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Collocation and Quadrature Methods for Solving Singular Integral Equations with Piecewise Continuous Coefficients

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Abstract. The computation schemes of collocation and mechanical quadrature methods for approximate solving of the complete singular integral equations with piecewise continuous coefficients and a regular kernel with weak singularity are elaborated. The case when the equations are defined on the unit circumference of the complex plane is examined. The sufficient conditions for the convergence of these methods in the space L_2 are obtained.

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1 The Problem Formulation

Let Γ_0 be a unit circumference of the complex plane $\mathbb{C}$ with the center at the origin, let D^+ be a domain bounded by Γ_0, $D^- = \mathbb{C} \setminus \{D^+ \cup \Gamma_0\}$, and let $L_2(\Gamma_0)$ be a space of all functions $f : \Gamma_0 \to \mathbb{C}$ that are Lebesgue measurable and square integrable on Γ_0.

We will denote by $PC(\Gamma_0)$ a Banach algebra of all functions $a : \Gamma_0 \to \mathbb{C}$ which are continuous on Γ_0 with exception of a finite number of points in such a way that at each point of discontinuity there exist unilateral finite limits $a(t-0)$, $a(t+0)$ and $a(t-0) = a(t)$.

To each element $a \in PC(\Gamma_0)$ we associate the function $\hat{a} : \Gamma_0 \times [0,1] \to \mathbb{C}$ in the following way $\hat{a}(t,\mu) = \mu a(t+0) + (1-\mu)a(t), t \in \Gamma_0,\ 0 \le \mu \le 1$. The set $\Gamma_{\hat{a}}$ of values of the function $\hat{a}(t,\mu)$ represents a closed curve. This curve is a union of the set of values of the function $a(t)$ and segments $\mu a(t_k+0) + (1-\mu)a(t_k)$ $(0 \le \mu \le 1, k = \overline{1,n})$, where $t_1, ..., t_n$ are all points of discontinuity of the function a. The curve $\Gamma_{\hat{a}}$ can be oriented in a natural way.

We say that the function $a \in PC(\Gamma_0)$ is 2-nonsingular if the curve $\Gamma_{\hat{a}}$ doesn't go through the origin. We denote the number of rotations of the curve $\Gamma_{\hat{a}}$ around the origin by index $ind_2 a$ of the 2-nonsingular function a.

In $L_2(\Gamma_0)$ we consider the following singular integral equation

$$(A\varphi \equiv)\, a_0(t)\varphi(t) + \frac{b_0(t)}{\pi i}\int_{\Gamma_0}\frac{\varphi(\tau)}{\tau - t}d\tau + \frac{1}{2\pi i}\int_{\Gamma_0} h(t,\tau)\varphi(\tau)d\tau = f(t),\ t \in \Gamma_0, \quad (1)$$

where $a_0, b_0, f : \Gamma_0 \to \mathbb{C}$, $h : \Gamma_0 \times \Gamma_0 \to \mathbb{C}$ are known functions, $a_0, b_0 \in PC(\Gamma_0)$, $h(t,\tau) = h_0(t,\tau)|\tau - t|^{-\gamma}\,(0 < \gamma < 1)$, $h_0 \in C(\Gamma_0 \times \Gamma_0)$, $f \in L_2(\Gamma_0)$ and $\varphi : \Gamma_0 \to \mathbb{C}$ is an unknown function.

It is known that operators $K, S : L_2(\Gamma_0) \to L_2(\Gamma_0)$, defined in the following way $(K\varphi)(t) = \dfrac{1}{2\pi i}\int_{\Gamma_0} h(t,\tau)\varphi(\tau)d\tau$, $(S\varphi)(t) = \dfrac{1}{\pi i}\int_{\Gamma_0} \frac{\varphi(\tau)}{\tau - t}d\tau$, are bounded [1,2]. Taking into account that $\|c\varphi\|_2 \le \|c\|_\infty \|\varphi\|_2$ for all functions $c \in PC(\Gamma_0)$, the operator $A = a_0 I + b_0 S + K$ which describes the left term of equation (1) is bounded in $L_2(\Gamma_0)$.

In [3, 4] the theoretical foundation of the collocation and quadrature methods for equation (1) in the norm of the space $L_2(\Gamma_0)$ was obtained in the case of coefficients that satisfy Holder condition on Γ_0 and in [5] the foundation was obtained in the case of continuous coefficients on Γ_0. In the present paper we will state conditions of convergence of these methods in $L_2(\Gamma_0)$ in the case when coefficients of the equation (1) belong to the space $PC(\Gamma_0)$.

2 The deduction of a computation schemes

We will denote by $\mathcal{P}_n$ the set of all trigonometric polynomials of the form $\sum\limits_{k=-n}^{n} r_k t^k$ $(t \in \Gamma_0)$, where r_k $(k = \overline{-n,n})$ are arbitrary complex numbers. We will consider on Γ_0 the following equidistant points

$$t_j = exp\,(2\pi i j/(2n+1)),\ j = \overline{-n,n}. \tag{2}$$

In the following it is convenient to write equation (1) in the equivalent form

$$(A\varphi \equiv) a(t)(P\varphi)(t) + b(t)(Q\varphi)(t) + (K\varphi)(t) = f(t), t \in \Gamma_0, \tag{3}$$

where $a(t) = a_0(t) + b_0(t), b(t) = a_0(t) - b_0(t), P = (I + S)/2, Q = I - P$, I is the identity operator, and S is a singular operator.

The presence of discontinuity in the kernel of the regular part of equation (1) implies essential difficulties in the practical realization of the calculation scheme of the collocation method applied to it, and the quadrature method cannot be applied.

In order to eliminate this drawback, in an analogous way to [3, 6], we introduce a new equation

$$(A_\rho\varphi \equiv) a(t)(P\varphi)(t) + b(t)(Q\varphi)(t) + (K_\rho\varphi)(t) = f(t), t \in \Gamma_0, \tag{4}$$

in which

$$(K_\rho\varphi)(t) = \frac{1}{2\pi i}\int_{\Gamma_0} h_\rho(t,\tau)\varphi(\tau)d\tau,$$

$$h_\rho(t,\tau) = \begin{cases} h_0(t,\tau)|\tau - t|^{-\gamma},\ for\ \ |\tau - t| \ge \rho \\ h_0(t,\tau)\rho^{-\gamma},\ for\ \ |\tau - t| < \rho \end{cases},\ \rho \in (0,1).$$

Equations (3) and (4) have the same characteristic part, and the kernel of the regular part of equation (4) is a continuous function on Γ_0 in both variables.

In the following the collocation and quadrature methods will be applied to equation (4). The obtained approximate solutions will be considered as the approximations of the exact solution of equation (3), and, thus, of equation (1).

According to the collocation method we will seek for an approximate solution of equation (4) in the form of the polynomial

$$\varphi_n(t) = \sum_{k=-n}^{n} \alpha_k^{(n)} t^k \in \mathcal{P}_n, \tag{5}$$

unknown coefficients of which $\alpha_k^{(n)} = \alpha_k \, (k = \overline{-n,n})$ will be determined from the following system of linear algebraic equations (SLAE)

$$a(t_j)\sum_{k=0}^{n} \alpha_k t_j^k + b(t_j)\sum_{k=-n}^{-1} \alpha_k t_j^k + \sum_{k=-n}^{n} \alpha_k \frac{1}{2\pi i} \int\limits_{\Gamma_0} h_\rho(t_j,\tau)\tau^k d\tau = f(t_j), \, j = \overline{-n,n}. \tag{6}$$

The proposed calculation scheme essentially simplifies the process of its numerical implementation.

