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Factorization theorems for some spaces of analytic functions

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Abstract. We provide several factorization theorems for different subspaces of the space of all analytic functions in the unit disk, in particular we prove a strong factorization theorem for Classical Hardy classes with Muckenhoupt weights. Proofs are based on a new weighted version of Coifman–Meyer–Stein theorem on factorization of tent spaces and on properties of an extremal outher function,which was constructed by E. Dynkin.

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1 Introduction

The aim of this note is to provide several factorization theorems for different subspaces of the $H(\mathbb{D})$ space, where $\mathbb{D}$ is the unit disk on the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ is the space of all holomorphic in the unit disk functions. Let us mention several vital known results in that direction.

In [1] such theorem was proved by Gorowitz for B^p_α-Bergman spaces. Much later similar result was proven in [2] by W. Cohn and in much more general form by W. Cohn and I. Verbitsky in [3]. Such theorems are playing very important role in different questions in the theory of analytic functions.

2 Definition and main results

In order to formulate the main results of the paper we will give several definitions. Let Z, X and Y be subspaces of $H(\mathbb{D})$. We will say that Z admits strong factorization g from Z can be represented as a product $g = f_1 f_2$, where $f_1 \in X$, $f_2 \in Y$, and the reverse is also true: for any $f_1 \in X$ and $f_2 \in Y$ we have $f_1 f_2 \in Z$, so $Z = XY$.

Let $T = \{z : |z| = 1\}$ be the boundary of $\mathbb{D}$,

$$\Gamma_\alpha(\xi) = \{z : |1 - \bar{\xi} z| \le \alpha(1 - |z|)\}, \quad \alpha > 1,$$

$dm(\xi)$ and $dm_2(z)$ are normalized Lebesgue measures on the boundary T and in the unit disk $\mathbb{D}$,

$$H^p(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \sup_{r \in (0,1)} M_p(f, r) < \infty, p \in (0, \infty] \right\}.$$

Hardy spaces for $p \in (0, \infty]$, where

$$M_p^p(f, r) = \int_T |\ f(r\xi)\ |^p\ dm(\xi), \quad r \in [0, 1),$$

$$\text{Let} : \Box I = \big\{ z = r\xi : \xi \in I, 1 - |I| \le r < 1 \big\},$$

where I is an arc on T, $|I|$ is a length of the arc. Let further $T(E) = \big\{(x,t) \in \mathbb{R}_+^{n+1}, B(x,t) \subset E\big\}$ be a tent on $E, E \subset \mathbb{R}^n$ (see [3]), for example E can be a ball $E = B(x, r)$ in $\mathbb{R}^n$ with center at x and radius r. Denote as usual by $A^p(\mathbb{R}^n)$, $p \in [1, \infty)$ all measurable functions in $\mathbb{R}^n w(x)$ such that w is belonging to Muckenhoupt class (see [5]). $A^p(T)$ is a Muckenhoupt class on T. Further let

$$(A_q f)(\xi) = \left(\int_{\Gamma_\alpha(\xi)} \frac{|f(z)|^q}{(1 - |z|)^2} dm_2(z) \right)^{\frac{1}{q}},$$

$$\big(\tilde{T}_q^\infty(w)\big) = \left\{ f \text{ measurable in } \mathbb{D} : \sup_{I \subset T} \left(\int_I w^{\frac{q}{q-p}} dy \right) \left(\int_{T(I)} \frac{|f(z)|^q}{1 - |z|} dm_2(z) \right) < \infty \right\},$$

$$\big(C_q^q f\big)(\xi) = \sup_{\xi \in I} \left(\frac{1}{|I|} \right) \left(\int_{\Box I} \frac{|f(z)|^q}{(1 - |z|)} dm_2(z) \right),$$

$$\big(A_\infty f\big)(\xi) = \sup_{\Gamma_\alpha(\xi)} \big\{ |f(z)| : z \in \Gamma_\alpha(\xi) \big\},\ I \subset T, I \text{ is an arc.}$$

