BULETINUL ACADEMIEI DE ŞTIINŢE
A REPUBLICII MOLDOVA. MATEMATICA
Number 3(52), 2006, Pages 3–16
ISSN 1024–7696

Lie algebras of operators and invariant $GL(2,\mathbb{R})$-integrals for Darboux type differential systems

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Abstract. In this article two-dimensional autonomous Darboux type differential systems with nonlinearities of the i^{th} $(i=\overline{2,7})$ degree with respect to the phase variables are considered. For every such system the admitted Lie algebra is constructed. With the aid of these algebras particular invariant $GL(2,\mathbb{R})$-integrals as well as first integrals of considered systems are constructed. These integrals represent the algebraic curves of the $(i-1)^{th}$ $(i=\overline{2,7})$ degree. It is showed that the Darboux type systems with nonlinearities of the 2^{nd}, the 4^{th} and the 6^{th} degree with respect to the phase variables do not have limit cycles.

Mathematics subject classification: 34C05, 34C14.
Keywords and phrases: The Darboux type differential system, comitant, invariant $GL(2,\mathbb{R})$-integrating factor, invariant $GL(2,\mathbb{R})$-integral, limit cycle.

Consider the system of differential equations

$$\frac{dx^j}{dt}=a^j_\alpha x^\alpha+a^j_{\alpha_1\alpha_2...\alpha_m}x^{\alpha_1}x^{\alpha_2}...x^{\alpha_m}\quad (j,\alpha,\alpha_1,\alpha_2,...,\alpha_m=1,2;\ m\geq 2),\tag{1}$$

where coefficient tensor $a^j_{\alpha_1\alpha_2...\alpha_m}$ is symmetrical in lower indices, in which the complete convolution holds. The system (1) will be considered with the action of the group $GL(2,\mathbb{R})$ of center-affine transformations [1].

We shall consider the following center-affine invariants and comitants [1] of the system (1) written in the tensorial form

$$I_1=a^\alpha_\alpha,\quad I_2=a^\alpha_\beta a^\beta_\alpha,\quad K_2=a^\alpha_\beta x^\beta x^\gamma\varepsilon_{\alpha\gamma},\ \widetilde{K}_{m-1}=a^\alpha_{\alpha\alpha_1\alpha_2...\alpha_{m-1}}x^{\alpha_1}x^{\alpha_2}...x^{\alpha_{m-1}},$$

$$\widetilde{K}_{m+1}=a^\alpha_{\alpha_1\alpha_2...\alpha_m}x^{\alpha_1}x^{\alpha_2}...x^{\alpha_m}x^\beta\varepsilon_{\alpha\beta},\tag{2}$$

where $\varepsilon_{\alpha\beta}$ is the unit bi-vector with coordinates $\varepsilon_{11}=\varepsilon_{22}=0,\ \varepsilon_{12}=-\varepsilon_{21}=1$.

It is easy to see that when the condition $\widetilde{K}_{m+1}\equiv 0$ holds, the system (1) takes the form

$$\frac{dx^j}{dt}=a^j_\alpha x^\alpha+mx^jR(x^1,x^2)\equiv P^j(x^1,x^2)\quad (j,\alpha=1,2),\tag{3}$$

where $R(x^1,x^2)$ is a homogeneous polynomial of the $(m-1)^{th}$ order. As it is well known, the system (3) is called a Darboux type differential system (see, for example, [2,3]).

A series of papers is devoted to the problem of the investigation of systems of the form (3) from different points of view (see, for example, [2–7]).

Note that the family of systems (3) is a subset of the family of systems (1) defined via center-affine invariant conditions. Indeed, one can verify easily that for the system (3) the conditions $\widetilde{K}_{m+1} \equiv 0$, $\widetilde{K}_{m-1} = (m+1)R(x^1, x^2)$ hold. Therefore we have the next

Lemma 1. *A system (1) belongs to a family of the Darboux type differential systems (3) with $R(x^1, x^2) \not\equiv 0$ if and only if $\widetilde{K}_{m+1} \equiv 0$, $\widetilde{K}_{m-1} \not\equiv 0$.*

For the system (1) with $\widetilde{K}_{m+1} \equiv 0$ and $m = 2, 3, ..., 7$ or, that is the same, for (3) with $m = 2, 3, ..., 7$, two algebraic curves of the form

$$\sum_{j=0}^{k} A_j (x^1)^{k-j} (x^2)^j = B_k \quad (k = 2, m-1), \tag{4}$$

where $B_2 = 0$, and $B_{m-1} \neq 0$ and A_j are polynomials in the coefficients of this system, are particular invariant $GL(2, \mathbb{R})$-integrals.

Remark 1. The construction of particular invariant $GL(2, \mathbb{R})$-integrals (4) is remarkable, because as it is shown in [2], the system (3) can have only one limit cycle and if it exists, it represents an algebraic curve of the form (4) with $k = m - 1$ and $B_k \neq 0$, surrounding the origin of coordinates.

Lemma 2. *If the factorization over $\mathbb{C}[x, y]$ of the left-hand side of the algebraic curve of the form (4) with $B_{m-1} \neq 0$ contains at least one real linear factor, then this algebraic curve cannot be of the ellipsoidal form.*

Proof. Suppose that some algebraic curve of the form (4) with $B_{m-1} \neq 0$ can be written as

$$(Ax^1 + Bx^2) \sum_{j=0}^{m-2} A'_j (x^1)^{m-j-2} (x^2)^j = B_{m-1},$$

where the linear factor $Ax^1 + Bx^2$ is real. Suppose that the last equation has the ellipsoidal form, surrounding the origin of coordinates, it means, that any line, passing through the origin, has to intersect the curve in two points. Particularly, this holds for the line $Ax^1 + Bx^2 = 0$. However, in this case we get the contradiction: at the intersection points we have $0 = B_{m-1}$, i.e. the assertion of Lemma 2 is true.

