

$$\begin{cases} (D^2-a^2)^2 W = Ra^2\Theta, \\ (D^2-a^2)\Theta = -RW, \end{cases}$$

which together with the boundary conditions (4), form the classical Bénard convection problem. Denote by R_c^2 the Rayleigh number for this two-point problem and by R_ε^2 the Rayleigh number for the two-point problem (5), (4) in which the gravity field is linearly decreasing across the layer. Then the following result holds.

Proposition 3. *The domain of stability in the convection problem (5), (4) increases with $\varepsilon > 0$, i.e. $R_c^2 \leq R_\varepsilon^2$.*

Proof. By multiplying (6) by Θ, integrating the obtained result between 0 and 1, (10) is rewritten in the form

$$I_2(\Psi) - 2a^2 I_1^2(\Psi) + a^4 I_0^2(\Psi) = R^2a^2\int_0^1 -\Theta\Psi dz + R^2a^2\varepsilon\int_0^1 z\Theta\Psi dz. \tag{11}$$

Taking into account (6) projected on Ψ, for $\varepsilon = 0$, from (10) it follows [1] that the lowest characteristic value of the Bénard problem is given as the minimum

$$R_c^2 = \min_{\Psi\in H_1,\Theta\in H_2} \frac{\displaystyle\int_0^1 [(D^2-a^2)^2\Psi dz]}{a^2\displaystyle\int_0^1 [(D\Theta)^2 + a^2\Theta^2]dz}. \tag{12}$$

Further let us come back to the case $\varepsilon \neq 0$. By (6) we have $\Psi = (D^2-a^2)\Theta$. Then the following equalities

$$\begin{aligned} &\int_0^1 -\Theta\Psi dz = \int_0^1 (D\Theta)^2 + a^2(\Theta)^2 dz = I_1^2(\Theta) + a^2 I_0^2(\Theta) > 0, \\ &\int_0^1 z\Theta\Psi dz = -\int_0^1 z[(D\Theta)^2 + a^2\Theta^2]dz < 0 \end{aligned} \tag{13}$$

hold. Consequently, from (11) it follows that the lowest characteristic value can be obtained by taking the minimum of the functional

$$R_e^2 = \min_{\Psi\in H_1,\Theta\in H_2} \frac{I_2(\Psi) + 2a^2I_1^2(\Psi) + a^4I_0^2(\Psi)}{\displaystyle\int_0^1 \Big\{(D\Theta)^2 + a^2\Theta^2 - z[(D\Theta)^2 + a^2\Theta^2]\Big\}dz}. \tag{14}$$

The comparison of (12) and (13) implies immediately that $R_\varepsilon^2 \geq R_c^2$. □

In Fig. 1 we present the neutral curve for the classical case ($\varepsilon = 0$) and some neutral curves for the variable gravity field case. These graphs illustrate the stability criteria too.

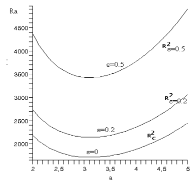

FIG. 1. The function $Ra(a)$

3 Conclusions

In this paper we presented two stability criteria (one from [2] and a new one) for a convection problem with a variable gravity field. They show that when the gravity field is linearly decreasing across the layer (in our case this means that $\varepsilon > 0$), the stability domain enlarges. The numerical results obtained in [2–4] sustain this conclusion.

In [3, 4] we obtained numerical evaluations of the Rayleigh number by using methods based on Fourier series expansions and these results agree very well with the ones obtained by Straughan in [7] using the energy method. In [2], using a variational method (in fact, isoperimetric inqualities), we also obtained numerical evaluations of the stability bounds for this convection problem. Obviously, these bounds are smaller than the limits (even approximate) obtained by methods based on Fourier series expansions. However, the advantage of applying the variational method is its easy use and the quick result obtained.

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