If for solving equation (4) the method of quadratures is applied, then we will seek for the approximate solution of this equation in the form (5) and we will determine coefficients $\alpha_k \, (k = \overline{-n,n})$ as solutions of SLAE

$$a(t_j)\sum_{k=0}^{n} \alpha_k t_j^k + b(t_j)\sum_{k=-n}^{-1} \alpha_k t_j^k + \frac{1}{2n+1}\sum_{k=-n}^{n} \alpha_k \sum_{s=-n}^{n} h_\rho(t_j,t_s) t_s^{k+1} = f(t_j), \, j = \overline{-n,n}. \tag{7}$$

Let a bounded and measurable function $f : \Gamma_0 \to \mathbb{C}$ be given. There exists a unique interpolation polynomial

$$(L_n f)(t) = \sum_{k=-n}^{n} \Lambda_k t^k \in \mathcal{P}_n, \;\; \Lambda_k = \frac{1}{2n+1} \sum_{j=-n}^{n} f(t_j) t_j^{-k} \tag{8}$$

such that $(L_n f)(t_j) = f(t_j)$ for each $j = \overline{-n,n}$ [7, p.151]. The operator L_n, for which $L_n^2 = L_n$, is a Lagrange interpolation projector. Besides this nonorthogonal projector, we consider an orthogonal projector $S_n : L_2(\Gamma_0) \to \mathcal{P}_n$, which for each function $\varphi \in L_2(\Gamma_0)$ puts into correspondence a partial sum of order n of the Fourier series after the system of functions $\{t^k\}_{k=-\infty}^{+\infty}$, $(S_n\varphi)(t) = \sum_{k=-n}^{n} \varphi_k t^k$. Taking into account that for functions of the form (5) the following equalities are true $(S_n\varphi_n)(t) = \varphi_n(t)$, we obtain that systems of equations (6), (7) are equivalent to the following operator equations

$$(A_{n,\rho}\varphi_n \equiv) \, L_n(aP + bQ + K_\rho)S_n\varphi_n = L_n f, \tag{9}$$

$$(A'_{n,\rho}\tilde{\varphi}_n \equiv) \, L_n(aP + bQ + \Delta_n)S_n\tilde{\varphi}_n = L_n f, \tag{10}$$

where $(\Delta_n\tilde{\varphi}_n)(t) = \frac{1}{2\pi i}\int_{\Gamma_0} L_n^\tau(h_\rho(t,\tau)\tilde{\varphi}_n(\tau))d\tau$. Notice that here and in what follows L_n^τ denotes the operator L_n, applied with respect to the variable τ. Therefore in the following instead of systems (6) and (7) we will study operator equations (9) and, respectively, (10) which are considered in the subspace $\mathcal{P}_n$, in which the same norm as in $L_2(\Gamma_0)$ is introduced.

In the case of an equation with coefficients from $PC(\Gamma_0)$, in order to apply the methods studied in the paper it is necessary to choose the right term f from a subclass of $L_2(\Gamma_0)$. As such a subclass the set $R(\Gamma_0)$ of all bounded, defined on Γ_0 and integrable by Riemann functions can be chosen. With the norm $\|g\|_\infty = \sup_{t\in\Gamma_0} |g(t)|$ the set $R(\Gamma_0)$ becomes the Banach space.

3 Some preliminary results

In this section we will state some relations between integral operators with kernel $h_0(t,\tau)|\tau - t|^{-\gamma}$ and $h_\rho(t,\tau)$, considered in the space $L_2(\Gamma_0)$. These results, as well as other results from this section, will be used for the theoretical foundation of the elaborated computational schemes.

We will denote by $\chi_\rho(t)$ the function defined on Γ_0 in the following way. If $\varphi(t) \in L_2(\Gamma_0)$, then

$$\chi_\rho(t) = \frac{1}{2\pi i}\int_{\Gamma_0} [h_0(t,\tau)|\tau - t|^{-\gamma} - h_\rho(t,\tau)]\varphi(\tau)d\tau,$$

where $h_0(t,\tau)$ and $h_\rho(t,\tau)$ are the defined above functions.

Lemma 1. *Let $h_0(t,\tau) \in C(\Gamma_0 \times \Gamma_0)$ (in both variables) and $\varphi(t) \in L_2(\Gamma_0)$. Then it is true that*

a) $\|\chi_\rho\|_2 \le d_1\rho^{\frac{1-\gamma}{2}}\|\varphi\|_2$;
b) $(K_\rho\varphi)(t) \in C(\Gamma_0)$;
c) The operator $K_\rho : L_2(\Gamma_0) \to C(\Gamma_0)$ is completely continuous.

Proof. Let $t \in \Gamma_0$ and $\Gamma_\rho := \{\tau \in \Gamma_0 : |\tau - t| < \rho\}$. Then, as $\chi_\rho(t) = 0$ for $|\tau - t| \ge \rho$, we have

$$\|\chi_\rho\|_2^2 = \frac{1}{2\pi}\int_{\Gamma_0} |\chi_\rho|^2|dt| = \frac{1}{2\pi}\int_{\Gamma_0}\left|\frac{1}{2\pi i}\int_{\Gamma_0}[h_0(t,\tau)|\tau - t|^{-\gamma} - h_\rho(t,\tau)]\varphi(\tau)d\tau\right|^2|dt| =$$

$$= \frac{1}{(2\pi)^3}\int_{\Gamma_0}\left|\int_{\Gamma_\rho} h_0(t,\tau)\left[|\tau - t|^{-\gamma} - \rho^{-\gamma}\right]\varphi(\tau)d\tau\right|^2|dt| \le$$

$$\le \frac{1}{(2\pi)^3}\int_{\Gamma_0}\left(\int_{\Gamma_\rho}|h_0(t,\tau)|\left||\tau - t|^{-\gamma} - \rho^{-\gamma}\right||\varphi(\tau)||d\tau|\right)^2|dt|.$$

Since $|\tau - t|^{-\gamma} - \rho^{-\gamma} > 0\,(\tau \in \Gamma_\rho)$, from the last relation we obtain

$$\|\chi_\rho\|_2^2 \le \frac{\|h_0\|_C^2}{(2\pi)^3}\int_{\Gamma_0}\left(\int_{\Gamma_\rho}\frac{|\varphi(\tau)||d\tau|}{|\tau - t|^\gamma}\right)^2|dt|.$$

Estimating the interior integral using the Holder inequality for integrals (see [8, p.496]), we obtain

$$\int\limits_{\Gamma_\rho} \frac{|\varphi(\tau)|}{|\tau-t|^\gamma}|d\tau| = \int\limits_{\Gamma_\rho} \frac{1}{|\tau-t|^{\gamma/2}}\frac{|\varphi(\tau)|}{|\tau-t|^{\gamma/2}}|d\tau| \le \left(\int\limits_{\Gamma_\rho} \frac{|d\tau|}{|\tau-t|^\gamma}\right)^{\frac{1}{2}} \left(\int\limits_{\Gamma_\rho} \frac{|\varphi(\tau)|^2}{|\tau-t|^\gamma}|d\tau|\right)^{\frac{1}{2}}.$$

Then

$$\|\chi_\rho\|_2^2 \le \frac{\|h_0\|_C^2}{(2\pi)^3}\int_{\Gamma_0}\left(\int_{\Gamma_\rho}\frac{|d\tau|}{|\tau-t|^\gamma}\right)\left(\int_{\Gamma_\rho}\frac{|\varphi(\tau)|^2}{|\tau-t|^\gamma}|d\tau|\right)|dt|.$$

We estimate integral $\int_{\Gamma_\rho}\dfrac{|d\tau|}{|\tau-t|^\gamma}$ using the following relation (see [9, p.10])

$$|d\tau| = |ds| \le \frac{\pi}{2}dr, \tag{11}$$

where ds is a length of the arc of the circumference $\breve{\tau t}$ (the smallest arc from two possible ones), and dr is a length of the chord that subtends the arc $\breve{\tau t}$ $(|\tau-t|=r)$. Then when τ passes the arc Γ_ρ, the value r passes the segment $[0;\rho]$. Using relation (11) we obtain

$$\int_{\Gamma_\rho}\frac{|d\tau|}{|\tau-t|^\gamma} \le \frac{\pi}{2}\int_0^\rho r^{-\gamma}dr = \frac{\pi}{2(1-\gamma)}\rho^{1-\gamma}.$$

Then we have

$$\|\chi_\rho\|_2^2 \le \frac{\|h_0\|_C^2}{16\pi^2}\frac{1}{(1-\gamma)}\rho^{1-\gamma}\int\limits_{\Gamma_0}\int\limits_{\Gamma_\rho}\frac{|\varphi(\tau)|^2}{|\tau-t|^\gamma}|d\tau||dt| =$$

$$= \frac{\|h_0\|_C^2}{16\pi^2}\frac{1}{1-\gamma}\rho^{1-\gamma}\int\limits_{\Gamma_\rho}|\varphi(\tau)|^2\int\limits_{\Gamma_0}\frac{|dt|}{|\tau-t|^\gamma}|d\tau|.$$

Repeating the above argumentation, we obtain for interior integral the following estimation

$$\int_{\Gamma_0}\frac{|dt|}{|\tau-t|^\gamma} \le \frac{\pi}{2}\int_0^2 r^{-\gamma}dr = \frac{\pi}{1-\gamma}2^{-\gamma}.$$

Taking this into account, we obtain

$$\|\chi_\rho\|_2^2 \le \frac{\|h_0\|_C^2}{16\pi}\frac{2^{-\gamma}}{(1-\gamma)^2}\rho^{1-\gamma}\int_{\Gamma_\rho}|\varphi(\tau)|^2|d\tau|,$$

from which results the inequality a), in which $d_1 = \dfrac{2^{(-2-\gamma/2)}}{(1-\gamma)\pi^{1/2}}\|h_0\|_C$.

