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Nearly simple elementary divisor domains

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Abstract. It is proved that a nearly simple Bezout domain is an elementary divisor ring if and only if it is 2-simple.

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1 Introduction

According to Kaplansky's definition [1], a ring R is an elementary divisor ring if every matrix over R is equivalent to a diagonal matrix with condition of complete divisibility of the diagonal elements. In [2] Zabavsky proved that a simple Bezout domain is an elementary divisor ring if and only if it is 2-simple. Nearly simple domains were constructed in [3–6]. We prove that a nearly simple Bezout domain is an elementary divisor ring if and only if it is 2-simple.

2 Definitions

Throughout R will always denote a ring (associative, but not necessarily commutative) with $1 \neq 0$. We shall write R_n for the ring of $n \times n$ matrices with elements in R. By a unit of ring we mean an element with two-sided inverse. We'll say that matrix is unimodular if it is the unit of R_n. We denote by $GL_n(R)$ the group of units of R_n. The Jacobson radical of a ring R is denoted by $J(R)$.

An n by m matrix $A = (a_{ij})$ is said to be diagonal if $a_{ij} = 0$ for all $i \neq j$. We say that a matrix A admits a diagonal reduction if there exist unimodular matrices $P \in GL_n(R)$, $Q \in GL_m(R)$ such that PAQ is a diagonal matrix. We shall call two matrices A and B over a ring R equivalent (and write $A \sim B$) if there exist unimodular matrices P, Q such that $B = PAQ$. If every matrix over R is equivalent to a diagonal matrix (d_{ij}) with the property that every d_{ii} is a total divisor of $d_{i+1,i+1}$ ($Rd_{i+1,i+1}R \subseteq d_{ii}R \cap Rd_{ii}$), then R is an elementary divisor ring. We recall that a ring R is said to be right (left) Hermite if every 1 by 2 (2 by 1) matrix admits a diagonal reduction, and if both, R is an Hermite ring. By a right (left) Bezout ring we mean a ring in which all finitely generated right (left) ideals are principal, and by a Bezout ring a ring which is both right and left Bezout [1].

In any simple ring the property $RaR = R$ holds for every element $a \in R \setminus \{0\}$ and some depends on a. As R is a ring with identity then there exist elements

$u_1, \dots u_k, v_1, \dots, v_k$ such that $u_1 a v_1 + \dots + u_k a v_k = 1$. If the same integer n can be chosen for all nonzero elements a with $u_1 a v_1 + \dots + u_n a v_n = 1$ we say that a ring R is n-simple [3]. For example the full n by n matrix ring over a field K (even a skew field) is n-simple. A nearly simple ring is a ring in which case R, $J(R)$ and (0) are its only ideals.

3 Main result

Main result is the next theorem.

Theorem. *Let R be a nearly simple Bezout domain. Then R is an elementary divisor domain if and only if R is 2-simple domain.*

Proof. If $J(R) = (0)$ then R is a simple domain and the result follows by [2].

If $J(R) \neq (0)$ and R is an elementary divisor domain then it is enough to consider the matrix A of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

where $a \in J(R) \setminus \{0\}$. Since R is an elementary divisor domain there exist matrices $P = (p_{ij}) \in GL_2(R)$ and $Q = (q_{ij}) \in GL_2(R)$ such that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} P = Q \begin{pmatrix} z & 0 \\ 0 & b \end{pmatrix}, \tag{1}$$

where $RbR \subseteq zR \cap Rz$ for some $z, b \in R$.

Let's consider the ideal RbR. Since R is a nearly simple domain, we obtain three chances:

1) $RbR = \{0\}$;
2) $RbR = R$;
3) $RbR = J(R)$.

1) Let $RbR = \{0\}$ then $b = 0$. From (1) we have

$$a p_{12} = q_{12} b, \quad a p_{22} = q_{22} b. \tag{2}$$

Since $b = 0$ and (2),

$$a p_{12} = 0, \quad a p_{22} = 0. \tag{3}$$

As $a \neq 0$ and R is a domain then $p_{12} = p_{22} = 0$, this case is impossible.

2) Let $RbR = R$. Since $RbR \subseteq zR \cap Rz$, z is a unit of the domain R. Then from (1) we obtain that $z \in RaR$. Since $a \in J(R)$, $z \in J(R)$. And this case is impossible too.

3) Let $RbR = J(R)$. Since $RbR \subseteq zR \cap Rz$, $a \in J(R)$ and (1), $z \in J(R)$. Then $J(R) = zR = Rz$. Also $z^2 R = R z^2$ takes place. Then $z^2 R = R z^2 = J(R) = zR = Rz$, that is impossible as R is a domain and $z \in J(R)$.

The proof is completed.

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