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Criterion of parametrical completeness in the 6-element non-chain extension of Intuitionistic logic of A. Heyting

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Abstract. The problem of parametrical completeness in the 6-element non-chain extension of Intuitionistic logic is considered. The conditions permiting to determine the parametrical completeness of an arbitrary system of formulas in mentionted logic are established in terms of 13 parametrical pre-complete classes of formulas.

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L.E. Brouwer [1] discarded the Tertiun non datur Law and proclaimed Classical logic doubtful. Gradually it became clear that Intuitionistic logic presents value in diverse aspects, including in the theory of algorithms. A. Heyting (1930) succeeded to represent it by means of well known nowadays Intuitionistic calculus [2].

A.V. Kuznetsov [3] introduced in consideration the notion of *parametrical expressibility* as a generalization of explicit expressibility. He found out the criterion of parametrical completeness in the classical logic, and put the problem to find out conditions for parametrical completeness in the Intuitionistic propositional logic [3, p. 28, problem 16]. In order to approach to the problems for Intuitionistic logic, it is more preferable to solve analogous problems, first, for some more simple logic which approximates it. A. Danil'chenko [4] obtained a criterion of parametrical completeness for the logic of *First Jaskowski's matrix*, generalized later by I. Cucu [5] for the case of the logic of any finite or countable chain.

In the present paper we give the necessary and sufficient conditions for parametrical completeness of any arbitrary system of formulas in the logic of 6-element pseudo-boolean algebra with one atom, and one penultimate element and two incomparable ones. This logic played an essential role in solving the problem of completeness with respect to explicit expressibility in the Intuitionistic logic realized by M. Rata [7] in 1970.

We construct the formulas in usual way [2] with the connectives $\&$, $\vee$, $\supset$, and $\neg$, starting with propositional variables $p, q, r, \ldots$, possibly with indices. The symbols 0, 1, $\perp p$, $(A \sim B)$ and $(A \oplus B)$ denote respective by the formulas

$(p\&\neg p)$, $(p \supset p)$, $(p \vee \neg p)$, $(A \supset B)\&(B \supset A)$ and $((\neg A\&B) \vee (A\&\neg B))$.

The result of substituting formulas $F_1, \ldots, F_n$ in a formula G, respectively, for the propositional variables $\pi_1, \ldots, \pi_n$ is denoted by the symbols $G[\pi_1/F_1, \ldots, \pi_n/F_n]$ or in short, $G[F_1, \ldots, F_n]$.

A formula F is said to be *explicitly expressible in the logic L by a system Σ of formulas* if F can be obtained from variables and formulas belonging to Σ by means of a finite numbers of *week substitutions* (i.e. transitions from B and C to $B[\pi/C]$, where π is a variable) and *replacements by equivalents in L* (i.e. transitions from B to C such that $(B \sim C) \in L$). If all the transitions consist only in applications of week substitution rule, then they say that F *is directly expressible by* Σ.

A formula F is said to be *parametrically expressible* (in short, p. expressible) in a logic L in terms of a system (of formulas) Σ if there exist numbers l and s, variables $\pi, \pi_1, \ldots, \pi_l$ not occurring in F, pairs a formulas A_i, B_i $(i = 1, \ldots, s)$ that are expressible in L in terms of Σ, and formulas $D_1, \ldots, D_l$ that do not contain the variables $\pi, \pi_1, \ldots, \pi_l$ such that take place the relations

$$L \vdash ((F \sim \pi) \supset (A_1 \sim B_1)\& \ldots \&(A_s \sim B_s)[\pi_1/D_1], \ldots, \pi_l/D_l]),$$

$$L \vdash ((A_1 \sim B_1)\& \ldots \&(A_s \sim B_s) \supset (F \sim \pi)).$$

The relation of *parametrical expressibility is transitive.* But the partial case of this relation when parameters are absent is called *implicit expressibility*, and in general case it is not transitive. A system (of formulas) Σ is said to be *parametrically complete* (in short, p. complete) in a logic L if all formulas of the language of L are p. expressible in L in terms of Σ.

Classical logic, Intuitionistic one, which is intermediate between logics, and also absolute contradictory logic can be united under the general notion of super-Intuitionistic logic. For any of these logic there exist some pseudo-boolean algebra in which the respective logic may be interpreted.

By a *pseudo-boolean algebra* [6] we mean a system $< M; \Omega >$, where $\Omega = \{\&, \vee, \supset, \neg\}$, which is a lattice with respect to & and $\vee$, with relative pseudo-complement $\supset$ and pseudo-complement $\neg$. They say that a formula F is true in a (pseudo-boolean) algebra Λ if F, as on function of Λ, is identically equal to the greatest element 1 of Λ. The set of all formulas true in Λ constitutes a super-intuitionistic logic, called the logic of the algebra Λ and denoted below by the expression $L\Lambda$.

