Nearly simple elementary divisor domains

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Abstract. It is proved that a nearly simple Bezout domain is an elementary divisor ring if and only if it is 2-simple.


Keywords and phrases: Bezout domain, elementary divisor ring, 2-simple domain.

1 Introduction

According to Kaplansky’s definition [1], a ring \( R \) is an elementary divisor ring if every matrix over \( R \) is equivalent to a diagonal matrix with condition of complete divisibility of the diagonal elements. In [2] Zabavsky proved that a simple Bezout domain is an elementary divisor ring if and only if it is 2-simple. Nearly simple domains were constructed in [3–6]. We prove that a nearly simple Bezout domain is an elementary divisor ring if and only if it is 2-simple.

2 Definitions

Throughout \( R \) will always denote a ring (associative, but not necessarily commutative) with \( 1 \neq 0 \). We shall write \( R_n \) for the ring of \( n \times n \) matrices with elements in \( R \). By a unit of ring we mean an element with two-sided inverse. We’ll say that matrix is unimodular if it is the unit of \( R_n \). We denote by \( GL_n(R) \) the group of units of \( R_n \). The Jacobson radical of a ring \( R \) is denoted by \( J(R) \).

An \( n \) by \( m \) matrix \( A = (a_{ij}) \) is said to be diagonal if \( a_{ij} = 0 \) for all \( i \neq j \). We say that a matrix \( A \) admits a diagonal reduction if there exist unimodular matrices \( P \in GL_n(R), Q \in GL_m(R) \) such that \( PAQ \) is a diagonal matrix. We shall call two matrices \( A \) and \( B \) over a ring \( R \) equivalent (and write \( A \sim B \)) if there exist unimodular matrices \( P, Q \) such that \( B = PAQ \). If every matrix over \( R \) is equivalent to a diagonal matrix \( (d_{ij}) \) with the property that every \( d_{ii} \) is a total divisor of \( d_{i+1,i+1} \) \( (Rd_{i+1,i+1} \subseteq d_{ii}R \cap Rd_{ii}) \), then \( R \) is an elementary divisor ring. We recall that a ring \( R \) is said to be right (left) Hermite if every 1 by 2 (2 by 1) matrix admits a diagonal reduction, and if both, \( R \) is an Hermite ring. By a right (left) Bezout ring we mean a ring in which all finitely generated right (left) ideals are principal, and by a Bezout ring a ring which is both right and left Bezout [1].

In any simple ring the property \( RaR = R \) holds for every element \( a \in R \setminus \{0\} \) and some depends on \( a \). As \( R \) is a ring with identity then there exist elements...
$u_1, \ldots, u_k, v_1, \ldots, v_k$ such that $u_1 a v_1 + \ldots + u_k a v_k = 1$. If the same integer $n$ can be chosen for all nonzero elements $a$ with $u_1 a v_1 + \ldots + u_n a v_n = 1$ we say that a ring $R$ is $n$-simple [3]. For example the full $n$ by $n$ matrix ring over a field $K$ (even a skew field) is $n$-simple. A nearly simple ring is a ring in which case $R$, $J(R)$ and $(0)$ are its only ideals.

3 Main result

Main result is the next theorem.

**Theorem.** Let $R$ be a nearly simple Bezout domain. Then $R$ is an elementary divisor domain if and only if $R$ is 2-simple domain.

**Proof.** If $J(R) = (0)$ then $R$ is a simple domain and the result follows by [2].

If $J(R) \neq (0)$ and $R$ is an elementary divisor domain then it is enough to consider the matrix $A$ of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},$$

where $a \in J(R) \setminus \{0\}$. Since $R$ is an elementary divisor domain there exist matrices $P = (p_{ij}) \in GL_2(R)$ and $Q = (q_{ij}) \in GL_2(R)$ such that

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} P = Q \begin{pmatrix} z & 0 \\ 0 & b \end{pmatrix},$$

(1)

where $RbR \subseteq zR \cap Rz$ for some $z, b \in R$.

Let’s consider the ideal $RbR$. Since $R$ is a nearly simple domain, we obtain three chances:

1) $RbR = \{0\}$;
2) $RbR = R$;
3) $RbR = J(R)$.

1) Let $RbR = \{0\}$ then $b = 0$. From (1) we have

$$ap_{12} = q_{12} b, \quad ap_{22} = q_{22} b.$$  \hspace{1cm} (2)

Since $b = 0$ and (2),

$$ap_{12} = 0, \quad ap_{22} = 0.$$  \hspace{1cm} (3)

As $a \neq 0$ and $R$ is a domain then $p_{12} = p_{22} = 0$, this case is impossible.

2) Let $RbR = R$. Since $RbR \subseteq zR \cap Rz$, $z$ is a unit of the domain $R$. Then from (1) we obtain that $z \in RaR$. Since $a \in J(R)$, $z \in J(R)$. And this case is impossible too.

3) Let $RbR = J(R)$. Since $RbR \subseteq zR \cap Rz$, $a \in J(R)$ and (1), $z \in J(R)$. Then $J(R) = zR = Rz$. Also $z^2R = Rz^2$ takes place. Then $z^2R = Rz^2 = J(R) = zR = Rz$, that is impossible as $R$ is a domain and $z \in J(R)$.

The proof is completed.
References


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