Theorem 3. *System (1) with $\widetilde{K}_{m+1} \equiv 0$ has the particular invariant $GL(2, \mathbb{R})$-integral*

$$K_2 = 0,$$

where K_2 is from (2).

Proof. According to Lemma 1, the system (1) with $\widetilde{K}_{m+1} \equiv 0$ has the form (3). Denote by Λ the operator

$$P^1(x^1, x^2)\frac{\partial}{\partial x^1} + P^2(x^1, x^2)\frac{\partial}{\partial x^2}, \tag{5}$$

where P^j $(j = 1, 2)$ is from (3). It is easy to see that

$$\Lambda(K_2) = K_2\left(I_1 + \frac{2m}{m+1}\widetilde{K}_{m-1}\right),$$

where I_1, K_2 and $\widetilde{K}_{m-1}$ are from (2). This identity shows that K_2 is a particular integral of the system (3) or, that is the same, of the system (1) with $\widetilde{K}_{m+1} \equiv 0$. Theorem 3 is proved.

Consider the differential operator

$$X = \xi^1\frac{\partial}{\partial x^1} + \xi^2\frac{\partial}{\partial x^2}, \tag{6}$$

where ξ^1 and ξ^2 are polynomials in variables x^1, x^2 and in coefficients of the system (3).

According to [8], we can show that the system (3) admits the operator (6) if and only if its coordinates satisfy the system of constitutive equations

$$\xi^j_{x^\alpha} P^\alpha = \xi^\beta P^j_{x^\beta} \quad (j, \alpha, \beta = 1, 2), \tag{7}$$

where $\xi^j_{x^\alpha} = \dfrac{\partial \xi^j}{\partial x^\alpha}$ and $P^j_{x^\beta} = \dfrac{\partial P^j}{\partial x^\beta}$.

As well, according to [8] we have that if the system (3) admits the operator (6), then we can apply Lie theorem on integrating factor: *The system (3) admits a group with the operator (6) if and only if the function μ of the form*

$$\mu^{-1} = \xi^1 P^2 - \xi^2 P^1 \tag{8}$$

is an integrating factor of the equation

$$P^2 dx^1 - P^1 dx^2 = 0. \tag{9}$$

In what follows we shall say that μ is an integrating factor of the system (3) if it is an integrating factor of the equation (9).

Theorem 4. *The system (1) with $m = 2$ and $\widetilde{K}_3 \equiv 0$ has the invariant $GL(2, \mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1} = K_2\Phi_1,$$

where K_2 *is from (2) and*

$$\Phi_1 \equiv 8I_1\widetilde{K}_1 - 12K_3 + 3(I_1^2 - I_2) = 0 \tag{10}$$

is a particular invariant $GL(2,\mathbb{R})$*-integral of this system.*

In (10) invariants and comitants $I_1,\ K_2,\ \widetilde{K}_1 = a^{\alpha}_{\alpha\beta}x^{\beta}$ *are taken from (2), and*

$$K_3 = a^{\alpha}_{\beta}a^{\beta}_{\alpha\gamma}x^{\gamma}$$

are defined in [1].

Proof. Consider the system (3) with $m = 2$ and $x^1 = x, x^2 = y$, written in the form

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + 2x(gx + hy) \equiv P^1(x,y),\\ \frac{dy}{dt} &= ex + fy + 2y(gx + hy) \equiv P^2(x,y). \end{aligned} \tag{11}$$

where $c, d, e, f, g, h \in \mathbb{R}$.

Considering (7) it is easy to verify that the system (11) admits the two-dimensional commutative Lie algebra of operators of the form

$$\begin{aligned} &Z_1 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)[2(fg - eh)x + 2(ch - dg)y + cf - de],\\ &Z_2 = \{[h(cf - de) + c(dg - ch)]\,x + d(dg - ch)y + 2\,[g(dg - ch) + h(fg -\\ &\quad -eh)x^2]\}\,\frac{\partial}{\partial x} + \{e(dg - ch)x + d(fg - eh)y + 2\,[g(dg - ch) + h(fg - eh)]\,xy\}\,\frac{\partial}{\partial y}. \end{aligned}$$

Using any one of these operators and (8) we obtain up to a constant factor an integrating factor of the system (11), in the form

$$\mu^{-1} = [2(fg - eh)x + 2(ch - dg)y + cf - de]\,[-ex^2 + (c - f)xy + dy^2]. \tag{12}$$

Remark 2. In what follows we will use invariants I_1 and I_2 and comitant K_2 from (2) for the system (3) with $a^1_1 = c,\ a^1_2 = d,\ a^2_1 = e,\ a^2_2 = f$

$$I_1 = c + f, \quad I_2 = c^2 + 2de + f^2, \quad K_2 = -ex^2 + (c - f)xy + dy^2. \tag{13}$$

Besides I_1, I_2, K_2, calculating for the system (11) the comitants $\widetilde{K}_1 = a^{\alpha}_{\alpha\beta}x^{\beta}$ and K_3 we obtain

$$\widetilde{K}_1 = 3(gx + hy), \quad K_3 = [g(2c + f) + eh]x + [h(c + 2f) + dg]y. \tag{14}$$

We observe that the second factor from (12) exactly coincides with K_2. Moreover, considering (13),(14) we obtain the first factor from (12) in the form (10) up to a constant factor.

Using the operator (5), one can verify that the first factor from (12) as well as the second one (Theorem 3) is a particular integral for the system (11), or, that is the same, for the system (1) with $m = 2$ and $\widetilde{K}_3 \equiv 0$. Theorem is proved.