The pseudo-boolean algebra whose diagram is represented in Fig. 1 is denoted by the expression $Z_2 + Z_5$. The logic $L(Z_2 + Z_5)$ played an essential role in solving the problem of completeness relative to explicit expressibility in the Intuitionistic logic and in its super-intuitionistic extensions realized by M.Rata [7].

Let us remark that the chains $\{0, \tau, \omega, 1\}$, $\{0, \rho, \omega, 1\}$ and $\{0, \sigma, \omega, 1\}$ with respect to operations &, $\vee$, $\supset$, $\neg$ constitute isomorphic subalgebras of the algebra $Z_2 + Z_5$, and any of them is the interpretation of one and the same (super-intuitionistic) logic denoted below by the symbol LZ_4.

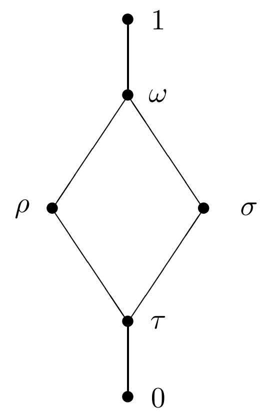

Fig. 1

Analogously, we denote by the symbol LZ_2 the logic of Boolean algebra $Z_2 =< \{0,1\}; \Omega >$.

Following A.V. Kuznetsov [3], we say that a formula $F(p_1, \dots p_n)$ *preserves the predicate* $R(x_1, \dots x_m)$ in the algebra Λ if, for any elements $\alpha_{ij} \in \Lambda$ ($i = 1, \dots, m$; $j = 1, \dots, n$), the truth of propositions

$$R[\alpha_{11}, \alpha_{21}, \dots, \alpha_{m1}], \dots, R[\alpha_{1n}, \alpha_{2n}, \dots, \alpha_{mn}]$$

implies

$$R[F[\alpha_{11}, \dots, \alpha_{1n}], \dots, F[\alpha_{m1}, \dots, \alpha_{mn}]].$$

Proposition 1 [3]. *A system of formulas Σ is p.complete in the classical logic LZ_2 if and only if there are formulas of Σ that do not preserve the predicates*

$$x = 0, x = 1, x = \neg y, x\&y = z, x \vee y = z, ((x \sim y) \sim z) = u \tag{1}$$

in the algebra Z_2.

Under *the formula centralizer [8] of a function F* we mean the set of formulas permutable with F in a given pseudo-boolean algebra. Let denote it by the symbol $< F >$.

Let us define seven functions $f_1, \dots, f_7$ by means of Tables 1 and 2, and note that these functions cannot be expressed by formulas.

p	0	τ	ω	1
f_1	0	0	-	1
f_2	0	τ	τ	1
f_3	0	1	τ	1

Table 1

p	0	τ	ρ	σ	ω	1
f_4	0	τ	σ	ρ	1	1
f_5	0	τ	τ	τ	1	1
f_6	0	τ	1	1	1	1
f_7	0	1	τ	τ	1	1

Table 2

Theorem 1. *In order that a system Σ of formulas be parametrically complete in the logic $L(Z_2 + Z_5)$ it is necessary and sufficient that Σ be parametrically complete* *in the classical logic LZ_2 and for every $i = 1, \ldots, 7$ there exist a formula F_i of Σ which does not belong to the formula centralizer $< f_i >$.*

Let's remind [3] that the formula centralizer $< F >$ coincides with the set of all formulas preserving the *predicate* $f(x_1, \ldots, x_n) = y$ in the considered algebra, where the variable y differs from $x_1, \ldots, x_n$. Let denote the classes of formulas preserving the predicates of line (1) in Z_2 , respectively, by the symbols $C_0, C_1, \ldots, C_5$. Analogously, for any $i = 1, 2, \ldots, 7$, we denote the class of formulas preserving the predicate $f_i(x) = y$ by the symbol C_{i+5}

On the base of Proposition 1 Theorem 1 is equivalent with the following

Theorem 2. *In order that a system of formulas Σ be p.complete in the logic $L(Z_2 + Z_5)$ it is necessary and sufficient that Σ be not included in one of the classes $C_0, \ldots, C_{12}$.*

The necessity follows from the fact that the classes $C_0, \ldots, C_{15}$ are closed with respect to p.expressibility, and they are incomparable two by two relative to the inclusion.

Sufficiency. If the condition holds, then for each $i = 1, 2, \ldots, 12$ there exists a formula F_i from system Σ not belonging to the class C_i. Note that the system of six formulas $\{F_0, F_1, \ldots, F_5\}$, in accordance with Proposition 1, is p.complete in the classical logic LZ_2.

In following we present twelve lemmas necessary for the proof of Theorem 2. Also we admit, for short, to use the symbol L_6 instead of the expression $L(Z_2 + Z_5)$.

Lemma 1. *The formulas 0 and 1 are explicitly expressible in L_6 by means of F_0, F_1 and F_2.*

Lemma 2. *At least one of three formulas*

$$\neg p, \neg\neg p, or \perp p \tag{2}$$

is explicitly expressible in L_6 by means of the formulas 0, 1 and F_6.