Now we will show that the function $(K_\rho\varphi)(t) = \dfrac{1}{2\pi i}\int_{\Gamma_0} h_\rho(t,\tau)\varphi(\tau)d\tau$ is continuous on Γ_0. For $\varphi \in L_2(\Gamma_0)$ we have $\|\varphi\|_2 < \alpha$. The function $h_\rho(t,\tau)$, being continuous on the compact $\Gamma_0 \times \Gamma_0$, is uniformly continuous. In such a way for $\varepsilon > 0$ there exists $\delta > 0$ such that the inequalities $|t_2 - t_1| < \delta$, $|\tau_2 - \tau_1| < \delta$ imply the relation $|h_\rho(t_2, \tau_2) - h_\rho(t_1, \tau_1)| < \varepsilon/\alpha$. Taking into account the last inequality and Holder inequalities, we obtain for $|t_2 - t_1| < \delta$

$$|(K_\rho\varphi)(t_2) - (K_\rho\varphi)(t_1)| \le \frac{1}{2\pi}\int_{\Gamma_0} |h_\rho(t_2,\tau) - h_\rho(t_1,\tau)||\varphi(\tau)||d\tau| \le$$

$$\le \frac{1}{2\pi}\left(\int_{\Gamma_0} |h_\rho(t_2,\tau) - h_\rho(t_1,\tau)|^2|d\tau|\right)^{1/2}\left(\int_{\Gamma_0}|\varphi(\tau)|^2|d\tau|\right)^{1/2} \le \frac{\varepsilon}{\alpha}\|\varphi\|_2 < \varepsilon. \quad (12)$$

In such a way, the function $(K_\rho\varphi)(t)$ is continuous.

The affirmation from point c) is stated using the Arzela-Ascoli theorem. The linearity of the operator K_ρ is evident. Let M be a bounded set in $L_2(\Gamma_0)$. In this way there exists $\alpha > 0$ such that $\|\varphi\|_2 < \alpha$ $(\varphi \in M)$. For every $\varphi \in M$, according to inequality (12), we obtain that inequality $|t_2 - t_1| < \delta$ implies $|(K_\rho\varphi)(t_2) - (K_\rho\varphi)(t_1)| < \varepsilon$. This means that the functions of the set $K_\rho(M)$ are equally continuous. Let us show that the set $K_\rho(M)$ is bounded in $C(\Gamma_0)$. Let $\beta = \max\limits_{t,\tau\in\Gamma_0} |h_\rho(t,\tau)|$. We have

$$|(K_\rho\varphi)(t)| \le \frac{1}{2\pi}\int_{\Gamma_0} |h_\rho(t,\tau)||\varphi(\tau)||d\tau| \le$$

$$\le \frac{1}{2\pi}\left(\int_{\Gamma_0}|h_\rho(t,\tau)|^2|d\tau|\right)^{1/2}\left(\int_{\Gamma_0}|\varphi(\tau)|^2|d\tau|\right)^{1/2} \le \beta\|\varphi\|_2.$$

So, for each $K_\rho\varphi \in K_\rho(M)$ we have $\|K_\rho\varphi\|_{C(\Gamma_0)} = \max\limits_{t\in\Gamma_0}|(K_\rho\varphi)(t)| < \alpha\beta$. Therefore, the set $K_\rho(M) \subset C(\Gamma_0)$ is uniformly bounded and functions of this set are equally bounded.

According to the Arzela-Ascoli theorem the set $K_\rho(M)$ is relatively compact in $C(\Gamma_0)$ and in such a way the operator K_ρ is completely continuous. The lemma is proved.

Lemma 2. *Let the operator A, defined by the left term of equation (3), be invertible in the space $L_2(\Gamma_0)$. Then for ρ such that*

$$\varepsilon_\rho := d_1\rho^{(1-\gamma)/2}\left\|A^{-1}\right\|_2 \le q_1 < 1, \quad (13)$$

the operator A_ρ, defined by the left term of equation (4), is invertible in $L_2(\Gamma_0)$ as well and the inequality $\|A_\rho^{-1}\|_2 \le (1-\varepsilon_\rho)^{-1}\|A^{-1}\|_2$ is true. For the solutions $\varphi = A^{-1}f$ and $\varphi_\rho = A_\rho^{-1}f$ of equations (3) and (4), respectively, we have

$$\|\varphi - \varphi_\rho\|_2 \le \varepsilon_\rho(1-\varepsilon_\rho)^{-1}\left\|A^{-1}\right\|_2\|f\|_2.$$

Proof. Using item a) from Lemma 1 we obtain the estimation $\|(A - A_\rho)x\|_2 = \|(K - K_\rho)x\|_2 = \|\chi_\rho\|_2 \le d_1\rho^{(1-\gamma)/2}\|x\|_2$, $\forall x \in L_2(\Gamma_0)$. Then $\|A - A_\rho\|_2 \le d_1\rho^{(1-\gamma)/2}$.

We will show that if inequality (13) holds, then the operator A_ρ is invertible for sufficiently small values of ρ. For this we will use the representation $A_\rho = A - (A - A_\rho) = A(I - A^{-1}(A - A_\rho))$. Since $\|A^{-1}(A - A_\rho\|_2 \le \|A^{-1}\|_2 d_1 \rho^{(1-\gamma)/2} = \varepsilon_\rho \le q_1 < 1$ is true, then according to Banach theorem about small perturbations of an invertible operator, results the existence of the inverse operator $A_\rho^{-1} = (I - A^{-1}(A - A_\rho))^{-1} A^{-1}$ the norm of which satisfies the inequality

$$\left\|A_\rho^{-1}\right\|_2 \le \left\|(I - A^{-1}(A - A_\rho))^{-1}\right\|_2 \left\|A^{-1}\right\|_2 = (1 - \varepsilon_\rho)^{-1} \left\|A^{-1}\right\|_2.$$

For solutions φ and φ_ρ of equations (3) and (4), respectively, we have

$$\|\varphi - \varphi_\rho\|_2 \le \left\|A^{-1} - A_\rho^{-1}\right\|_2 \|f\|_2 \le \left\|A^{-1}\right\|_2 \|A_\rho - A\|_2 \left\|A_\rho^{-1}\right\|_2 \|f\|_2 \le$$

$$\le (1 - \varepsilon_\rho)^{-1} \left\|A^{-1}\right\|_2^2 d_1 \rho^{(1-\gamma)/2} \|f\|_2 = \varepsilon_\rho (1 - \varepsilon_\rho)^{-1} \left\|A^{-1}\right\|_2 \|f\|_2.$$

The lemma is proved.

Remark 1. As $\varepsilon_\rho \to 0$ when $\rho \to 0$, it results that $\|\varphi - \varphi_\rho\|_2 \to 0$ when $\rho \to 0$. This fact justifies the made convention with relation to the possibility of approximation of the exact solution of equation (3) with the approximate solution of equation (4), obtained according to the collocation method. In the following we will consider that ρ satisfies condition (13). This is true if ρ is sufficiently small.

It is known from [10, p.5; 11, p.12], that the operator L_n that acts in the space $L_2(\Gamma_0)$ is unbounded, but being looking for as an operator that acts from the space $R(\Gamma_0)$ to $L_2(\Gamma_0)$ it is bounded, and in [11] it is shown that

$$\|L_n f - f\|_2 \to 0, \; \forall f \in R(\Gamma_0). \tag{14}$$

Lemma 3. *Let $\{t_j\}_{j=-n}^{n}$ be the system of points (2). Then for each integer number m, such that $|m| \le 2n$ the following relation is true:*

$$\frac{1}{2n+1} \sum_{j=-n}^{n} t_j^m = \begin{cases} 1, \; if \;\; m = 0 \\ 0, \; if \;\; m \ne 0 \end{cases}. \tag{15}$$

Proof. For $m = 0$ relation (15) is evident. For $m \ne 0$, $|m| \le 2n$, we have $t_m \ne 1$ and $t_m^{2n+1} = 1$. In such a way we obtain $\sum\limits_{j=-n}^{n} t_j^m = \sum\limits_{j=-n}^{n} t_m^j = \dfrac{1 - t_m^{2n+1}}{t_m^n (1 - t_m)} = 0$. The lemma is proved.