Theorem 5. *The system (1) with $m = 3$ and $\widetilde{K}_4 \equiv 0$ has an invariant $GL(2, \mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1} = K_2 \Phi_2,$$

and

$$\Phi_2 \equiv 3(4I_1 Q_2 - 3I_1^2 \widetilde{K}_2 + 2J_7 K_2) - 4I_1(I_1^2 - I_2) = 0 \tag{15}$$

is a particular invariant $GL(2, \mathbb{R})$-integral of this system.

In the last expression the invariants and comitants $I_1,\ I_2, K_2,\ \widetilde{K}_2 = a^\alpha_{\alpha\beta\gamma} x^\beta x^\gamma$ are taken from (2), and

$$J_7 = a^\alpha_p a^\beta_{\alpha\beta q} \varepsilon^{pq},\ Q_2 = a^\alpha_\beta a^\beta_{\alpha\gamma\delta} x^\gamma x^\delta \tag{16}$$

are defined in [9] ($\varepsilon^{11} = \varepsilon^{22} = 0,\ \varepsilon^{12} = -\varepsilon^{21} = 1$).

Proof. Consider the system (3) with $m = 3$ and $x^1 = x, x^2 = y$, written in the form

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + 3x(gx^2 + hxy + iy^2) \equiv P^1(x, y), \\ \frac{dy}{dt} &= ex + fy + 3y(gx^2 + hxy + iy^2) \equiv P^2(x, y), \end{aligned} \tag{17}$$

where $c, d, e, f, g, h, i \in \mathbb{R}$.

Then it is easy to verify with the aid of constitutive equations (7) that this system admits the two-dimensional commutative Lie algebra of operators

$$Z_1 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \right) \{(c + f)(cf - de) + 3[(cf - de)g + f^2 g - efh + e^2 i]x^2 -$$

$$-6(dfg - cfh + cei)xy + 3[d^2 g + c(c + f)i - d(ch + ei)]y^2\},$$

$$Z_2 = \{[defh + c(-2dfg + deh - f^2 h) + c^2(fh - 2ei)]x + 3[dg(-2fg + eh) + cg(fh - 2ei) +$$
$$+ h(-f^2 g + efh - e^2 i)]x^3 - 2d(dfg - cfh + cei)y + 3[-d^2 gh + ci(-ch + fh - 2ei) + d(ch^2 -$$
$$-2fgi + ehi)]xy^2\} \frac{\partial}{\partial x} + \{-2e(dfg - cfh + cei)x + (d(-2f^2 g + ceh + efh) + cf(-ch +$$
$$+ fh - 2ei))y + 3(dg(-2fg + eh) + cg(fh - 2ei) + h(-f^2 g + efh - e^2 i))x^2 y + 3(-d^2 gh +$$
$$+ ci(-ch + fh - 2ei) + d(ch^2 - 2fgi + ehi))y^3\} \frac{\partial}{\partial y}.$$

Using any of these operators and equality (8) we obtain up to a constant an integrating factor of the system (17) in the form

$$\mu^{-1} = \{(c+f)(cf-de)+3[(cf-de)g+f^2g-efh+e^2i]x^2-6(dfg-cfh+cei)xy+ +3[d^2g+c(c+f)i-d(ch+ei)]y^2\}[-ex^2+(c-f)xy+dy^2]. \quad (18)$$

Calculating J_7, Q_2 and $\widetilde{K}_2 = a^{\alpha}_{\alpha\beta\gamma}x^{\beta}x^{\gamma}$ for the system (17) we obtain

$$\widetilde{K}_2 = 4(gx^2+hxy+iy^2),\ J_7 = -4dg+2ch-2fh+4ei,\ Q_2 = (3cg+fg+eh)x^2+ +2(dg+ch+fh+ei)xy+(dh+ci+3fi)y^2. \quad (19)$$

We observe that the second factor from (18) exactly coincides with K_2. Moreover, considering (13) and (19) we obtain the first factor from (18) in form (15) up to a constant factor.

Using the operator (5), one can verify that the first factor from (18) as well as the second one (Theorem 3) is a particular integral for the system (17) or, that is the same, for the system (1) with $m = 3$ and $\widetilde{K}_4 \equiv 0$. Theorem is proved.

Remark 3. In [7] it is shown that for the existence of a limit cycle for the system (1) with $m = 3$ and $\widetilde{K}_4 \equiv 0$, surrounding the origin, it is necessary and sufficient that the following conditions hold

$$2I_2 - I_1^2 < 0; \quad I_1^2 J_4 + 2J_7^2 > 0; \quad I_1(4I_1Q_2 - 3I_1^2\widetilde{K}_2 + 2J_7K_2)|_{y=0} > 0,$$

where $I_1, I_2, J_7, K_2, \widetilde{K}_2, Q_2$ are from (2) and (16) and $J_4 = a^{\alpha}_{\alpha pr}a^{\beta}_{\beta qs}\varepsilon^{pr}\varepsilon^{rs}$. Moreover, the limit cycle is unique and it is stable (unstable) if $I_1 > 0$ ($I_1 < 0$) and has the form (15).

Theorem 6. *Differential system (1) with $m = 4$ and $\widetilde{K}_5 \equiv 0$ has the invariant $GL(2,\mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1} = K_2\Phi_3,$$

and

$$\Phi_3 \equiv 8(5I_1^2-I_2)(4I_1\widetilde{K}_3-5M_1)+96K_2(M_3-2I_1M_2)+15(5I_1^2-I_2)(I_1^2-I_2) = 0 \quad (20)$$

is a particular invariant $GL(2,\mathbb{R})$-integral of this system.