Lemma 3. *The formula $\neg p$ is implicitly expressible in L_6 by means of the formulas 0, 1, F_3, F_4 and F_6.*

Lemma 4. *The formula $\neg\neg(p\&q)$ is explicitly expressible in L_6 by means of the formulas 0, 1, $\neg p$ and F_5.*

Lemma 5. *The formulas $\perp p$ and $\neg p \& \perp q$ are p.expressible in L_6 by means of the formulas 0, 1, $\neg p$, F_9 and F_{11}.*

Lemma 6. *The formulas $\neg p \& q$, $\neg p \vee q$ and $p \oplus q$ are implicitly expressible in L_6 by means of the formulas*

$$\neg p, \neg\neg(p\&q), \neg p \& \perp q. \tag{3}$$

Lemma 7 [3]. *The conjunction $p\&q$ is implicitly expressible in any super-intuitionistic logic by means of the implication $p \supset q$.*

Lemma 8. *At least one of the following three formulas*

$$p\&q, p \sim q, p \supset q \tag{4}$$

is p.expressible in L_6 by means of the formulas of the list

$$0, 1, \neg p, \perp p, \neg p\&q, \neg p \vee q, p \oplus q \tag{5}$$

and the formulas (plus)

$$F_7, F_8, F_9, \dots, F_{11}. \tag{6}$$

Lemma 9. *The formula $p \supset q$ is p.expressible in L_6 by means of the formulas of list (5) plus the list*

$$p \sim q, F_{12}. \tag{7}$$

Lemma 10. *At least one of three formulas*

$$p \vee q, p \sim q, p \supset q \tag{8}$$

is p.expressible in L_6 by means of the formulas (5) and the formulas

$$p\&q, F_7, F_8, F_9, F_{10}. \tag{9}$$

Lemma 11. *The formula $p \supset q$ is p.expressible in L_6 by means of the formulas of list (5) and the formulas*

$$p\&q, p \vee q, F_7, F_8. \tag{10}$$

Lemma 12. *The formula $p \vee q$ is p.expressible in L_6 by means of the formulas of list (5) and the formulas*

$$p \supset q, F_9. \tag{11}$$

Let us return to the proof of the theorem. We sum up that the formulas of list (5) because of Lemmas 1–6 are p.expressible in L_6 by means of the formulas $F_0, \dots, F_6, F_9$, and F_{11}.

On the base of Lemma 8 at least one of the formulas of the line (4) is p.expressible in L_6 by means of the formulas of the lists (5) and (6).

In dependence of this fact there are three cases.

CASE 1. Let the formula $p \supset q$ be p.expressible in L_6 by means of the lists (5) and (6). Then in virtue of Lemma 7 the formula $p\&q$ also is p.expressible in L_6 through formulas (5) and (6). It remain to say in analyzed case that third formula $p \sim q$ from line (4) is explicitly expressible in L_6 by means of $p\&q$ and $p \supset q$, because it takes place that $(p \sim q) \sim ((p \supset q)\&(q \supset p))$.

CASE 2. Let the formula $p \sim q$ be p. expressible in L_6 via the formulas of lists (5) and (6). Then on the base of Lemma 9 the formula $p \supset q$ is p. expressible in L_6 by means of formulas from list (7). But in virtue of Lemma 7 the third formula $p\&q$ of list (4) is implicitly expressible in L_6 via the implication $p \supset q$.

CASE 3. Let the formula $p\&q$ be p. expressible in L_6 by means of the formulas of lists (5) and (6). Then on base of Lemma 10 at least one of three formulas of line

(8) is p. expressible in L_6 via formulas of lines (5) and (9). If $p \supset q$ is p.expressible then the subcase falls under the case 1. If $p \sim q$ is p.expressible then it falls under the case 2. Let $p \vee q$ be p. expressible by means of formulas of lists (5) and (9). Then, in accordance with Lemma 11, the formula $p \supset q$ is p.expressible in L_6 via formulas of lines (5) and (10).

So, we can say that all three formulas of list (4) are p. expressible in L_6 by means of the formulas of line (5) and formulas $F_7, \dots, F_{12}$. On the base of Lemma 12, the formula $p \vee q$ also is p. expressible in L_6 by means of the formulas of lines (5) and (11).

It remained to sum up that any formula of the following system $\{\neg p, p\&q, p \vee q, p \supset q\}$ is p. expressible in L_6 by means of formulas from the hypothesis of theorem, and add that this system is explicitly complete in the logic L_6.

The theorem is proved.

A system (of formulas) Σ is said to be *parametrically pre-complete in a logic* L if Σ is not complete in L, but, for any formula F not belonging to Σ, the system $\Sigma \cup \{F\}$ is p. complete in L.

Theorem 3. *There exist exactly 13 parametrically pre-complete in* $L(Z_2 + Z_5)$ *classes of formulas.*

Theorem 4. *There exists non-complex algorithm which, for any finite system of formulas, enables to determine whether this system is parametrically complete in the* $L(Z_2 + Z_5)$.

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