Lemma 4. *For each measurable and bounded function $g : \Gamma_0 \to \mathbb{C}$ and each polynomial $p_n \in \mathcal{P}_n$ the following relation is true*

$$\|L_n g \, p_n\|_2 \le \|g\|_\infty \|p_n\|_2, \tag{16}$$

where $\|g\|_\infty = \sup\limits_{t \in \Gamma_0} |g(t)|$.

Proof. Taking into account the fact that the functions t^n, $t = e^{i\theta}$, $n \in \mathbb{Z}$, form an orthogonal basis in $L_2(\Gamma_0)$ and relations (8) and (15), the norm of the polynomial $L_n f$ (f is measurable and bounded in $L_2(\Gamma_0)$) can be calculated:

$$\|L_n f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|(L_n f)(e^{i\theta})\right|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{k=-n}^{n}\Lambda_k e^{i\theta k}\right|^2 d\theta = \sum_{k=-n}^{n}|\Lambda_k|^2 =$$

$$= \frac{1}{(2n+1)^2}\sum_{k=-n}^{n}\left(\sum_{j=-n}^{n} f(t_j)t_j^{-k}\right)\left(\sum_{l=-n}^{n}\overline{f(t_l)}t_l^k\right) =$$

$$= \frac{1}{(2n+1)^2}\sum_{j=-n}^{n} f(t_j)\left(\sum_{l=-n}^{n}\overline{f(t_l)}\left(\sum_{k=-n}^{n} t_k^{l-j}\right)\right) = \frac{1}{2n+1}\sum_{j=-n}^{n}|f(t_j)|^2 .$$

In this way we obtain:

$$\|L_n g\, p_n\|_2^2 = \frac{1}{2n+1}\sum_{j=-n}^{n}|g(t_j)|^2\,|p_n(t_j)|^2 \le \|g\|_\infty^2 \|L_n\, p_n\|_2^2 = \|g\|_\infty^2\|p_n\|_2^2,$$

which implies relation (16). The lemma is proved.

Lemma 5. *Each 2-nonsingular function $a \in PC(\Gamma_0)$ can be represented in the form $a(t) = r_n(t)h(t)$, where r_n is a trigonometric polynomial from $\mathcal{P}_n$ and $h \in PC(\Gamma_0)$ (h is 2-nonsingular and with the same discontinuities as a) such that $\|h-1\|_\infty = \sup_{t\in\Gamma_0}|h(t)-1| \le q < 1$.*

Proof. The 2-nonsingular function $a \in PC(\Gamma_0)$ with discontinuity points $t_1, ..., t_n$ can be represented in the form $a(t) = |a(t)|\exp(i\theta(t))$. We set $\rho(t) = |a(t)|$. From the hypothesis it results that $\rho \in PC(\Gamma_0)$ and there exists $\delta > 0$ such that $\rho(t) \ge \delta$ for all $t \in \Gamma_0$. In such a way we can include ρ in the factor h and so we can assume, without loosing generality, that $a(t) = \exp(i\theta(t))$.

We choose an arbitrary point $t_0 \in \Gamma_0$, $t_0 \ne t_j$ ($j = \overline{1,n}$) as an initial point from which the calculation of argument begins. The fact that $\hat{a}(t,\mu) \ne 0$ for all $(t,\mu) \in \Gamma_0 \times [0,1]$ allows us to choose the function θ with real values in such a way that θ is continuous at all points $t \in \Gamma_0$ which are different from t_j ($j = \overline{0,n}$), is left continuous at $t_0, t_1, ..., t_n$ and for $\delta > 0$ the relations $|\theta(t_j) - \theta(t_j+0)| < \pi - \delta$ ($j = \overline{1,n}$) are true while $a(t_0) - a(t_0+0)$ is multiple of 2π. We define the functions $b, c \in PC(\Gamma_0)$ with real values in the following way: $b(t_j) = \theta(t_j)$, $b(t_j+0) = \theta(t_j+0)$, $j = \overline{0,n}$, $c(t_0) = \theta(t_0)$, $c(t_0+0) = \theta(t_0+0)$, $c(t_j) = c(t_j+0) = \frac{1}{2}(\theta(t_j)+\theta(t_j+0))$, $j = \overline{1,n}$, and on residual arcs of Γ_0, $b(t)$ and $c(t)$ are defined by linear interpolation. Then the following inequality is true

$$\sup_{t\in\Gamma_0}|b(t)-c(t)| < \frac{1}{2}(\pi - \delta). \tag{17}$$

The mode of choice of functions $b(t)$ and $c(t)$ implies the fact that the functions $\theta(t) - b(t)$ and $\exp(ic(t))$ are continuous on Γ_0. So, the following function

$$f(t) = \exp(i(\theta(t) - b(t) + c(t))) \tag{18}$$

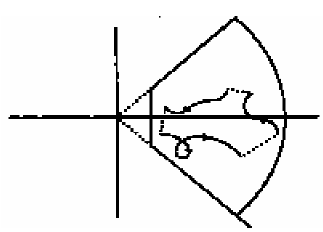
is continuous on Γ_0. It is evident that $|f(t)| = 1$ for all $t \in \Gamma_0$. Therefore, according to the Wierstrass second theorem of approximation, there exists trigonometric polynomial $p_n(t) = \sum_{k=-n}^{n} a_k t^k$ such that $p_n(t) \neq 0$ on Γ_0 and which approximates uniformly the function f, such that f can be represented in the following way $f = p_n(1 - m)$, and for $m \in C(\Gamma_0)$ the following relations are true:

$$\sup_{t \in \Gamma_0} |m(t)| < \frac{1}{2}, \tag{19}$$

$$-\frac{1}{4}\delta < \arg(1 - m(t)) < \frac{\delta}{4}. \tag{20}$$

We mention the fact that relation (20) can be obtained by choosing the polynomial p_n in such a way that for the function m the value $\sup_{t \in \Gamma_0} |m(t)|$ is sufficiently small.

We define the function u in the following way $u(t) = (1 - m(t)) \exp(i(b(t) - c(t)))$, $t \in \Gamma_0$. Then $u \in PC(\Gamma_0)$. As the function f from relation (18) is equal to $p_n(1 - m)$, we conclude that $a(t) = \exp(i\theta(t)) = f(t) \exp(i(b(t) - c(t))) = p_n(t)u(t)$. Since $p_n \in C(\Gamma_0)$, $p_n(t) \neq 0$ on Γ_0, and the function a is 2-nonsingular, from the last relation it results that the function u is 2-nonsingular and it has the same discontinuities as a. From relations (17) and (20) we obtain $|\arg u(t)| < \pi/2 - \delta/4$, and from (19) we obtain $|u(t)| \geq 1/2$. In such a way values of the function u are situated in a semi-plane of the line $Re\, u(t) \geq \delta_0 > 0\,(t \in \Gamma_0)$. More exactly, the values $u(t)$ for all $t \in \Gamma_0$ are situated in the triangular sector as it is indicated on the figure.



Evidently, by the similarity transformation with the coefficient $\gamma\,(> 0)$ this sector can be translated into a sector all points of which are distant from point 1 with the distance which is less than 1. So, a number $\gamma > 0$ can be chosen such that for all $t \in \Gamma_0$ the values $\gamma u(t)$ belong to the unit circle and $\|1 - \gamma u\|_\infty = \sup_{t \in \Gamma_0} |1 - \gamma u| \leq q < 1$. Now we set $r_n(t) = \gamma^{-1} p_n(t)$, $h(t) = \gamma u(t)$. As $a(t) = r_n(t)h(t)$, it results that the lemma is proved.

Corollary 1. *According to Lemma 5, each 2-nonsingular function $a(t) \in PC(\Gamma_0)$ can be represented in the form*

$$a(t) = r_n(t)(g(t) + 1), \tag{21}$$

where $r_n \in \mathcal{P}_n$, and the function $g \in PC(\Gamma_0)$ satisfies the condition $\|g\|_\infty = \sup_{t\in\Gamma_0} |g(t)| \le q < 1$. So, we have $ind_2(g(t) + 1) = 0$, and, as $r_n(t)$ and $g(t) + 1$ do not have common discontinuity points, we obtain that $ind_2 a(t) = ind\, r_n(t)$.