Here invariants and comitants $I_1,\ I_2, K_2,\ \widetilde{K}_3 = a^{\alpha}_{\alpha\beta\gamma\delta}x^{\beta}x^{\gamma}x^{\delta}$ are from (2), and

$$M_1 = a^{\alpha}_{\beta}a^{\beta}_{\alpha\gamma\delta\mu}x^{\gamma}x^{\delta}x^{\mu},\ M_2 = a^{\alpha}_{\beta}a^{\gamma}_{\delta\alpha\gamma\mu}x^{\mu}\varepsilon^{\beta\delta},\quad M_3 = a^{\alpha}_{\beta}a^{\gamma}_{\delta}a^{\mu}_{\mu\gamma\nu}x^{\delta}\varepsilon^{\beta\nu}.$$

Proof. Consider the system (3) with $m = 4$ and $x^1 = x,\ x^2 = y$, written in the form

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + 4x(gx^3 + hx^2y + ixy^2 + jy^3), \\ \frac{dy}{dt} &= ex + fy + 4y(gx^3 + hx^2y + ixy^2 + jy^3), \end{aligned} \tag{21}$$

where $c, d, e, f, g, h, i, j \in \mathbb{R}$.

This system admits a two-dimensional commutative Lie algebra with one of operators in the form

$$Z_1 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \times \varphi_3(x, y),$$

where

$$\varphi_3(x,y) = \{-(cf-de)[2(c+f)^2 + (cf-de)] - 4[2c^2fg + de(-5fg+eh)-$$
$$-c[2deg+f(-5fg+eh)]+2(f^3g-ef^2h+e^2fi-e^3j)]x^3-12[d^2eg-d(cfg+2f^2g+ceh)+$$
$$+c(cfh+2f^2h-2efi+2e^2j)]x^2y-12[2d^2fg+c(2c+f)(fi-ej)+d[-2cfh+e(-fi+$$
$$+ej)]]xy^2+4[2d^3g-d^2(2ch+ei)-c(2c^2+5cf+2f^2)j+d(2c^2i+cfi+5cej+2efj)]y^3\}.$$

Using this operator and equality (8) we obtain up to a constant an integrating factor of the system (21) in the form

$$\mu^{-1} = \varphi_3(x,y) \times [-ex^2 + (c-f)xy + dy^2]. \tag{22}$$

Calculating $M_1,\ M_2,\ M_3$ and $\widetilde{K}_3$ for the system (21) we obtain

$$\begin{aligned} &\widetilde{K}_3 = 5(gx^3 + hx^2y + ixy^2 + jy^3),\ M_1 = (4cg + fg + eh)x^3 + (3dg + 3ch + 2fh+ \\ &+ 2ei)x^2y + (2dh + 2ci + 3fi + 3ej)xy^2 + (di + cj + 4fj)y^3,\ M_2 = -\frac{5}{3}(3dg- \\ &ch + fh - ei)x - \frac{5}{3}(dh - ci + fi - 3ej)y,\ M_3 = \frac{5}{3}(-3cdg + c^2h - deh - cfh+ \\ &+ 2cei - efi + 3e^2j)x + \frac{5}{3}(-3d^2g + cdh - 2dfh + dei + cfi - f^2i + 3efj)y. \end{aligned} \tag{23}$$

The second factor from (22) exactly coincides with K_2. Moreover considering (13) and (23) we obtain the first factor from (22) in the form (20) up to a constant.

Using the operator (5), one can verify that the first factor from (22) as well as the second one (Theorem 3) is a particular integral for the system (21) or, that is the same, for the system (1) with $m = 4$ and $\widetilde{K}_5 \equiv 0$. The theorem is proved.

Theorem 7. *The differential system (1) with $m=5$ and $\widetilde{K}_6 \equiv 0$ has the invariant $GL(2,\mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1} = K_2\Phi_4,$$

and

$$\Phi_4 \equiv 5(5I_1^2-2I_2)(5I_1^2\widetilde{K}_4-6I_1N_1)-60K_2(3I_1^2N_2-2I_1N_3-K_2N_4)+$$
$$+12I_1(I_1^2-I_2)(5I_1^2-2I_2)=0 \tag{24}$$

is a particular invariant $GL(2,\mathbb{R})$-integral of this system.

Here invariants and comitants $I_1,\ I_2, K_2,\ \widetilde{K}_4 = a^\alpha_{\alpha\beta\gamma\delta\mu}x^\beta x^\gamma x^\delta x^\mu$ are from (2), and

$$N_1 = a^\alpha_\beta a^\beta_{\alpha\gamma\delta\mu\nu}x^\gamma x^\delta x^\mu x^\nu,\ \ N_2 = a^\alpha_p a^\beta_{q\alpha\beta\gamma\delta}x^\gamma x^\delta \varepsilon^{pq},\quad N_3 = a^\alpha_p a^\beta_\delta a^\gamma_{\alpha\beta\gamma\mu q}x^\delta x^\mu \varepsilon^{pq},$$

$$N_4 = a^\alpha_p a^\beta_r a^\gamma_{\alpha\beta\gamma s q}\varepsilon^{pq}\varepsilon^{rs}.$$

Proof. Consider the system (3) with $m=5$ and $x^1=x,\ x^2=y$ in the form

$$\begin{aligned}\frac{dx}{dt} &= cx+dy+5x(gx^4+hx^3y+ix^2y^2+jxy^3+ky^4),\\ \frac{dy}{dt} &= ex+fy+5y(gx^4+hx^3y+ix^2y^2+jxy^3+ky^4),\end{aligned} \tag{25}$$

where $c,d,e,f,g,h,i,j,k\in\mathbb{R}$. This system admits a two-dimensional commutative Lie algebra with one of operators

$$Z_1 = \left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\times\varphi_4(x,y),$$