Theorem 1. *Let* $p < q$, $s > 0$, $w_1 = w^{\frac{q}{q-p}}$, $w_1 \in L^1_{loc}$ *and* $w_1 \in A^1(T)$. *Then* $(HT^p_{s,q})(w) = (H^p(w_1))(HT^\infty_{s,q}(w))$ *where*

$$\big(HT^p_{s,q}\big)(w) = \big\{ f \in H(\mathbb{D}) : \|A_q\big(f(z)(1 - |z|)^s\big)\|_{L^p(w)} < \infty \big\},$$

$$\big(HT^\infty_{s,q}\big)(w) = \big\{ f \in H(\mathbb{D}) : f(z)(1 - |z|)^s \in \tilde{T}_q^\infty(w) \big\},$$

$$\big(H^p\big)(w_1) = \left\{ f \in H(\mathbb{D}) : \int_T |\big(A_\infty f\big)(\xi)|^p w_1(\xi) d\xi < \infty, \quad 0 < p < \infty \right\},$$

and moreover if $F = F_1 F_2$, *then* $\|F_1\|_{H^p(w_1)} \le c\|F\|_{HT^p_{s,q}}$ *and* $\|F_2\|_{HT^\infty_{s,q}} \le 1$.

Remark 1. The pair $\big(\omega^{-\frac{p}{q-p}}, \omega^{\frac{q}{q-p}}\big)$ can be changed in Theorem 1 to $\big(\omega^{\tau_1}, \omega^{\tau_2}\big)$, $\tau_1 + \tau_2 = 1$.

The proof of Theorem 1 relies on the following extension of Coifman–Meyer–Stein theotem on factorization of tent spaces and some ideas from the article of W. Cohn and I. Verbitsky.

Let $\Gamma(\xi)$ be Luzin cone in $\mathbb{R}^n$ [3].

Theorem 2. *Let* $0 < p < q$, $\omega_1 = \omega^{\frac{q}{q-p}}$, $\omega_1 \in L^1_{loc}$, $\omega_1 \in A^1(\mathbb{R}^n)$. *Then the following equality holds*

$$\tilde{T}^p_q(\omega) = \tilde{T}^p_\infty\big(\omega^{\frac{q}{q-p}}\big)\big(\tilde{T}^\infty_q(\omega)\big),$$

where

$$\tilde{T}^\infty_q(\omega) = \Big\{ f \quad \textit{is measurable in} \quad \mathbb{R}^n :$$

$$\sup_B \left(\int_B \omega^{\frac{q}{q-p}} dy\right)^{-1} \int_{T(B)} \frac{|f(x,t)|^q}{t} dx\ dt < \infty \Big\},$$

$$\tilde{T}^p_\infty(\phi) = \Big\{ f \quad \textit{is measurable in} \quad \mathbb{R}^n : \big\| \big(A_\infty f\big)(x)\big(\phi(x)\big)^{\frac{1}{p}} \big\|_{L^p(\mathbb{R}^n)} < \infty \Big\},$$

$$\tilde{T}^p_q(\omega) = \Big\{ f \quad \textit{is measurable in} \quad \mathbb{R}^n :$$

$$\int_{\mathbb{R}^n} \left(\int_{\Gamma(\xi)} \frac{|f(y,t)|^q}{t^{n+1}} dy\ dt\right)^{\frac{p}{q}} \omega(\xi) d\xi < \infty, \quad 0 < p, q < \infty \Big\},$$

where ω *is a locally integrable function,* $\omega \in L^1_{loc}(\mathbb{R}^n)$.

Remark 2. Theorem 0.2 for $\omega = const$ is known and was proved in [3] and [4].

Remark 3. Note that many known spaces of holomorphic functions can be represented by T^p_q, $0 < p, q < \infty$, spaces. So such factorization are very useful in different problems, connected with the theory of spaces of analytic functions [4, 5].

We are going to formulate two theorems in similar direction. The proof relies on the existence of extremal outer function, that was constructed by Dynkin in [6].