4 The formulation and the proof of the convergence theorems

Let equation (3) have a unique solution, i.e. the operator A that describes the left term of the given equation is invertible in $L_2(\Gamma_0)$. We will show that this condition is sufficient for the convergence of the collocation and quadrature methods applied to this equation.

The integral operator K with the weak singularity (see equation (1)) is completely continuous in the space $L_2(\Gamma_0)$ [1].

Let the operator $M = aP + bQ \in L(L_2(\Gamma_0))$ be invertible. Then M is nöetherian and $Ind\, M = 0$, that implies the nöetherian character of the operator $A = M + K$ and the condition $Ind\, A = Ind\, M = 0$ [2, p.145]. Let $\dim\ker A = 0$. Then, as $Ind\, A = \dim\ker A - \dim co\ker A$, we obtain that $\dim co\ker A = 0$, and thus $Im A = L_2(\Gamma_0)$, that implies the invertibility of the operator A in $L_2(\Gamma_0)$.

Taking into account all the mentioned above and the necessary and sufficient conditions of invertibility of the operator M (see [12, 13]), the following results about convergence of the collocation and quadrature methods can be formulated:

Theorem 1. *Let the following conditions be true:*

1) $a_0(t), b_0(t) \in PC(\Gamma_0)$, $f(t) \in R(\Gamma_0)$, $h_0(t,\tau) \in C(\Gamma_0 \times \Gamma_0)$*;*

2) (i) $b(t \pm 0) \neq 0$, $t \in \Gamma_0$*; (ii)* $\hat{c}(t,\mu) \neq 0$, $(t,\mu) \in \Gamma_0 \times [0,1]$, where $c = ab^{-1}$;

3) *The number* $k := ind_2 c(t) = 0$*;*

4) $\dim\ker A = 0$*;*

5) *Nodes t_j $(j = \overline{-n,n})$ are calculated according to formula (2).*

Then, for sufficiently small ρ $(\varepsilon_\rho \le q_1 < 1)$ and for sufficiently large n $(n \ge n_0)$, system (6) has a unique solution α_k $(k = \overline{-n,n})$. The approximate solutions $\varphi_n(t)$, constructed according to formula (5), converge when $\rho \to 0$ and $n \to \infty$ to exact solution $\varphi(t)$ of equation (1) in the norm of the space $L_2(\Gamma_0)$ $\lim_{\rho\to 0}\lim_{n\to\infty} \|\varphi - \varphi_n\|_2 = 0$.

Theorem 2. *Let all conditions of Theorem 1 be true with the exception of $h_0(t,\tau) \in H_\alpha(\Gamma_0 \times \Gamma_0)$, where H_α is the Banach space of all functions that satisfy Hölder condition on Γ_0 (see, for example [4, 6]). Then the affirmations of Theorem 1 are true with the condition that SLAE (6) is changed with SLAE (7).*

Proof of Theorem 1. According to condition (ii) we have that the function $c \in PC(\Gamma_0)$ is 2-nonsingular. Then according to the above corollary c is represented in the form (21) and $ind_2 c(t) = ind\, r_n(t)$. From condition 3) it results that $ind\, r_n(t) = 0$.

As the trigonometric polynomial $r_n(t) \neq 0$ on Γ_0, it can be represented in the form (see [14, p.30])

$$r_n(t) = \prod_{j=1}^{n+k} \left(1 - t_j^+ t^{-1}\right) t^k \prod_{j=1}^{n-k} \left(t - t_j^-\right), \tag{22}$$

where $k = ind\, r_n(t)$, and t_j^+ $(j = \overline{1, n+k})$ $(t_j^-$ $(j = \overline{1, n-k}))$ are all zeroes (taking into account their multiplicity) of the polynomial $r_n(t)$ which belong to the domain D^+ (the domain D^-). As polynomials

$$r_{n+k}^-(t) = \prod_{j=1}^{n+k} \left(1 - t_j^+ t^{-1}\right), \;\; r_{n-k}^+(t) = \prod_{j=1}^{n-k} \left(t - t_j^-\right) \tag{23}$$

satisfy conditions $r_{n+k}^-(t) \neq 0,\, t \in D^- \cup \Gamma_0,\, r_{n-k}^+(t) \neq 0,\, t \in D^+ \cup \Gamma_0$, and $(r_{n+k}^-)^{\pm 1}$ (respectively $(r_{n-k}^+)^{\pm 1}$) are analytical in D^- (respectively in D^+), and $k = ind\, r_n(t) = 0$, we obtain that equality (22) is a canonic factorization of the polynomial r_n with respect to the closed contour Γ_0

$$r_n(t) = r_n^-(t) r_n^+(t). \tag{24}$$

Taking into account properties of polynomials (23) (for $k = 0$) and the equality $P + Q = I$, we obtain that $P r_n^- Q \varphi_n = Q (r_n^+)^{-1} P \varphi_n = 0$. From this we have

$$P (r_n^+)^{\pm 1} P = (r_n^+)^{\pm 1} P; \;\; P (r_n^-)^{\pm 1} P = P (r_n^-)^{\pm 1}. \tag{25}$$

The condition (i) implies the existence of the inverse of the function $b \in PC(\Gamma_0)$. We have that $b^{-1} \in PC(\Gamma_0)$. Relations (21), (24) imply for the 2-nonsingular function $c = ab^{-1} \in PC(\Gamma_0)$ (a, b are coefficients of equation (3)) the representation

$$c(t) = r_n^-(t) r_n^+(t) h(t), \tag{26}$$

where $h(t) = g(t) + 1$ is a function that possesses properties described above (in Lemma 5).

According to equality (26) equation (4) is equivalent to the equation

$$h(t) r_n^-(t) (P\varphi)(t) + (r_n^+(t))^{-1} (Q\varphi)(t) + (r_n^+(t))^{-1} b^{-1}(t) (K_\rho \varphi)(t) = f_1(t), \tag{27}$$

where $f_1 = (r_n^+)^{-1} b^{-1} f \in R(\Gamma_0)$. Thus, system (6) is equivalent to the system $h(t_j) r_n^-(t_j) (P\varphi_n)(t_j) + (r_n^+(t_j))^{-1} (Q\varphi_n)(t_j) + (r_n^+(t_j))^{-1} b^{-1}(t_j) (K_\rho \varphi_n)(t_j) = f_1(t_j), j = \overline{-n, n}$. As the last system is equivalent to the following operator equation

$$L_n (h r_n^- P + (r_n^+)^{-1} Q + (r_n^+)^{-1} b^{-1} K_\rho) S_n \varphi_n = L_n f_1, \; \varphi_n \in \mathcal{P}_n, \tag{28}$$

equations (9) and (28) are equivalent. Thus, the invertibility of the operator $L_n(aP+bQ+K_\rho)S_n$ implies the invertibility of $L_n(hr_n^-P+(r_n^+)^{-1}Q+(r_n^+)^{-1}b^{-1}K_\rho)S_n$ and vice versa.

Using relations (25), we obtain

$$hr_n^-P+(r_n^+)^{-1}Q = hPr_n^-P+hQr_n^-P+P(r_n^+)^{-1}Q+Q(r_n^+)^{-1}Q =$$

$$= hPr_n^- + Q(r_n^+)^{-1}+hQr_n^-P+P(r_n^+)^{-1}Q,$$

and, as

$$hPr_n^- + Q(r_n^+)^{-1} = (hP+Q)(Pr_n^-+Q(r_n^+)^{-1}) = (I+gP)(Pr_n^-+Q(r_n^+)^{-1})$$

is true, it results that equation (28) has the form

$$L_n\Big((I+gP)(Pr_n^-+Q(r_n^+)^{-1})+hQr_n^-P+P(r_n^+)^{-1}Q+(r_n^+)^{-1}b^{-1}K_\rho\Big)S_n\varphi_n = L_nf_1.$$

Introducing notations $V = (I+gP)(Pr_n^-+Q(r_n^+)^{-1})$, $K_1 = (r_n^+)^{-1}b^{-1}K_\rho$, $K_2 = hQr_n^-P+P(r_n^+)^{-1}Q$, the last equation is written in the following form

$$L_n(V+K_1+K_2)S_n\varphi_n = L_nf_1. \tag{29}$$

We will show that for sufficiently large n, the operator $L_n(V+K_1+K_2)S_n$, defined by the left term of equation (29), is invertible as an operator that acts from $\mathcal{P}_n$ to $\mathcal{P}_n$, and approximate solutions φ_n converge to the solution φ_ρ of equation (4). Toward this end we will show that for sufficiently large values n all conditions of the following known affirmation about the relation between convergence manifolds of operators C and $C+T$, where T is a complete continuous operator (see [4, p.22; 15, p.432]) are true.