where

$$\varphi_4(x,y) = (c+f)(cf-de)[3(c+f)^2+4(cf-de)]+5[3d^2e^2g+3c^3fg-$$
$$-c^2[3deg+f(-13fg+eh)]-de(13f^2g-4efh+e^2i)+c[de(-16fg+eh)+f(13f^2g-$$
$$-4efh+e^2i)]+3(f^4g-ef^3h+e^2f^2i-e^3fj+e^4k)]x^4+20[df(4de-3f^2)g+c^3fh-c^2(dfg+$$
$$+deh-4f^2h+efi)+c(d^2eg-4df^2g-4defh+3f^3h+de^2i-3ef^2i+3e^2fj-3e^3k)]x^3y-$$
$$-10[3d^3eg-d^2[9f^2g+3c(fg+eh)+e^2i]-c(3c+f)(cfi+3f^2i-3efj+3e^2k)+d[3c^2(fh+$$
$$+ei)+cf(9fh+2ei)+3e(f^2i-efj+e^2k)]]x^2y^2-20[3d^3fg-d^2f(3ch+ei)-c(3c^2+4cf+$$
$$+f^2)(fj-ek)+d[3c^2fi+ef(fj-ek)+c(f^2i+4efj-4e^2k)]]xy^3+5[3d^4g-d^3(3ch+$$
$$+ei)+c(3c^3+13c^2f+13cf^2+3f^3)k+d^2(3c^2i+cfi+4cej+efj+3e^2k)-d(c+f)(3c^2j+$$
$$+cfj+13cek+3efk)]y^4.$$

Using this operator and equality (8) we obtain up to a constant an integrating factor of the system (25), in the form

$$\mu^{-1} = \varphi_4(x,y) \times [-ex^2 + (c-f)xy + dy^2]. \tag{26}$$

Calculating $I_1,\ I_2,\ K_2,\ \widetilde{K}_4 = a^\alpha_{\alpha\beta\gamma}x^\beta x^\gamma$ and $N_1,\ N_2,\ N_3,\ N_4$ for the system (25) we obtain the expression (26) in the invariant form. Theorem is proved.

In the same way for the system (3) with $m = 6$ and $x^1 = x,\ x^2 = y$ written in the form

$$\begin{aligned} \frac{dx}{dt} &= cx + dy + 6x(gx^5 + hx^4y + ix^3y^2 + jx^2y^3 + kxy^4 + ly^5),\\ \frac{dy}{dt} &= ex + fy + 6y(gx^5 + hx^4y + ix^3y^2 + jx^2y^3 + kxy^4 + ly^5) \end{aligned} \tag{27}$$

a two-dimensional commutative Lie algebra is obtained with one of operators in the form

$$Z_1 = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) \times \varphi_5(x,y),$$

where

$$\begin{aligned}
\varphi_5(x,y) &= (cf-de)[4(c+f)^2+9(cf-de)][6(c+f)^2+cf-de]+6\,\big[24c^4fg + d^2e^2(97fg-\\
&-9eh) - 2c^3[12deg + f(-77fg+3eh)] + c^2[2de(-103fg+3eh) + f(269f^2g - 37efh+\\
&+4e^2i)]+2de(-77f^3g+29ef^2h-11e^2fi+3e^3j)+2c[26d^2e^2g+de(-183f^2g+23efh-\\
&-2e^2i)+f(77f^3g-29ef^2h+11e^2fi-3e^3j)]+24(f^5g-ef^4h+e^2f^3i-e^3f^2j+e^4fk-\\
&-e^5l)\,\big]\,x^5+30\,[\,d(-9d^2e^2+58def^2-24f^4)g+6c^4fh-c^3[6d(fg+eh)+f(-37fh+4ei)]+\\
&+c^2(6d^2eg-37df^2g-46defh+58f^3h+4de^2i-22ef^2i+6e^2fj)+c[d^2e(46fg+9eh)-\\
&-2d(29f^3g+29ef^2h-11e^2fi+3e^3j)+24(f^4h-ef^3i+e^2f^2j-e^3fk+e^4l)]]x^4y+\\
&+30[8c^4fi-2c^3[4d(fh+ei)+f(-23fi+6ej)]+c^2[8d^2(fg+eh)+59f^3i-51ef^2j-\\
&-4d(11f^2h+12efi-3e^2j)+48e^2fk-48e^3l]+d[-44d^2efg+d(48f^3g+11e^2fi-3e^3j)+\\
&+12e(-f^3i+ef^2j-e^2fk+e^3l)]-2c[4d^3eg-d^2(22f^2g+22efh+e^2i)+df(24f^2h+11efi-\\
&-3e^2j)+6f(-f^3i+ef^2j-e^2fk+e^3l)]]x^3y^2+30[12d^4eg-3d^3[16f^2g+4c(fg+eh)+e^2i]+\\
&+d^2[12c^2(fh+ei)+2ef(6fi+ej)+c(48f^2h+6efi+11e^2j)]+c(12c^2+11cf+2f^2)(cfj+\\
&+4f^2j-4efk+4e^2l)-d[12c^3(fi+ej)+c^2f(51fi+22ej)+8ef(f^2j-efk+e^2l)+4c(3f^3i+\\
&+12ef^2j-11e^2fk+11e^3l)]]x^2y^3+30[24d^4fg-6d^3f(4ch+ei)+c(24c^3+58c^2f+37cf^2+\\
&+6f^3)(fk-el)-2d[12c^3fj+3ef^2(fk-el)+c^2(11f^2j+29efk-29e^2l)+cf(2f^2j+\\
&+23efk-23e^2l)]+d^2[24c^2fi+2cf(3fi+11ej)+e(4f^2j+9efk-9e^2l)]]xy^4-6[24d^5g-
\end{aligned}$$

$-6d^4(4ch+ei)+d^3(24c^2i+6cfi+22cej+4efj+9e^2k)-c(24c^4+154c^3f+269c^2f^2+$
$+154cf^3+24f^4)l+d[24c^4k+24ef^3l+2c^3(29fk+77el)+2cf^2(3fk+103el)+c^2f(37fk+$
$+366el)]-d^2[24c^3j+c^2(22fj+58ek)+2ef(3fk+26el)+c(4f^2j+46efk+97e^2l)]]y^5.$