Let

$$\big(F^{\infty,q}_{s,p,k}\big) = \big\{ f \in H(\mathbb{D}) : |\tilde{\mathbb{D}}^k f(z)|^q \big(1 - |z|\big)^{(k-s)q-1} - p \ \text{ is Carleson measure } \big\},$$

where $k \in \mathbb{R}$, $k > s$, $s \in \mathbb{R}$, $q \in (0, \infty)$,

$$\big(\tilde{\mathbb{D}}^\alpha f\big)(z) = \sum_{k \geq 0} (k+1)^\alpha a_k z^k, \quad \alpha \in \mathbb{R}, \qquad f(z) = \sum_{k \geq 0} a_k z^k$$

is a fractional derivate and a positive Borel measure μ in $\mathbb{D}$ is a p-Carleson measure if

$$\Big\| \sup_{\xi \in I} \frac{1}{|I|^p} \int_{\square I} d\mu(z) \Big\|_{L^\infty(T)} = \big\|\phi(\xi)\big\|^{\infty(T)} < \infty, \quad 0 < p \leq 1.$$

Below $L^{p,q}(T)$ are Lorentz spaces on T. In order to formulate our next theorem we need the following notation. We will write $\|f\|_X \subseteq Y \cdot Z$, X, Y, Z are subspaces of $H(\mathbb{D})$, if any functions f, $\|f\|_X < \infty$ can be writen in the following form $f = (f_1)(f_2)$, $f_1 \in Y$, $f_2 \in Z$.

Let

$$\Lambda^s \overset{def}{=} \Big\{ f \in H(\mathbb{D}) : \sup_{|z|<1} |f'(z)| \big(1-|z|\big)^{1-s} < \infty \Big\}, \quad s \in (0,1) \text{ be the Goelder space.}$$

Theorem 3. *Let Y, Z be subspaces of $H(\mathbb{D})$.*

$$\textit{Let}\ \ (Y)\Big(F^{\infty,q}_{-\frac{s}{q},1-q,k}\Big) \subset Z, \quad q \in (0,1),\ s \in (0,1),$$

then $Y \subset (\Lambda^s)(Z)$.

Theorem 4.

(*i*) *Let $s > 0$, $q > 1$, $f \in H(\mathbb{D})$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then*

$$\left\| \left(\int_{\Gamma_\alpha(\xi)} \Big| f(z) \Big|^{2q'} \big(1-|z|\big)^{s-2)} dm_2(z) \right)^{\frac{1}{2}} \right\|_{L^{q,1}} \subseteq (\Lambda^s)\Big(\tilde{S}^{2(q')^{-1},2(q')^{-1}}_{\frac{sq'}{2}}\Big), \quad s < 1.$$

(*ii*) *Let $v > 0$, $q > 1$, $f \in H(\mathbb{D})$, $t \geq 0$ and $v - t = s \in (0,1)$. Then*

$$\left\| \sup_{z \in \Gamma_\alpha(\xi)} |f(z)|^{q'} \big(1-|z|\big)^t \right\|_{L^{q,1}} \subseteq (\Lambda^s)\Big(\tilde{S}^{(q')^{-1},(q')^{-1}}_{vq'}\Big),$$

where

$$\tilde{S}^{p,q}_s = \Big\{ f \in H(\mathbb{D}) : \int_0^1 \big(M_p(f,|z|)\big)^q \big(1-|z|\big)^{sq-1} d|z| < \infty \Big\}, \quad p,q,s, \in (0,\infty).$$

The proofs of these theorems will be presented elsewhere. Here we indicate that some ideas from [3] are being used.

References

[1] Gorowitz Ch. *Factorization theorems for functions in the Bergman space.* Duke Math. Journal, 1977, **44**, p. 201–213.

[2] Cohn W. *A factorization theorem for the derivative of a function on Hp.* Proc. AMS, 1999, **127**, N 2, p. 509–517.

[3] Cohn W., Verbitsky I. *Factorization of Tent spaces and Hankel operators.* Journal of Functional Analysis, 2000, **175**, p. 308–329.

[4] Coifman R.., Meyer Y., Stein E. *Some new functional spaces and their application to harmonic analysis.* Journal of Functional Analysis, 1985, p. 304–335.

[5] Grafakos L. *Classical and Modern Fourier Analysis.* Prentice Hall, 2003.

[6] Dynkin E. *Sets of Free interpolation for Hoelder classes.* Mat. Sbornik, 1979, **109**, N 1, p. 107–128.

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