Let X, Y be Banach spaces, and $\{P_n\}, \{Q_n\}\,(n=1,2,\ldots)$ are two sequences of projectors with domains $D(P_n)\subset X$, $D(Q_n)\subset Y$ and closed images $Im\,P_n\subset X$, $Im\,Q_n\subset Y$. By $L(X,Y)$ we will denote the Banach algebra of all linear and bounded operators that acts from X to Y, and by $\mathcal{K}(X,Y)$ - the ideal of all complete continuous operators that acts from X to Y. By $GL(X,Y)$ we denote the set of all invertible elements of $L(X,Y)$.

Lemma 6. *Let the operator $C\in GL(X,Y)$, for $n\ge n_0$ the relation $C(Im\,P_n)\subset D(Q_n)$ be true and operators $Q_nCP_n\in GL(Im\,P_n, Im\,Q_n)$. Let Z be a Banach space that is continuously embedded in Y, such that $Z\subset \mathcal{L}(C,P_n,Q_n) := \{f\in Y : f\in D(Q_n),\, n\ge n_1(f),\, \|C^{-1}f-(Q_nCP_n)^{-1}Q_nf\|_X\to 0\}$ - the convergence manifold of the operator C after the system of projectors Q_n and P_n. Also, let $T\in\mathcal{K}(X,Z)$ and the following two conditions be true:*

$$1)\dim Ker(C+T)=0; \qquad 2)\ Q_n|_Z\in L(Z,Y).$$

Then the operators $Q_n(C+T)P_n\in GL(Im\,P_n, Im\,Q_n)$ for $n\ge n_2$ and the equality $\mathcal{L}(C,P_n,Q_n)=\mathcal{L}(C+T,P_n,Q_n)$ holds.

We set $X = Y = L_2(\Gamma_0)$, $Q_n = L_n$ $P_n = S_n$ $C = V$ $D(Q_n) = R(\Gamma_0)$, $Z = R(\Gamma_0)$, $T = K_1 + K_2$. Let us show that all conditions of the lemma take place.

Using relations (23) and (25), it can be easily verified that the operator $B = Pr_n^- + Q(r_n^+)^{-1}$ is invertible in $L_2(\Gamma_0)$, with the inverse operator $B^{-1} = P(r_n^-)^{-1} + Qr_n^+$.

As $\|S\|_2 = 1$ and $P = (I + S)/2$ is a projector, it results that $\|P\|_2 = 1$. From here we have $\|gP\|_2 \le \|g\|_2 \le \|g\|_\infty$, and, as $\|g\|_\infty < 1$ (see Corollary 1), it results that the operator $D = I + gP$ is invertible in $L_2(\Gamma_0)$. In such a way the invertibility of operators B and D implies the invertibility of the operator V in $L_2(\Gamma_0)$.

Lemma 7. *The inclusion $B\mathcal{P}_n \subseteq \mathcal{P}_n$ takes place.*

Proof. Let $x_n(t) = \sum\limits_{k=-n}^{n} q_k t^k$ be an arbitrary polynomial from $\mathcal{P}_n$. As $r_n^-(t) = \sum\limits_{k=-n}^{0} l_k t^k$ and $\left(r_n^+(t)\right)^{-1} = \sum\limits_{k=0}^{\infty} m_k t^k$. It results that $r_n^-(t)x_n(t) = \sum\limits_{k=-n}^{0} l_k t^k \sum\limits_{j=-n}^{n} q_j t^j = \sum\limits_{k=-2n}^{n} n_k t^k$, and $\left(r_n^+(t)\right)^{-1} x_n(t) = \sum\limits_{k=0}^{\infty} m_k t^k \sum\limits_{j=-n}^{n} q_j t^j = \sum\limits_{k=-n}^{\infty} s_k t^k$. Then we have $P(r_n^- x_n) = \sum\limits_{k=0}^{n} n_k t^k$ and $Q\left((r_n^+)^{-1} x_n\right) = \sum\limits_{k=-n}^{-1} s_k t^k$, from which we obtain $P(r_n^- x_n) + Q\left((r_n^+)^{-1} x_n\right) = \sum\limits_{k=0}^{n} n_k t^k + \sum\limits_{k=-n}^{-1} s_k t^k \in \mathcal{P}_n$. Thus, $B\mathcal{P}_n \subseteq \mathcal{P}_n$ takes place, and the lemma is proved.

On the basis of this result and thanks to the fact that $PC(\Gamma_0)$ is an algebra, we obtain $\forall x_n \in \mathcal{P}_n$, $Vx_n = DBx_n = (I + gP)y_n \in PC(\Gamma_0) \subset R(\Gamma_0) = D(Q_n)$. As the operators $L_n V S_n$ are linear and $dim \mathcal{P}_n < \infty$, it results that they are bounded as operators that act in $\mathcal{P}_n$.

We consider the operator $D_n = L_n(I + gP)S_n \in L(\mathcal{P}_n)$. Using the evident relations $S_n x_n = x_n$, $Px_n \in \mathcal{P}_n$, and $\|Px_n\|_2 \le \|x_n\|_2$, where $x_n \in \mathcal{P}_n$, as well as relation (16), we obtain $\|L_n(I + gP)S_n x_n\|_2 = \|(S_n + L_n gPS_n)x_n\|_2 = \|x_n + L_n gPx_n\|_2 \ge \|x_n\|_2 - \|L_n gPx_n\|_2 \ge \|x_n\|_2 - \|g\|_\infty \|Px_n\|_2 \ge \|x_n\|_2 - \|g\|_\infty \|x_n\|_2 = (1 - \|g\|_\infty)\|x_n\|_2$, $\forall x_n \in \mathcal{P}_n$. The constant $C = 1 - \|g\|_\infty > 0$, because $\|g\|_\infty \le q < 1$, therefore the operator D_n is bounded below in $\mathcal{P}_n$. As $Im\, D_n = \mathcal{P}_n$, according to the known criterion of invertibility (see [16, p.209]), the operator D_n is invertible in $\mathcal{P}_n$. At the same time the following inequality is true:

$$\|x_n\|_2 \le \frac{1}{1 - \|g\|_\infty} \|(S_n + L_n gPS_n)x_n\|_2,\; x_n \in \mathcal{P}_n. \tag{30}$$

The relation $Bx_n \in \mathcal{P}_n\, (x_n \in \mathcal{P}_n)$, implies the representation $L_n V S_n = L_n(I + gP)S_n(Pr_n^- + Q(r_n^+)^{-1}) = D_n B$, and the invertibility of the operators D_n and B in $\mathcal{P}_n$ implies the invertibility of the operator $L_n V S_n$ in $\mathcal{P}_n$.

The Banach space $R(\Gamma_0)$ is included continuously in $L_2(\Gamma_0)$. We will show that $R(\Gamma_0) \subset \mathcal{L}(V, S_n, L_n)$.