Using this operator and equality (8) we obtain up to a constant an integrating factor of the system (27), in the form

$$\mu^{-1}=\varphi_5(x,y)\times[-ex^2+(c-f)xy+dy^2].$$

Therefore we have the next

Theorem 8. *Differential system (1) with $m=6$ and $\widetilde{K}_7\equiv 0$ has the invariant $GL(2,\mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1}=K_2\Phi_5,$$

and

$$\Phi_5\equiv 12(17I_1^2-9I_2)(13I_1^2-I_2)(6I_1\widetilde{K}_5-7O_1)-480(13I_1^2-I_2)K_2(4I_1O_2-3O_3)+$$
$$+5760K_2^2(3I_1O_4-O_5)+35(I_1^2-I_2)(17I_1^2-9I_2)(13I_1^2-I_2)=0 \qquad (28)$$

is a particular invariant $GL(2,\mathbb{R})$-integral of this system.

Here invariants and comitants $I_1,\ I_2,K_2,\ \widetilde{K}_5=a^\alpha_{\alpha\beta\gamma\delta\mu\nu}x^\beta x^\gamma x^\delta x^\mu x^\nu$ are from (2), and

$$O_1=a^\alpha_\beta a^\beta_{\alpha\gamma\delta\mu\nu\eta}x^\gamma x^\delta x^\mu x^\nu x^\eta,\ O_2=a^\alpha_p a^\beta_{q\alpha\beta\gamma\delta\mu}x^\gamma x^\delta x^\mu\varepsilon^{pq},\ \ O_3=a^\alpha_p a^\beta_\delta a^\gamma_{\alpha\beta\gamma\mu\nu q}x^\delta x^\mu x^\nu\varepsilon^{pq},$$
$$O_4=a^\alpha_p a^\beta_r a^\gamma_{\alpha\beta\gamma\delta qs}x^\delta\varepsilon^{pq}\varepsilon^{rs},\quad O_5=a^\alpha_p a^\beta_\mu a^\gamma_r a^\delta_{\alpha\beta\gamma\delta qs}x^\mu\varepsilon^{pq}\varepsilon^{rs}.$$

For the system (3) with $m=7$ and $x^1=x,\ x^2=y$ written in the form

$$\begin{aligned}\frac{dx}{dt}&=cx+dy+7x(gx^6+hx^5y+ix^4y^2+jx^3y^3+kx^2y^4+lxy^5+ny^6),\\ \frac{dy}{dt}&=ex+fy+7y(gx^6+hx^5y+ix^4y^2+jx^3y^3+kx^2y^4+lxy^5+ny^6)\end{aligned} \qquad (29)$$

a two-dimensional commutative Lie algebra is also found , for which one of operators has the form

$$Z_1=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\times\varphi_6(x,y),$$

where

$\varphi_6(x,y)=(c+f)(cf-de)[2(c+f)^2+cf-de][5(c+f)^2+16(cf-de)]+7[-10d^3e^3g+$
$+10c^5fg+c^4[-10deg+f(87fg-2eh)]+d^2e^2(101f^2g-16efh+2e^2i)+c^3[2de(-61fg+$
$+eh)+f(227f^2g-17efh+e^2i)]+c^2[35d^2e^2g-de(353f^2g-23efh+e^2i)+f(227f^3g-$

$-42ef^2h+8e^2fi-e^3j)]+de(-87f^4g+37ef^3h-17e^2f^2i+7e^3fj-2e^4k)+c[2d^2e^2(68fg-$
$-3eh)+de(-328f^3g+58ef^2h-10e^2fi+e^3j)+f(87f^4g-37ef^3h+17e^2f^2i-7e^3fj+$
$+2e^4k)]+10(f^6g-ef^5h+e^2f^4i-e^3f^3j+e^4f^2k-e^5fl+e^6n)]x^6+42[df(-16d^2e^2+$
$+37def^2-10f^4)g+2c^5fh-c^4[2d(fg+eh)+f(-17fh+ei)]+c^3[2d^2eg+d(-17f^2g-$
$-23efh+e^2i)+f(42f^2h-8efi+e^2j)]+c^2[d^2e(23fg+6eh)-d(42f^3g+58ef^2h-10e^2fi+$
$+e^3j)+f(37f^3h-17ef^2i+7e^2fj-2e^3k)]+c[-6d^3e^2g+2d^2e(29f^2g+8efh-e^2i)+$
$+d(-37f^4g-37ef^3h+17e^2f^2i-7e^3fj+2e^4k)+10(f^5h-ef^4i+e^2f^3j-e^3f^2k+e^4fl-$
$-e^5n)]]x^5y+21[5c^5fi-c^4[5d(fh+ei)+f(-41fi+5ej)]+c^3[5d^2(fg+eh)+d(-40f^2h-$
$-52efi+5e^2j)+f(93f^2i-36efj+10e^2k)]+c^2[-5d^3eg+67f^4i+d^2(40f^2g+50efh+$
$+11e^2i)-57ef^3j+52e^2f^2k-d(85f^3h+103ef^2i-37e^2fj+10e^3k)-50e^3fl+50e^4n]+$
$+d[10d^3e^2g-d^2(85ef^2g+2e^3i)+d(50f^4g+17e^2f^2i-7e^3fj+2e^4k)-10e(f^4i-ef^3j+$
$+e^2f^2k-e^3fl+e^4n)]-c[10d^3e(5fg+eh)+d^2(-85f^3g-85ef^2h-12e^2fi+e^3j)+$
$+2df(25f^3h+17ef^2i-7e^2fj+2e^3k)-10f(f^4i-ef^3j+e^2f^2k-e^3fl+e^4n)]]x^4y^2+$
$+14[10c^5fj-c^4[10d(fi+ej)+f(-77fj+20ek)]+2c^3[5d^2(fh+ei)+75f^3j-57ef^2k-$
$-2d(18f^2i+21efj-5e^2k)+50e^2fl-50e^3n]+c^2[-10d^3(fg+eh)+d^2(70f^2h+74efi+$
$+7e^2j)-2df(57f^2i+50efj-14e^2k)+f(77f^3j-72ef^2k+70e^2fl-70e^3n)]+df[70d^3eg-$
$-2d^2(50f^2g+7e^2i)+de(20f^2i+7efj-2e^2k)+10e(-f^3j+ef^2k-e^2fl+e^3n)]+$
$+2c[5d^4eg-d^3(35f^2g+35efh+e^2i)+d^2(50f^3h+14ef^2i+25e^2fj-7e^3k)+5f^2(f^3j-$
$-ef^2k+e^2fl-e^3n)+d(-10f^4i-42ef^3j+37e^2f^2k-35e^3fl+35e^4n)]]x^3y^3-21[10d^5eg-$
$-2d^4[25f^2g+5c(fg+eh)+e^2i]+d^3[10c^2(fh+ei)+c(50f^2h+4efi+7e^2j)+e(10f^2i+efj+$
$+2e^2k)]-c(10c^3+17c^2f+8cf^2+f^3)(cfk+5f^2k-5efl+5e^2n)-d^2[10c^3(fi+ej)+$
$+cf(10f^2i+37efj+12e^2k)+c^2(52f^2i+14efj+17e^2k)+e(5f^3j+11ef^2k-10e^2fl+$
$+10e^3n)]+d[10c^4(fj+ek)+c^3f(57fj+34ek)+5ef^2(f^2k-efl+e^2n)+cf(5f^3j+$
$+52ef^2k-50e^2fl+50e^3n)+c^2(36f^3j+103ef^2k-85e^2fl+85e^3n)]]x^2y^4-42[10d^5fg-$
$-2d^4f(5ch+ei)+d^3f(10c^2i+2cfi+7cej+efj+2e^2k)-c(10c^4+37c^3f+42c^2f^2+17cf^3+$
$+2f^4)(fl-en)+d(c+f)[10c^3fk+2ef^2(fl-en)+c^2(7f^2k+37efl-37e^2n)+cf(f^2k+$
$+21efl-21e^2n)]-d^2[10c^3fj+c^2f(7fj+17ek)+ef(f^2k+6efl-6e^2n)+c(f^3j+$
$+10ef^2k+16e^2fl-16e^3n)]]xy^5+7[10d^6g-2d^5(5ch+ei)+d^4(10c^2i+2cfi+7cej+efj+$
$+2e^2k)+c(10c^5+87c^4f+227c^3f^2+227c^2f^3+87cf^4+10f^5)n-d(c+f)[10c^4l+10ef^3n+$
$+3c^3(9fl+29en)+2cf^2(fl+56en)+c^2f(15fl+241en)]-d^3[10c^3j+c^2(7fj+$

$$+17ek) + c(f^2j + 10efk + 16e^2l) + e(f^2k + 6efl + 10e^2n)] + d^2(c + f)[10c^3k+ \\ +c^2(7fk + 37el) + ef(2fl + 35en) + c(f^2k + 21efl + 101e^2n)]]y^6.$$

Using this operator and equality (8) we obtain up to a constant an integrating factor of the system (29), in the form

$$\mu^{-1} = \varphi_6(x, y) \times [-ex^2 + (c - f)xy + dy^2]. \tag{30}$$

Analogously to previous cases we have the next

Theorem 9. *Differential system (1) with $m = 7$ and $\widetilde{K}_8 \equiv 0$ has the invariant $GL(2,\mathbb{R})$-integrating factor μ of the form*

$$\mu^{-1} = K_2\Phi_6,$$

and

$$\Phi_6 \equiv 7I_1(13I_1^2 - 8I_2)(5I_1^2 - I_2)(7I_1\widetilde{K}_6 - 8S_1) - 210I_1(5I_1^2 - I_2)K_2(5I_1S_2 - 4S_3)+ \\ +840K_2^2(6I_1^2S_4 - 3I_1S_5 - K_2S_6) + 24I_1(I_1^2 - I_2)(13I_1^2 - 8I_2)(5I_1^2 - I_2) = 0 \tag{31}$$

is a particular invariant $GL(2,\mathbb{R})$-integral of this system.

Here invariants and comitants $I_1,\ I_2, K_2,\ \widetilde{K}_6 = a^{\alpha}_{\alpha\beta\gamma\delta\mu\nu\eta}x^{\beta}x^{\gamma}x^{\delta}x^{\mu}x^{\nu}x^{\eta}$ are from (2), and

$$S_1 = a^{\alpha}_{\beta}a^{\beta}_{\alpha\gamma\delta\mu\nu\eta\varphi}x^{\gamma}x^{\delta}x^{\mu}x^{\nu}x^{\eta}x^{\varphi},\ \ S_2 = a^{\alpha}_{p}a^{\beta}_{q\alpha\beta\gamma\delta\mu\nu}x^{\gamma}x^{\delta}x^{\mu}x^{\nu}\varepsilon^{pq},$$
$$S_3 = a^{\alpha}_{p}a^{\beta}_{\delta}a^{\gamma}_{\alpha\beta\gamma\mu\nu\eta q}x^{\delta}x^{\mu}x^{\nu}x^{\eta}\varepsilon^{pq},\quad S_4 = a^{\alpha}_{p}a^{\beta}_{r}a^{\gamma}_{\alpha\beta\gamma\delta\mu qs}x^{\delta}x^{\mu}\varepsilon^{pq}\varepsilon^{rs},$$
$$S_5 = a^{\alpha}_{p}a^{\beta}_{\mu}a^{\gamma}_{r}a^{\delta}_{\alpha\beta\gamma\delta\nu qs}x^{\mu}x^{\nu}\varepsilon^{pq}\varepsilon^{rs},\quad S_6 = a^{\alpha}_{p}a^{\beta}_{r}a^{\gamma}_{k}a^{\delta}_{\alpha\beta\gamma\delta qsl}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}.$$