As it was shown above, the operators V and L_nVS_n are invertible respectively in $L_2(\Gamma_0)$ and in $\mathcal{P}_n$, with the inverse operators $V^{-1} = (P(r_n^-)^{-1} + Qr_n^+)(I+gP)^{-1}$ and $(L_nVS_n)^{-1} = (P(r_n^-)^{-1} + Qr_n^+)(L_n(I+gP)S_n)^{-1}$. Then for $f_1 \in R(\Gamma_0)$ we have

$$\|V^{-1}f_1 - (L_nVS_n)^{-1}L_nf_1\|_2 \leq$$
$$\leq \|P(r_n^-)^{-1} + Qr_n^+\|_2\|(I+gP)^{-1}f_1 - (L_n(I+gP)S_n)^{-1}L_nf_1\|_2. \tag{31}$$

Let

$$\psi = (I+gP)^{-1}f_1 \in L_2(\Gamma_0), \tag{32}$$
$$\psi_n = (L_n(I+gP)S_n)^{-1}L_nf_1 \in \mathcal{P}_n. \tag{33}$$

Evidently, the following relation is true:

$$\|\psi_n - \psi\|_2 \leq \|S_n\psi - \psi_n\|_2 + \|S_n\psi - \psi\|_2. \tag{34}$$

As $S_n\psi - \psi_n \in \mathcal{P}_n$, using consecutively inequalities (30) and (33), we obtain for the first term from the right term of inequality (34)

$$\|S_n\psi - \psi_n\|_2 \leq \frac{1}{1-\|g\|_\infty}\|(S_n + L_ngPS_n)(S_n\psi - \psi_n)\|_2 =$$
$$= \frac{1}{1-\|g\|_\infty}\|(S_n + L_ngPS_n)\psi - L_nf_1\|_2. \tag{35}$$

From relation (32) we obtain $\psi = f_1 - gP\psi$. Then we have $(S_n + L_ngPS_n)\psi = S_nf_1 - S_ngP\psi + L_ngPS_n\psi$, but in relation (35)

$$\|S_n\psi - \psi_n\|_2 \leq \frac{1}{1-\|g\|_\infty}\|S_nf_1 - S_ngP\psi + L_ngPS_n\psi - L_nf_1\|_2 \leq \frac{1}{1-\|g\|_\infty}\times$$
$$\times(\|L_ngPS_n\psi - gP\psi\|_2 + \|S_ngP\psi - gP\psi\|_2 + \|S_nf_1 - f_1\|_2 + \|L_nf_1 - f_1\|_2). \tag{36}$$

Lemma 8. *For every $x \in L_2(\Gamma_0)$*

$$\|L_ngPS_nx - gPx\|_2 \to 0. \tag{37}$$

Proof. For the proof of the lemma we will use the Banach-Steinhaus theorem [16, p.271]. Consecutively using relations (16), $\|Px_n\|_2 \leq \|x_n\|_2$ ($x_n \in \mathcal{P}_n$) and $\|S_n\|_2 = 1$, we obtain for every $x \in L_2(\Gamma_0)$, $\|L_ngPS_nx\|_2 \leq \|g\|_\infty\|PS_nx\|_2 \leq \|g\|_\infty\|S_nx\|_2 \leq \|g\|_\infty\|x\|_2 := c_x < \infty$. Such sequence of operators $L_ngPS_n : L_2(\Gamma_0) \to \mathcal{P}_n$ is simply bounded. As $L_2(\Gamma_0)$ is the Banach space, it results that the sequence L_ngPS_n is uniformly bounded (see [16, p.269, Theorem 1]) $\|L_ngPS_n\|_2 \leq const$, $n = 1, 2, \ldots$.

If $\mathcal{P}_m = \{x_m(t) = \sum\limits_{k=-m}^{m} s_kt^k | s_k \in \mathbb{C}\}$ is the set of trigonometrical polynomials of order m ($m \geq 0$), defined on Γ_0, the set $\bigcup_{m=0}^\infty \mathcal{P}_m$ is dense in $L_2(\Gamma_0)$. If $x \in \bigcup_{k=0}^\infty \mathcal{P}_k$, then there exists m such that $x = x_m \in \mathcal{P}_m$, and it is true that $S_nx_m = x_m$ for

$n \geq m$. The inclusions $g \in PC(\Gamma_0) \subset R(\Gamma_0)$ and $Px_m \in \mathcal{P}_m \subset R(\Gamma_0)$ imply the fact that $gPx_m \in R(\Gamma_0)$. Then according to relation (14) it results:

$$\|L_n gPS_n x_m - gPx_m\|_2 = \|L_n gPx_m - gPx_m\|_2 \to 0, \; \forall x_m \in \bigcup_{k=0}^{\infty} \mathcal{P}_k.$$

On the basis of all the mentioned above, according to the Banach-Steinhaus theorem, we have that

$$\|L_n gPS_n x - gPx\|_2 \to 0, \; \forall x \in L_2(\Gamma_0).$$

The lemma is proved.

As $\psi \in L_2(\Gamma_0)$, relation (37) implies

$$\|L_n gPS_n \psi - gP\psi\|_2 \to 0. \tag{38}$$

Let $L_\infty(\Gamma_0)$ be a Banach algebra of all essentially bounded functions on Γ_0. An alternative characterization for this space is $L_\infty(\Gamma_0) = \{\varphi \in L_2(\Gamma_0) : \varphi f \in L_2(\Gamma_0), \forall f \in L_2(\Gamma_0)\}$ [17, p.39]. Then, as $g \in PC(\Gamma_0) \subset L_\infty(\Gamma_0)$ and $P\psi \in L_2(\Gamma_0)$, we have $gP\psi \in L_2(\Gamma_0)$, and thus (see [11, 16])

$$\|S_n gP\psi - gP\psi\|_2 \to 0. \tag{39}$$

Analogously, as $f_1 \in R(\Gamma_0) \subset L_2(\Gamma_0)$, we have

$$\|S_n f_1 - f_1\|_2 \to 0, \tag{40}$$

and according to relation (14),

$$\|L_n f_1 - f_1\|_2 \to 0. \tag{41}$$

Using relations (38)–(41), we obtain from (36) $\|S_n\psi - \psi_n\|_2 \to 0$. The last relation with $\|S_n\psi - \psi\|_2 \to 0$, implies in (34) $\|\psi_n - \psi\|_2 \to 0$, i.e. $\|(L_n(I + gP)S_n)^{-1}L_n f_1 - (I + gP)^{-1} f_1\|_2 \to 0$. As the operator $P(r_n^-)^{-1} + Qr_n^+$ is bounded in $L_2(\Gamma_0)$, from (31) we obtain:

$$\|V^{-1} f_1 - (L_n V S_n)^{-1} L_n f_1\|_2 \to 0, \; \forall f_1 \in R(\Gamma_0).$$

In such a way the inclusion $R(\Gamma_0) \subset \mathcal{L}(V, S_n, L_n)$ takes place.

It is easy to verify (see [18, p.96]) that for every $x \in L_2(\Gamma_0)$, the functions $Qr_n^- Px$, $P(r_n^+)^{-1}Qx$ are continuous on Γ_0. Taking into account item b) from Lemma 1 we obtain that the bounded operators $K_1 = (r_n^+)^{-1} b^{-1} K_\rho$ and $K_2 = hQr_n^- P + P(r_n^+)^{-1}Q$, where $(r_n^+)^{-1}b^{-1}$, $h \in PC(\Gamma_0)$, act from $L_2(\Gamma_0)$ to $PC(\Gamma_0) \subset R(\Gamma_0)$. As the operator K_ρ is completely continuous (see item c) of Lemma 1) and equalities $Qr_n^- P = \frac{1}{2}(r_n^- S - Sr_n^-)$, $P(r_n^+)^{-1}Q = -\frac{1}{2}((r_n^+)^{-1}S - S(r_n^+)^{-1})$, are true, and r_n^-, $(r_n^+)^{-1}$ are continuous functions on Γ_0, we obtain that the operators K_1, K_2 are completely continuous (see [2, p.33]) $K_1, K_2 \in \mathcal{K}(L_2(\Gamma_0), R(\Gamma_0))$.

The conditions 1)–4) of the convergence theorem assure the invertibility of the operator A of equation (3). Then, according to Lemma 2, if relation (13) is true, the operator A_ρ is invertible as well, which implies the relation $dimKerA_\rho = 0$. As $A_\rho = V + K_1 + K_2$, we obtain that $dimKer(V + K_1 + K_2) = 0$. As it was mentioned above, the operator $L_n \in L(R(\Gamma_0), L_2(\Gamma_0))$.

In such a way all conditions of Lemma 6 about the lineal of convergence of the operator $V + K_1 + K_2$ are verified, and according to it we obtain that the operators $L_n(V + K_1 + K_2)S_n : \mathcal{P}_n \to \mathcal{P}_n$ are invertible for sufficiently large n and $\mathcal{L}(V, S_n, L_n) = \mathcal{L}(V+K_1+K_2, S_n, L_n)$ is true. Then $R(\Gamma_0) \subset \mathcal{L}(V+K_1+K_2, S_n, L_n)$ and, as equations (29), (9) and, respectively, (27), (4), are equivalent, we obtain that equation (9) for sufficiently large n has a unique solution, and the approximate solutions φ_n converge to the exact solution φ_ρ of equation (4) $\|\varphi_\rho - \varphi_n\|_2 \to 0$. From here and from Lemma 2, using the relation $\|\varphi-\varphi_n\|_2 \le \|\varphi-\varphi_\rho\|_2 + \|\varphi_\rho-\varphi_n\|_2$, we obtain $\|\varphi - \varphi_n\|_2 \to 0$ when $n \to \infty$ and $\rho \to 0$.