It is shown in [2,3] that all singular points of Darboux type differential system (3), different from the origin, are located on its integral straight lines, coinciding with integral straight lines of this system for $R_{m-1} \equiv 0$. Therefore the necessary condition for the existence of a limit cycle for the Darboux type differential system (3) is the condition that the eigenvalues of the matrix of linear terms should be imaginary, i.e. the condition [1] $2I_2 - I_1^2 < 0$.

We observe that the expression Φ_{m-1} from (10), (15), (20), (24), (28) and (31) with $m = \overline{2,7}$ are only algebraic integrals of the form (4) for the Darboux type system (3) with $m = \overline{2,7}$. To prove this remark it is sufficient to examine the explicit form of first integrals for the system (3) with $m = \overline{2,7}$.

One can verify easily that holds

Theorem 10. *The Darboux type differential system (3) with $2I_2 - I_1^2 < 0$ has the first real integral in the form*

$$\frac{I_1}{\sqrt{I_1^2 - 2I_2}}\arctan\frac{2a_1^2x^1 + (a_2^2 - a_1^1)x^2}{|x^2|\sqrt{I_1^2 - 2I_2}} + \frac{1}{2}\ln|K_2| - \frac{1}{m-1}\ln|\Phi_{m-1}| = C\ \ (m = \overline{2,7}), \tag{32}$$

where K_2 is from (2), Φ_{m-1} $(m = \overline{2,7})$ are from (10), (15), (20), (24), (28) and (31).

It is clear from (32) that Φ_{m-1} $(m=\overline{2,7})$ is the only one algebraic integral of the form (4).

As for differential systems (11), (21) and (27) the corresponding algebraic invariant integrals (10), (20) and (28) have the homogeneities of odd degree with respect to x^1 and x^2, than with the aid of Remark 1 and Lemma 2 we prove

Theorem 11. *The differential system (1) with* $m=2l$ *and* $\widetilde{K}_{2l+1}\equiv 0,\ (l=1,2,3)$ *does not have limit cycles.*

The main idea of this theorem allow us to suppose that systems of the form (1) with $m=2l$ and $\widetilde{K}_{2l+1}\equiv 0$ where $l\geq 4$ also do not have limit cycles.

It is easy to prove the next

Theorem 12. *For a system (1) with* $K_2\not\equiv 0$ *and* $\widetilde{K}_{m+1}\equiv 0, (m=\overline{2,7})$ *to have a first invariant* $GL(2,\mathbb{R})$*-integral of the Darboux type [10] in the form*

$$K_2^{1-m}\Phi_{m-1}^2=C\quad (m=\overline{2,7})$$

it is necessary and sufficient that $I_1=0$*, where* $K_2,\ \widetilde{K}_{m+1},I_1$ *are from (2), and* Φ_{m-1} $(m=\overline{2,7})$ *are from (10), (15), (20), (24), (28) and (31).*

The proof of Theorem 12 results from the identity

$$\Lambda(K_2^{1-m}\Phi_{m-1}^2)=(1-m)I_1K_2^{1-m}\Phi_{m-1}^2\quad (m=\overline{2,7}),$$

where Λ is from (5).

There exists the supposition that Theorem 12 holds for $m\geq 8$.

The following question remains open: Are all first invariant $GL(2,\mathbb{R})$-integrals of the differential system (3) with $(m=\overline{2,7})$ encapsulated by Theorem 12 or not?

The authors tender thanks to Professor N.I.Vulpe for effective discussions of the results of this paper.

References

[1] Sibirsky K.S. *Introduction to the algebraic theory of invariants of differential equations.* Kishinev, Shtiintsa, 1982 (Manchester University Press, Manchester, 1988).

[2] Lucashevich N.A. *The integral curves of Darboux equations.* Differential equations, 1966, **2(5)**, p. 628–633 (in Russian).

[3] Vulpe N.I., Costas S.I. *The center-affine invariant conditions of topological distinctions of the Darboux differential systems with cubic non-linearities.* Preprint, Chisinau, 1989 (in Russian).

[4] Dedok N.N. *On the singular points of differential equation Darboux.* Differential Equations, 1972, **8**, N 10, p. 1880–1881 (in Russian).

[5] Amelkin V.V., Lucashevich N.A., Sadovski A.P. *Nonlinear variation in systems of the second order.* Minsk, 1982 (in Russian).

[6] Gorbuzov V.N., Samodurov A.A. *Darboux equation and its analogues: Optional course manual*. Grodno, 1985 (in Russian).

[7] Vulpe N.I., Costas S.I. *The center-affine invariant conditions for existence of the limit cycle for one Darboux system*. Mat. issled., 1987, vyp. 92, p. 38–42 (in Russian).

[8] Popa M.N. *Algebraic methods for differential systems*. Seria Matematica Aplicata si Industriala, Nr. 15, Flower Power Edit., Pitesti Univers., Romania, 2004 (in Romanian).

[9] Chebanu V.M. *Minimal polynomial basis of comitants of cubic differential systems*. Differential equations, 1985, **21**, N 3, p. 541–543 (in Russian).

[10] Darboux G. *Memoire sur les equations differentielles algebriques du premier order et du premier degree*. Bull. Sciences Math., 1878, Ser.(2), 2(1), p. 60–96, 123–144, 151–200.

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Received August 21, 2006