In such a way Theorem 1 is proved.

Proof of Theorem 2. It is easy to verify that equation (10) is equivalent to the following operator equation

$$(A_n + \Gamma_n)\tilde{\varphi}_n = L_n f, \tag{42}$$

where $A_n\tilde{\varphi}_n = L_n(aP + bQ + K_\rho)\tilde{\varphi}_n$, $\Gamma_n\tilde{\varphi}_n = -L_n(K_\rho - \Delta_n)\tilde{\varphi}_n$, and the operators K_ρ and Δ_n were defined above.

Of course, $A_n\tilde{\varphi}_n = L_n f$ is an operator equation of the collocation method studied above. So, equation (42), which describes the quadrature method, can be interpreted as the perturbation of the equation of the collocation method.

In such a way to state the convergence of the quadrature method we will use the following lemma about the stability in the sense of Mikhlin of the approximation method [4, p.31; 15, p.438].

Let X, Y be Banach spaces, and $\{P_n\}, \{Q_n\}$ be sequences of projectors considered in Lemma 6.

Lemma 9. *Let $A \in GL(X, Y)$ and $A_n := Q_nAP_n \in GL(ImP_n, ImQ_n)$ $(n \ge n_0)$, and Z is a Banach space which is continuously embedded in Y in such a way that $ImQ_n \subset Z \subset \mathcal{L}(A, P_n, Q_n), Q_n|_Z \in L(Z, Y)$, and let $y \in Z$.*

Then there exist positive constants p, γ which do not depend on n and y in such a way that for the operator $R_n \in L(ImP_n, ImQ_n)$, which verifies the relation

$$\|R_n\|_{X \to Z} < \gamma, \tag{43}$$

we have

1) The equation

$$(A_n + R_n)\tilde{x}_n = Q_n y \tag{44}$$

has the unique solution $(n \ge n_0)$;

2) For solutions $\tilde{x}_n, x_n \in ImP_n$ of equation (44) and, respectively $A_n x_n = Q_n y$, the estimation

$$\|\tilde{x}_n - x_n\|_X \leq p\|y\|_Z \|R_n\|_{X\to Z} \tag{45}$$

holds.

We set $X = Y = L_2(\Gamma_0),\ Z = R(\Gamma_0)4,\ Q_n = L_n, P_n = S_n,\ A = A_\rho,\ R_n = \Gamma_n$, $y = f$. The conditions of the last lemma with the exception of relation (43) were already verified in the proof of Theorem 1. Evidently $\Gamma_n \in L(ImS_n, ImL_n)$. We will show that for Γ_n condition (43) holds and even more, $\|\Gamma_n\|_{L_2\to R} \to 0$ when $n \to \infty$. In such conditions, taking into account estimation (45), we have $\|\tilde{x}_n - x_n\|_2 \to 0$ when $n \to \infty$.

Taking into account the identity $\int\limits_{\Gamma_0} L_n^\tau(h_\rho(t,\tau)\varphi_n(\tau))d\tau = \int\limits_{\Gamma_0} \frac{1}{\tau} L_n^\tau[\tau h_\rho(t,\tau)]\varphi_n(\tau)d\tau$, $\forall \varphi_n \in \mathcal{P}_n$ (see [18, p.72]), and the fact that $(\Delta_n\varphi_n)(t) - (K_\rho\varphi_n)(t) \in C(\Gamma_0)$, as well as Hölder inequalities, we obtain $\|(K_\rho\varphi_n)(t) - (\Delta_n\varphi_n)(t)\|_C = (2\pi)^{-1}\max\limits_{t\in\Gamma_0}\left|\int\limits_{\Gamma_0}\tau^{-1}(\tau h_\rho(t,\tau) - L_n^\tau[\tau h_\rho(t,\tau)])\varphi_n(\tau)d\tau\right| \leq (2\pi)^{-1}\max\limits_{t\in\Gamma_0}\int\limits_{\Gamma_0}|\tau h_\rho(t,\tau) - L_n^\tau[\tau h_\rho(t,\tau)]|\,|\varphi_n(\tau)|\,|d\tau| \leq (2\pi)^{-1}\max\limits_{t\in\Gamma_0}\|\tau h_\rho(t,\tau) - L_n^\tau[\tau h_\rho(t,\tau)]\|_2\|\varphi_n\|_2$. As $t \mapsto \|\tau h_\rho(t,\tau) - L_n^\tau[\tau h_\rho(t,\tau)]\|_2$ is a continuous function on $\Gamma_0, \exists\, t_n \in \Gamma_0$ such that $\max\limits_{t\in\Gamma_0}\|\tau h_\rho(t,\tau) - L_n^\tau[\tau h_\rho(t,\tau)]\|_2 = \|\tau h_\rho(t_n,\tau) - L_n^\tau[\tau h_\rho(t_n,\tau)]\|_2$. As $\tau h_\rho(t,\tau) \in C(\Gamma_0)$ by τ, using the relation $\|g - L_n g\|_2 \leq 2E_n(g), \forall g \in C(\Gamma_0)$ (see [19, p.63]), we obtain $\|\tau h_\rho(t_n,\tau) - L_n^\tau[\tau h_\rho(t_n,\tau)]\|_2 \leq 2E_n^\tau(\tau h_\rho(t_n,\tau))$. According to Jackson theorem in $C(\Gamma_0)$ (see [19, p.43]) we have $E_n^\tau(\tau h_\rho(t_n,\tau)) \leq 12\omega^\tau(\tau h_\rho; \dfrac{1}{n+1}) \leq 12\omega^\tau(h_\rho; \dfrac{1}{n+1})$, where $\omega(g;\delta)$ is the modulus of continuity of the function $g(t)$.

In such a way we have $\|(K_\rho\varphi_n)(t) - (\Delta_n\varphi_n)(t)\|_C \leq \dfrac{12}{\pi}\omega^\tau(h_\rho; \dfrac{1}{n+1})\|\varphi_n\|_2$. Then taking into account the estimation $\|L_n\|_C \leq d_1 \ln n$ (see [19, p.49]), we have $\|\Gamma_n\varphi_n\|_C \leq \|L_n\|_C\|(K_\rho\varphi_n)(t) - (\Delta_n\varphi_n)(t)\|_C \leq d_2 \ln n\, \omega^\tau(h_\rho; \dfrac{1}{n+1})\|\varphi_n\|_2$.

Taking into account that $h_\rho(t,\tau) \in H_\delta(\Gamma_0\times\Gamma_0), \delta = \min(\alpha,\gamma)$ (see [20, p.22; 6, p.10]), we have $\omega^\tau\left(h_\rho; \dfrac{1}{n+1}\right) = \sup\limits_{|\tau'-\tau''|\leq\frac{1}{n+1}}|h_\rho(t,\tau') - h_\rho(t,\tau'')| \leq \sup\limits_{|\tau'-\tau''|\leq\frac{1}{n+1}} d_3|\tau'-\tau''|^\delta = d_3\dfrac{1}{(n+1)^\delta}$. Consequently, as $\Gamma_n\varphi_n \in C(\Gamma_0)$, we have $\|\Gamma_n\varphi_n\|_{R(\Gamma_0)} = \|\Gamma_n\varphi_n\|_{C(\Gamma_0)}$ and then $\|\Gamma_n\|_{L_2\to R} \leq d_4 n^{-\delta}\ln n \to 0$ when $n \to \infty$.

In virtue of Theorem 1, the equation $A_n\varphi_n = L_n f$, which describes the collocation method, for all sufficiently large n, has the unique solution φ_n and $\|\varphi_n - \varphi\| \to 0$ when $n \to \infty$ (φ is the solution of equation (1)). Applying Lemma 9 we obtain that equation (42) (equivalent to equation (10)) has the unique solution $\tilde{\varphi}_n$ and $\|\tilde{\varphi}_n - \varphi_n\| \to 0$ $(n \to \infty)$.

Then $\|\tilde{\varphi}_n - \varphi\| \leq \|\tilde{\varphi}_n - \varphi_n\| + \|\varphi_n - \varphi\| \to 0$ when $n \to \infty$ and in such a way we convince of the verity of statements of Theorem 2.

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